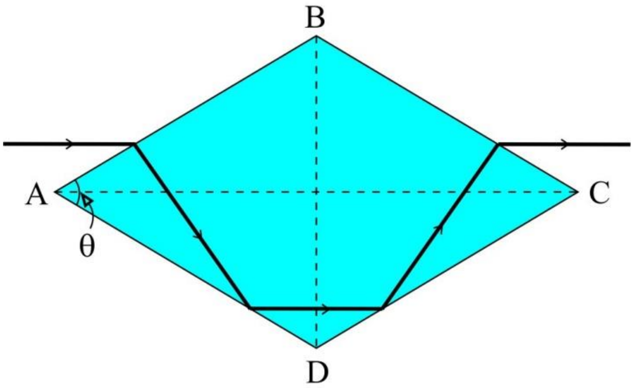
Problem I
Reflection and refraction of light
A. An interesting prism

The main section of a glass prism, situated in air $(n' = 1.00)$, has the form of a rhomb with $\sphericalangle BAD = \sphericalangle BCD \equiv \theta$. A thin yellow beam of monochromatic light, propagating towards the prism, parallel with the diagonal AC of the rhomb, is incident on the face AB (Fig. 1). The beam is totally reflected on the faces AD and DC, then emerges through its face BC. For the yellow radiation, the refraction index of the glass is $n = 1.60$.

Fig. 1



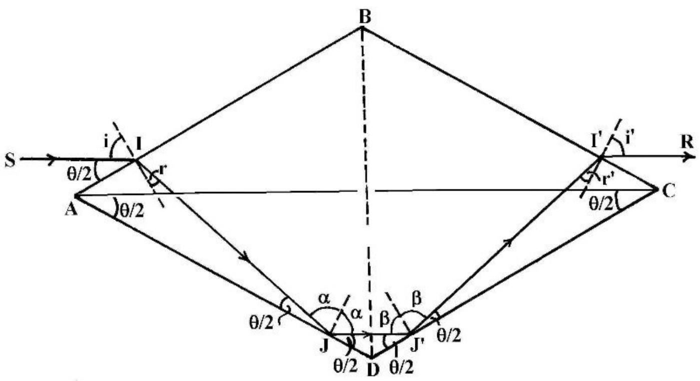
A1	Derive the mathematical expression for the angle θ as a function of the refraction index n of the prism, such that the total deviation of the beam that exits the prism to be zero. Under the above condition, calculate the numerical value of θ in degrees and minutes, if $n = 1.60$.	2.25 p

Solution:

The total deviation is zero only if I'R || SI. This means that $i' = i$ and $r' = r$ (symmetrical path inside of the rhombic prism). Therefore JJ' || AC || SI || I'R..(**0.25 p)**

From the figure below, $m(\widehat{J'JD}) = m(\widehat{JJ'D}) = \frac{\theta}{2}$ and $m(\widehat{AJI}) = m(\widehat{CJ'I'}) = \frac{\theta}{2}$.

In the ΔAIJ : $\theta + \pi/2 + r + \theta/2 = \pi$, so that $r = \pi/2 - 3\theta/2$ …………..……..........…**(0.25 p)**



On the other hand, $i = \pi/2 - \theta/2$ ……………………………………………………………**(0,25 p)**

The refraction law (Snell-Descartes) is $\sin(\pi/2-\theta/2)=n\sin(\pi/2-3\theta/2)$, or $\cos(\theta/2)=n\cos(3\theta/2)$...**(0,50 p)**

Since

$$\cos(3\theta/2)=\cos(\theta+\theta/2)=\cos\theta\cos(\theta/2)-\sin\theta\sin(\theta/2)=$$

$$=[2\cos^2(\theta/2)-1]\cos(\theta/2)-2\sin^2(\theta/2)\cos(\theta/2)=...=4\cos^3(\theta/2)-3\cos(\theta/2).$$

the refraction law becomes $\cos(\theta/2)\left[1-n[4\cos^2(\theta/2)-3]\right]=0$. The solution $\theta=\pi$ (i.e. 180°) cannot be accepted as unphysical, so, the physical solution is

$$\cos(\theta/2)=\sqrt{\frac{3n+1}{4n}}=0.952,$$ ....**(0.75p)**

i.e., $\theta=35°40'$......................................**(0.25 p)**

The prism with θ determined above and the direction of the incident beam remain fixed, but the nature of the light radiation changes, being formed now of the yellow doublet of the mercury. The two wavelengths have the values 579.1 nm, respectively 577 nm. The refraction indices of the glass for these wavelengths are $n=1.60$, respectively $n+\Delta n$, where $\Delta n=1.3\cdot10^{-4}$. The light rays that exit the prism enter longitudinally into an astronomical telescope adjusted for infinite distance.

A2	Derive the mathematical expression for „the angular distance” ε between the two images seen through the telescope (first as a function of θ and Δn, then as a function of n and Δn) and calculate its numerical value.	2.00 p

Solution:

We have a fixed angle of incidence (*i* = constant) and two angles of refraction *r* and *r* - d*r*, corresponding to *n* and *n* + d*n*. From sin *i*= *n* sin *r* we can write 0 = d*n* sin*r* + *n* cos*r* d*r*.

On the other hand, from sin *i*’ = *n* sin *r*’, we can write cos *i*’d*i*’ = d*n* sin *r*’ + *n* cos *r*’d*r*’. Because d*n* (or Δn, as in the text of the question) is a very small (infinitesimal) quantity, we can approximate *i*’= *i* and *r*’= *r*, but with $di'\neq di$ $(=0)$ and $dr'\neq dr$ $(\neq 0)$. .... **(0.50 p)**

We will demonstrate later (see ***Note*** below) that d*r*’= - d*r*.

Now, let us determine the angle between the two emergent rays of light, namely $\varepsilon\equiv di'$, obtaining

$$\varepsilon=\frac{\sin r}{\cos i}dn+n\frac{\cos r}{\cos i}(-dr)\ \text{ with } dr=-\mathrm{tg}r\frac{dn}{n}.$$

The final result is

$$\varepsilon=2\frac{\sin r}{\cos i}dn=2\left[\frac{\cos\left(3\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\right]dn.$$ ....**(0,50 p)**

We know the mathematical expression of cos (θ/2) and, after a little algebra, ε can be expressed in another form, namely

$$\varepsilon=2\frac{dn}{n}\sqrt{\frac{3n+1}{n-1}}$$ ....**(0,50 p)**

Numerically: $\varepsilon = 5.1 \cdot 10^{-4}$ rad (or 1.7')……………………….……………………………**(0.25 p)**
Note: In the triangle JDJ' we write that the sum of inner angles is π (radians), namely (π/2-α) + (π-θ) + (π/2-β) = π, i.e. α+β=π-θ = constant (for a given rhombic prism, with fixed θ). Therefore dα = -dβ, where α = θ+ r (see the triangle AIJ) and β = θ + r' (see the triangle CI'J'). Since θ is fixed (our situation) dα = dr and dβ = dr' so that dr =-dr'..................**(0.25 p)**

A3	If the focal distance of the telescope's objective is $f_{ob} = 0.40\,\mathrm{m}$, derive the linear distance y between the two images, seen in the focal plane of the objective and calculate its numerical value.	0.50 p

Solution:

$y = f_{ob} tg\varepsilon \approx \varepsilon f_{ob} = 2(\frac{dn}{n}) f_{ob} \sqrt{\frac{3n+1}{n-1}}$. Numerically: $y = 0.20$ mm………..……..…….**(0.50 p)**

B. Refraction, but…mostly reflection

B1. Total reflection in geometrical optics

Total reflection occurs when light travels from a medium with refractive index n_1 to another one, with the refractive index $n_2 < n_1$, at an incidence angle $\theta_1 \geq l$, where l is the critical value of the incidence angle, called limit angle, beyond which there will be no refracted light. At total reflection, the entire energy of the incident light beam goes to the reflected beam.

B1	Derive the mathematical expression for the limit angle	0.25 p

Solution:
When the incidence angle θ_1 reaches the limit angle l, the refraction angle will be $\theta_2 = 90^\circ$. In this case, the second Snell's refraction law gives

$$n_1 \sin l = n_2 ,$$

so

$$\boxed{l = \arcsin\frac{n_2}{n_1}.}$$

B2. Total reflection in electromagnetic optics

Electromagnetic optics proves that, besides being totally reflected, the incident light beam also penetrates the less refringent medium as an *evanescent wave*.
The characteristics of the reflected and the refracted light beams depend on the angle of incidence, as well as on the orientation of the electric field of light wave (called polarization). For simplicity, let us consider that the electric field intensity is perpendicular on the incidence plane, as represented in Fig. 2. The indices i, r, and t refer to the incident, reflected and transmitted properties of light wave, while $\vec{k}$ is the wave vector, giving the light propagation orientation. Moreover, $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors of the chosen Cartesian reference frame.

Physical note: The perturbation produced by a plane, monochromatic wave in a point in space at a certain moment of time can be written as $\vec{E}(\vec{r},t)=\vec{E}_0\cos(\omega t-\vec{k}\cdot\vec{r})$, or, to simplify calculations, in the complex form $\vec{e}(\vec{r},t)=\vec{E}_0 e^{i(\omega t-\vec{k}\cdot\vec{r})}$, where $i=\sqrt{-1}$, and then taking only the real part of the result.¶

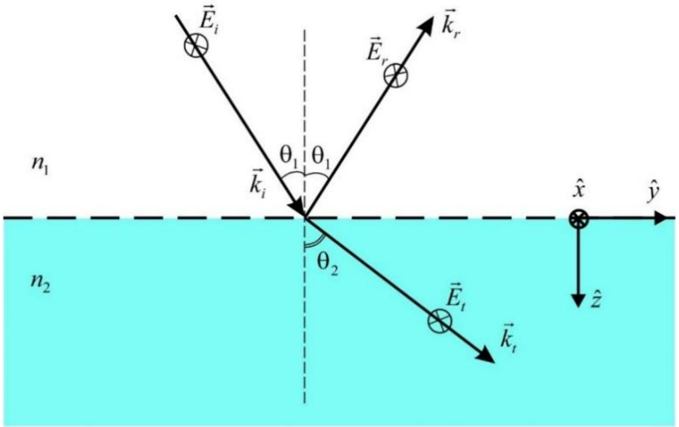


Fig. 2

Mathematical note: For the complex number $z=a\pm ib$, a is the real part and b is the imaginary part. It can be written as $z=\underbrace{\sqrt{a^2+b^2}}_{=|z|}\left(\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{=\cos\varphi}\pm i\underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{=\sin\varphi}\right)=|z|e^{\pm i\varphi}$, where $|z|$ is the modulus of the complex number z and $\tan\varphi=b/a$.

B2.1. Evanescent wave

B2.1	Knowing that the incident wave is a plane and monochromatic one, characterized by the equation $\vec{e}_i(\vec{r},t)=\vec{E}_{0i}e^{i(\omega t-\vec{k}_i\cdot\vec{r})}$, prove that the mathematical expression for the evanescent wave is $\vec{e}_t(\vec{r},t)=\vec{E}_{0t}e^{-\alpha z}e^{i\varphi}$ and derive the exact expression for the *attenuation coefficient* α as a function of the incidence angle θ_1, the limit angle l, and the wavelength λ of the incident wave. Also, derive the exact expression for *the phase* φ of the evanescent wave.	1.50 p

Solution:

For the transmitted wave

$$\vec{e}_t(\vec{r},t)=\vec{E}_{0t}e^{i(\omega t-\vec{k}_t\cdot\vec{r})},$$

where

$$\vec{k}_t\cdot\vec{r}=(\hat{y}k_t\sin\theta_2+\hat{z}k_t\cos\theta_2)\cdot(\hat{y}y+\hat{z}z)=k_t(y\sin\theta_2+z\cos\theta_2).$$

Since

$$n_1\sin\theta_1=n_2\sin\theta_2,$$

then

$$\cos\theta_2 = \pm\sqrt{1-\sin^2\theta_2} = \pm\sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_1} = \pm i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1},$$

because $\theta_1 > l$.

Under these conditions the electric field of the transmitted wave can be written as

$$\vec{e}_t(\vec{r},t) = \vec{E}_{0t} e^{\pm k_t z\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}} e^{i\left(\omega t - k_t y\frac{n_1}{n_2}\sin\theta_1\right)}.$$

The + sign in the first exponential has no physical significance because there is no wave at appreciable distances from the interface. In conclusion, the electric field of the transmitted (evanescent) wave has the form

$$\vec{e}_t(\vec{r},t) = \vec{E}_{0t} e^{-\alpha z} e^{i\varphi},$$

where

$$\alpha = k_t\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}$$

is the *attenuation coefficient* of the evanescent wave

and

$$\varphi = \omega t - k_t y\frac{n_1}{n_2}\sin\theta_1$$

is *the wave's phase*.

This result shows that the wave travels along the interface (along y direction) and that it is attenuated in the z direction (perpendicular on the interface).

Because

$$k_t = \frac{\omega}{v_2} = \frac{\omega}{v_1}\frac{v_1}{v_2} = k_i\frac{\frac{c}{n_1}}{\frac{c}{n_2}} = k_i\frac{n_2}{n_1},$$

then

$$\boxed{\alpha = k_i\sqrt{\sin^2\theta_1 - \frac{n_2^2}{n_1^2}} = \frac{2\pi}{\lambda}\sqrt{\sin^2\theta_1 - \frac{n_2^2}{n_1^2}} = \frac{2\pi}{\lambda}\sqrt{\sin^2\theta_1 - \sin^2 l}.}$$

B2.2. Penetration depth

B2.2	Derive the mathematical expression of the distance Δz from the interface at which the amplitude of the evanescent wave is e times smaller than at the interface, as a function of the incident wavelength λ and calculate its numerical value. The first medium is glass $(n_1 = 1.6)$, the second is air $(n_2 = 1.0)$, and the incidence angle of light is $\theta_1 = 40^\circ$.	0.50 p

Solution:

$$\boxed{\Delta z = \frac{1}{\alpha} = 1.1\lambda .}$$

B2.3. The phase speed of the evanescent wave

B2.3	Derive the mathematical expression for the ratio $\frac{v_e}{v_1}$, where v_e is *the phase speed of the evanescent wave* and v_1 - the phase speed of the incident wave, and compute its numerical value for the case of the incidence angle of light of $\theta_1 = 40^\circ$.	0.75 p

Solution:
Considering the phase of the evanescent wave,

$$\varphi = \omega t - k_t y \frac{n_1}{n_2} \sin\theta_1$$

its phase speed is

$$v_e = \frac{\omega}{k_t \frac{n_1}{n_2} \sin\theta_1} = \frac{\omega}{k_i \sin\theta_1} = \frac{v_1}{\sin\theta_1},$$

so, the requested ratio is

$$\boxed{\frac{v_e}{v_1} = \frac{1}{\sin\theta_1} = 1.6 .}$$

B2.4. The energy transferred from the incident wave to the totally reflected wave

For any value of the incidence angle, the relationship between the amplitude of field of the reflected wave and that of the incident wave was derived by the French physicist Augustin Fresnel (1788 – 1829):

$$E_{0r} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} E_{0i} .$$

Physical note: If the perturbation produced by a wave in a point in space at a given moment is expressed using complex numbers, then the wave intensity has the mathematical expression $I = \frac{1}{2}\varepsilon_0 c E_0^* E_0 = \frac{1}{2}\varepsilon_0 c |E_0|^2$, where $E_0^* = a - ib$ is the complex conjugate of the complex number $E_0 = a + ib$. Here ε_0 is the vacuum permittivity and c is the speed of light in vacuum.

B2.4	Prove that the totally reflected wave has the same intensity as the incident wave.	0.50 p

Solution:
Since

$$n_2\cos\theta_2 = -n_2\sqrt{1-\sin^2\theta_2} = -n_2\sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_1} = -in_2\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1} = -i\frac{n_1\alpha}{k_i},$$

then

$$E_{0r} = \frac{n_1\cos\theta_1 + i\frac{n_1\alpha}{k_i}}{n_1\cos\theta_1 - i\frac{n_1\alpha}{k_i}}E_{0i} = \frac{\cos\theta_1 + i\frac{\alpha}{k_i}}{\cos\theta_1 - i\frac{\alpha}{k_i}}E_{0i}.$$

For any complex number of the same form we can write

$$\frac{a+ib}{a-ib} = \frac{\sqrt{a^2+b^2}\,e^{i\varphi_0}}{\sqrt{a^2+b^2}\,e^{-i\varphi_0}} = e^{2i\varphi_0},$$

where

$$\tan\varphi_0 = \frac{b}{a}.$$

In conclusion

$$E_{0r} = e^{2i\varphi_0}E_{0i},$$

where

$$\tan\varphi_0 = \frac{\alpha}{k_i\cos\theta_1} = \frac{1}{\cos\theta_1}\sqrt{\sin^2\theta_1 - \sin^2 l}.$$

Under these conditions, the intensity of the totally reflected wave will be

$$\boxed{I_r = \frac{1}{2}\varepsilon_0 c\left|E_{0r}\right|^2 = \frac{1}{2}\varepsilon_0 c\left|E_{0i}\right|^2 = I_i.}$$

B2.5. The Goos – Hänchen effect

When an incident wave beam with a finite cross section undergoes total reflection at an interface between two media, the totally reflected wave beam is laterally displaced, on a distance *D* (see Fig. 3), that was measured for the first time by Goos and Hänchen in 1947. In Fig. 3, the displacement along the surface is *s*, and the Goos – Hänchen shift is the lateral shift *D* indicated in the diagram. This is the Goos – Hänchen effect. The explanation of this lateral shift is based on the appearance of the evanescent wave at the interface and its propagation parallel to the interface.

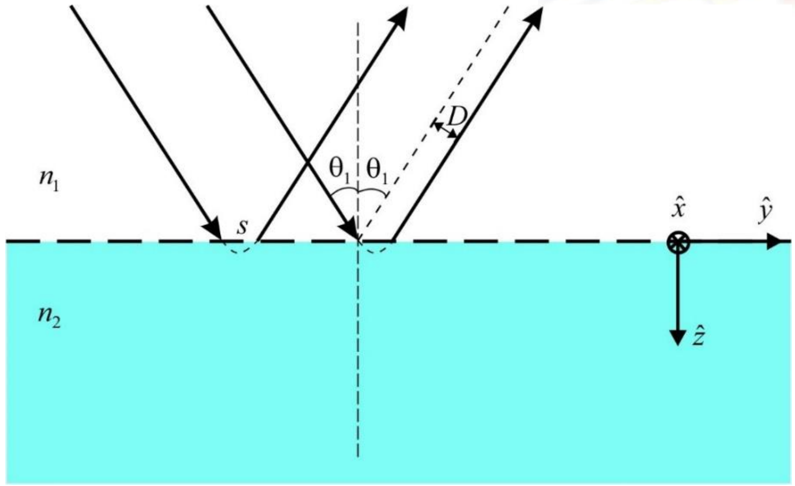


Fig. 3

B2.5.1. The lateral shift

B2.5.1	Derive the mathematical expression for the Goos – Hänchen lateral shift *D*, admitting that the phase difference between the totally reflected wave and the incident one is zero at the interface. Consequently, compute the numerical value of the displacement *s* along the interface as a function of the wavelength λ of the incident light, if the first medium is glass $(n_1 = 1.6)$, the second is air $(n_2 = 1.0)$, and the incidence angle of light is $\theta_1 = 40^\circ$.	1.00 p

Solution:
From Fig. 3, it follows that

$$D = s\cos\theta_1 .$$

If in the incidence point for the incident wave, on the interface, the phase of its electric field is Φ_i, then, at the starting point, for the totally reflected wave its phase is

$$\Phi_r = \Phi_i - k_{iy}s + 2\varphi_0 .$$

With $\Phi_r = \Phi_i$, and knowing that

$$k_{iy} = k_i \sin\theta_1 .$$

we obtain

$$s = \frac{2\varphi}{k_i \sin\theta_1} = \frac{\lambda}{\pi}\frac{1}{\sin\theta_1}\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_1 - \sin^2 l}}{\cos\theta_1}\right).$$

Finally,

$$D = \frac{\lambda}{\pi} \frac{\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_1 - \sin^2 l}}{\cos\theta_1}\right)}{\tan\theta_1}.$$

The numerical value of *s* is $s = 5.5\lambda$.

B2.5.2. Time needed for the total reflection

An alternative explanation of the Goos – Hänchen shift can be given in terms of the time delay associated with the scattering of a radiation pulse at the interface. The incident radiation pulse is not scattered instantaneously by the surface, but reemerges into medium 1 after a time delay τ, during which the pulse propagates parallel to the surface and is displaced by the distance *s*.

B2.5.2.	Derive the mathematical expression for the time delay τ and calculate its value if the first medium is glass $(n_1 = 1.6)$, the second is air $(n_2 = 1.0)$, the incidence angle of light is $\theta_1 = 40^\circ$, and the monochromatic radiation has the wavelength $\lambda = 579.1\,\mathrm{nm}$. The light speed in vacuum is $c = 3.0\cdot 10^8$ m/s.	0.75 p

Solution:

$$\tau = \frac{s}{v_{1y}} = \frac{s}{v_1\sin\theta_1} = \frac{sn_1}{c\sin\theta_1} = \frac{\lambda n_1}{\pi c}\frac{1}{\sin^2\theta_1}\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_1 - \sin^2 l}}{\cos\theta_1}\right) = 2.6\cdot 10^{-14}\ \mathrm{s}.$$

proposed by

Prof. Florea ULIU, PhD
Department of Physics, University of Craiova, ROMANIA

Assoc. Prof. Sebastian POPESCU, PhD
Faculty of Physics, Alexandru Ioan Cuza University of Iași, ROMANIA