Problem 1. *On the propagation and generation of gravitational waves*
Point distribution

The scope of this problem is to help you understand the properties of gravitational waves, whose discovery has recently been rewarded with the 2017 Nobel Prize in Physics. Part I of the problem will analyze the propagation of gravitational waves in close analogy with electromagnetic waves, while Part II will be concerned with estimating the amount of energy gravitational waves carry from the merger of two black holes.

Part I. [6 pts.] Numerous phenomena that occur when studying gravitational waves closely resemble those in the world of electromagnetism. This is due to the similarity between solutions to the Maxwell equations in vacuum,

$$\vec{\nabla}\cdot\mathbf{E} = \frac{\partial\vec{\mathbf{E}}_x}{\partial x} + \frac{\partial\vec{\mathbf{E}}_y}{\partial y} + \frac{\partial\vec{\mathbf{E}}_z}{\partial z} = 0\,, \tag{1}$$

$$\vec{\nabla}\cdot\vec{\mathbf{B}} = \frac{\partial\vec{\mathbf{B}}_x}{\partial x} + \frac{\partial\vec{\mathbf{B}}_y}{\partial y} + \frac{\partial\vec{\mathbf{B}}_z}{\partial z} = 0\,, \tag{2}$$

$$\vec{\nabla}\times\mathbf{E} = \left(\frac{\partial\vec{\mathbf{E}}_z}{\partial y} - \frac{\partial\vec{\mathbf{E}}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial\vec{\mathbf{E}}_x}{\partial z} - \frac{\partial\vec{\mathbf{E}}_z}{\partial x}\right)\hat{y} + \left(\frac{\partial\vec{\mathbf{E}}_y}{\partial x} - \frac{\partial\vec{\mathbf{E}}_x}{\partial y}\right)\hat{z} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}\,, \tag{3}$$

$$\vec{\nabla}\times\vec{\mathbf{B}} = \left(\frac{\partial\vec{\mathbf{B}}_z}{\partial y} - \frac{\partial\vec{\mathbf{B}}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial\vec{\mathbf{B}}_x}{\partial z} - \frac{\partial\vec{\mathbf{B}}_z}{\partial x}\right)\hat{y} + \left(\frac{\partial\vec{\mathbf{B}}_y}{\partial x} - \frac{\partial\vec{\mathbf{B}}_x}{\partial y}\right)\hat{z} = \frac{1}{c^2}\frac{\partial\vec{\mathbf{E}}}{\partial t}\,, \tag{4}$$

which describe electromagnetic waves propagating freely, and solutions to the equation for gravitational waves propagating in vacuum,

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{h}_{ij} + \nabla^2\mathbf{h}_{ij} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{h}_{ij} + \left(\frac{\partial^2\mathbf{h}_{ij}}{\partial x^2} + \frac{\partial^2\mathbf{h}_{ij}}{\partial y^2} + \frac{\partial^2\mathbf{h}_{ij}}{\partial z^2}\right) = 0 \tag{5}$$

Above, c is the speed of light and $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are the electric and magnetic field, respectively. $\mathbf{h}$ can be understood as a 3×3 matrix whose elements represent perturbations in the "fabric" of space. The matrix needs to be symmetric ($\mathbf{h}_{ij} = \mathbf{h}_{ji}$), with each line and column representing a direction in space: specifically, $i, j = \overline{1,3}$ and 1 corresponds to x, 2 to y and 3 to z. Thus, for instance, the element $\mathbf{h}_{11}$ is associated to the xx component of the wave, $\mathbf{h}_{12}$ is associated to the xy component of the wave, etc. The gravitational wave equation (5) thus applies independently to every element of the matrix $\mathbf{h}$.

a) [2 pts.] Show that for an electromagnetic wave propagating in the z-direction, $\vec{\mathbf{E}}(\vec{r},t) = E_0 e^{i(\omega t - kz)}\hat{x}$ is a solution to Maxwell's equations in vacuum (1–4), where $\hat{x}$ is the unit vector in the x-direction, and find the associated magnetic field $\vec{\mathbf{B}}(\vec{r},t)$. Along the way, prove that $\vec{\mathbf{B}}(\vec{r},t)$ it needs to be pointing in the y-direction. Furthermore show that there are solutions for which $\vec{k} \times \vec{\mathbf{E}}(\vec{r}, t) = \omega \vec{\mathbf{B}}(\vec{r}, t)$.

Points: Showing that $\vec{\mathbf{E}}(\vec{r}, t) = E_0 e^{i(\omega t - kz)} \hat{x}$ satisfies Maxwell's first equation [0.4pts]. Finding the associated magnetic field,

$$\vec{\mathbf{B}}(\vec{r}, t) = \left(\frac{k}{\omega} E_0 e^{i(\omega t - kz)} \right) \hat{y} + \vec{C} \tag{6}$$

is worth [0.5 pts]. Dealing with the constant vector $\vec{C}$ correctly will be worth [0.1 pts] of the [0.5 pts]. Checking that all other Maxwell equations are satisfied is worth [0.6 pts] ([0.2 pts] per remaining Maxwell equation). Proving that the magnetic field points in the $\hat{y}$ direction is worth [0.2pts]. Proving that $\vec{k} \times \vec{\mathbf{E}}(\vec{r}, t) = \omega \vec{\mathbf{B}}(\vec{r}, t)$ is worth [0.3pts].

b) [1 pt.] Prove that the equation for gravitational waves also has a similar oscillating solution propagating in the z-direction, $\mathbf{h} = \mathbf{h}_0 e^{i(\omega t - kz)}$, where $\mathbf{h}_0$ is a 3×3 matrix of constants.

Points: Obtaining $\partial^2 \mathbf{h} / \partial^2 t$ correctly is worth [0.4 pts]. Obtaining $\nabla^2 \mathbf{h}$ correctly is worth [0.4 pts]. Arriving to the correct conclusion is worth another [0.2 pts].

c) [0.5 pts.] Show that in vacuum both electromagnetic waves and gravitational waves propagate at the speed of light.

Points: Interpreting ω/k as the wave velocity gives [0.3pts]. Arriving to the correct conclusions for electromagnetic and gravitational waves are worth [0.1 pts] each.

d) [1 pt.] Similar to how Maxwell's equations impose a constraint that relates the electric field, $\vec{\mathbf{E}}$, to the magnetic field $\vec{\mathbf{B}}$, there are supplementary constrains that can be imposed for the matrix $\mathbf{h}$. Specifically, for a wave propagating in the z-direction one can impose that any component of the matrix related to the z-direction has to vanish ($\mathbf{h}_{i3} = \mathbf{h}_{3i} = 0$ for $i = \overline{1, 3}$), and that the matrix is traceless ($\mathbf{h}_{11} + \mathbf{h}_{22} + \mathbf{h}_{33} = 0$). Use these properties, together with the fact that the matrix is symmetric, to show that $\mathbf{h}_0$ only has two independent components. Show that $\mathbf{h}_0$ can be expressed as,

$$\mathbf{h}_0 = h_+ \epsilon_+ + h_\times \epsilon_\times \ , \tag{7}$$

where h_+ and $h_\times$ are two independent constants and ϵ_+ and $\epsilon_\times$ are given by,

$$\epsilon_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \epsilon_\times = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{8}$$

Points: Counting the number of degrees of freedom correctly is worth [0.6 pts] ([0.2 pts] for each step of the three steps in the solution). Identifying the correct form of the matrix is worth [0.4 pts].

e) [1.5 pts.] In quantum mechanics, the spin of a particle is given by a number s that describes the symmetry of the wave solution associated with the propagation of this particle. Specifically, the spin is the maximum number, s, for which the wave solution is invariant under rotations of $360^0/s$ around the direction of propagation. The three dimensional rotation matrix that performs a rotation of angle θ around the z-axis is given by,

$$\mathbf{U} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

and transforms the electric and magnetic fields as, $\vec{\mathbf{E}} \to \mathbf{U} \cdot \vec{\mathbf{E}}$, and, $\vec{\mathbf{B}} \to \mathbf{U} \cdot \vec{\mathbf{B}}$, respectively.[1] The matrix $\mathbf{h}$ transforms as, $\mathbf{h} \to \mathbf{U}\,\mathbf{h}\,\mathbf{U}^T$, where $\mathbf{U}^T$ is the transpose of $\mathbf{U}$.[2] Find the spin of the photon (the particle associated to electromagnetic waves) and that the graviton (associated to gravitational waves).

Points: Computing $\mathbf{U} \cdot \vec{\mathbf{E}}$ correctly for general θ is worth [0.4 pts]. Computing $\mathbf{U}\,\mathbf{h}\,\mathbf{U}^T$ correctly for general θ is worth [0.8 pts]. Arriving to the right conclusion, that the spin of the photon is 1 and that of the graviton is 2 is worth [0.3pts]. If one only shows that there is a symmetry for $\theta = 2\pi$ for electromagnetic waves and $\theta = \pi$ for gravitational waves but does not study the symmetries for general θ the solution will only receive [1 pt].

Part II. [4 pts.] The black holes, whose merger was recently detected by the winners of the 2017 Nobel Prize in Physics, are extremely massive objects with strange physical properties. The first detection of gravitational waves came from the merger of two black holes each with a mass equal to $M = 30$ solar masses located $d = 1.3 \times 10^9$ light-years from earth (1 solar mass is 1.98×10^{30} kg and 1 light year is 9.46×10^{15} meters).

a) [1.5 pts.] Assuming that the merger of the two black holes occurs while the black holes are close to being at rest and generates only gravitational waves, use the mass-energy relation to express the energy-flux detected on Earth from those gravitational waves. Express your answer both analytically (in terms of the mass M of each black hole, the distance d and the speed of light c) and numerically.

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[1] Note that when taking the product between the matrix $\mathbf{U}$ and a vector $\vec{\mathbf{E}}$, each element of the resulting vector $\vec{\mathbf{E}}'$ is given by, $(\vec{\mathbf{E}}')_i = \sum_{j=1}^{3} \mathbf{U}_{ij} (\vec{\mathbf{E}})_j$.

[2] Note that when taking the transpose $(U^T)_{ij} = U_{ji}$ for all $i, j = \overline{1,3}$. When taking the product between two 3×3 matrices, $\mathbf{A}$ and $\mathbf{B}$, $(\mathbf{AB})_{ij} = \sum_{l=1}^{3} \mathbf{A}_{il} \mathbf{B}_{lj}$, for every $i, j = \overline{1,3}$.

Points: Using the mass energy formula correctly is worth [0.6 pts]. Obtaining the flux correctly is worth another [0.6 pts]. Finally obtaining the numerical answer correctly is worth [0.3 pts] of which [0.1 pts] are accorded for giving the correct units.

b) [2.5 pts.] The assumption that all the energy from the merger is converted only to gravitational waves can be easily refined using the laws of thermodynamics. Gravitational waves hold only negligible entropy while black holes carry huge amounts of entropy. The entropy of a single black hole of mass M is given by, $S_{BH} = sM^2$, where s is related to fundamental constants. Argue that if two black holes violently collide to make one bigger black hole, then at most 29% of their initial rest energy can be radiated in gravitational waves. Thus, assuming that the two black holes are almost static during the collision, give a numerical upper bound for the energy flux determined on Earth from the black hole collision. Note that the assumption that the two black holes are static should only apply to the numerical result and no to the rest of the problem.

Points: Using the second law of thermodynamics correctly is worth [1.2 pts]. Using the mass energy relation together with the second law to obtain an inequality is worth [0.8 pts]. Arriving to the right conclusion about the energy percentage of gravitational waves is worth another [0.2 pts]. Obtaining the correct numerical bounds is worth [0.3 pts] of which [0.1 pts] are accorded for giving the correct units.