PROBLEM 1: MECHANICS

A. Determining the radius of curvature of a planar curve by means of Mechanics

a. 1.0 point

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$$

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = f'(x)\cdot\dot{x}$$

b. 1.0 point

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \dot{v}_x = \ddot{x}$$

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}\left[f'(x)\cdot\dot{x}\right] = f''(x)\cdot\dot{x}^2 + f'(x)\cdot\ddot{x}$$

$$\vec{a}_\mathrm{t} = \frac{\vec{a}\vec{v}}{v}\cdot\frac{\vec{v}}{v} = \frac{a_x v_x + a_y v_y}{\sqrt{v_x^2+v_y^2}}\cdot\frac{v_x\vec{i}+v_y\vec{j}}{\sqrt{v_x^2+v_y^2}} = \frac{v_x\left(a_x v_x + a_y v_y\right)\vec{i} + v_y\left(a_x v_x + a_y v_y\right)\vec{j}}{v_x^2+v_y^2}$$

c. 1.0 point

$$\vec{a}_\mathrm{n} = \vec{a} - \vec{a}_\mathrm{t} =$$

$$= a_x\vec{i} + a_y\vec{j} - \frac{v_x\left(a_x v_x + a_y v_y\right)\vec{i} + v_y\left(a_x v_x + a_y v_y\right)\vec{j}}{v_x^2+v_y^2} =$$

$$= \frac{\left[a_x\left(v_x^2+v_y^2\right) - v_x\left(a_x v_x + a_y v_y\right)\right]\vec{i} + \left[a_y\left(v_x^2+v_y^2\right) - v_y\left(a_x v_x + a_y v_y\right)\right]\vec{j}}{v_x^2+v_y^2} =$$

$$= \frac{\left(a_x v_y^2 - a_y v_x v_y\right)\vec{i} + \left(a_y v_x^2 - a_x v_x v_y\right)\vec{j}}{v_x^2+v_y^2} =$$

$$= \frac{\left(a_x v_y - a_y v_x\right)\left(v_y\vec{i} - v_x\vec{j}\right)}{v_x^2+v_y^2}$$

$$\left|\vec{a}_\mathrm{n}\right| = \frac{\left|a_x v_y - a_y v_x\right|}{\sqrt{v_x^2+v_y^2}}$$

d. **0.5 point**

$$R=\frac{v^2}{a_n}=$$

$$=\frac{\sqrt{v_x^2+v_y^2}^{\,3}}{\left|a_x v_y-a_y v_x\right|}=$$

$$=\frac{\left[\dot{x}^2+\left(f'(x)\cdot\dot{x}\right)^2\right]^{3/2}}{\left|\ddot{x}\cdot f'(x)\cdot\dot{x}-\left(f''(x)\cdot\dot{x}^2+f'(x)\cdot\ddot{x}\right)\cdot\dot{x}\right|}=$$

$$=\frac{\left|\dot{x}\right|^3\left[1+f'^2(x)\right]^{3/2}}{\left|\dot{x}^3\cdot f''(x)\right|}=$$

$$=\frac{\left[1+f'^2(x)\right]^{3/2}}{\left|f''(x)\right|}$$

e. **0.5 point**

$$\left.\begin{aligned}f'(x_0)&=2Ax_0\\ f''(x_0)&=2A\end{aligned}\right\}\Rightarrow R=\frac{\left(1+4A^2x_0^2\right)^{3/2}}{2A}$$

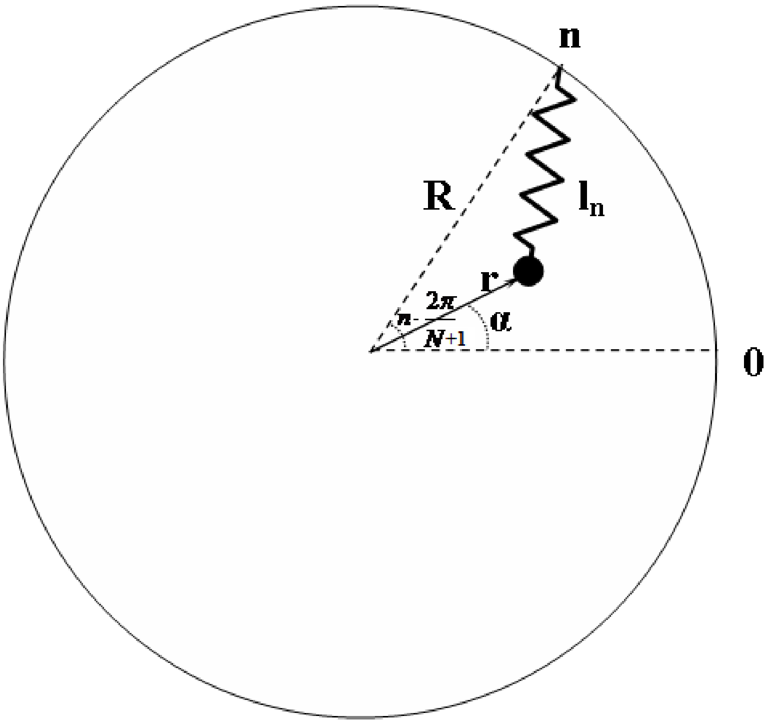
f. **0.5 point**

$$f(x_0)=-1\Rightarrow\sin 2x_0=-1\Rightarrow x_0=\frac{3}{4}\pi$$

$$\left.\begin{aligned}f'(x_0)&=2\cos\left(2\cdot\frac{3}{4}\pi\right)=0\\ f''(x_0)&=-4\sin\left(2\cdot\frac{3}{4}\pi\right)=4\end{aligned}\right\}\Rightarrow R=\frac{1}{4}=0.25\,\mathrm{m}$$

$$T=2\pi\sqrt{\frac{R}{g}}\approx 1\,\mathrm{s}$$

B. Springs on a circle



g. 0.5 point

According to the cosine rule:

$$l_n^2 = R^2 + r^2 - 2Rr\cos\left(\frac{2n\pi}{N+1} - \alpha\right)$$

h. 0.25 point + 0.25 point

$$E_{\text{kin}} = \frac{m\left(\dot{r}^2 + r^2\dot{\alpha}^2\right)}{2}$$

$$E_{\text{pot}} = \sum_{n=0}^{N} \frac{kl_n^2}{2} =$$

$$= \frac{k}{2}\sum_{n=0}^{N}\left[R^2 + r^2 - 2Rr\cos\left(\frac{2n\pi}{N+1} - \alpha\right)\right]$$

i. 0.25 point + 0.25 point

One can use the sum of a geometric progression having complex terms.

$$\sum_{n=0}^{N}\left[\cos\left(\frac{2n\pi}{N+1}\right) + i\sin\left(\frac{2n\pi}{N+1}\right)\right] = \sum_{n=0}^{N} e^{i\frac{2n\pi}{N+1}} =$$

$$= \frac{\left(e^{i\frac{2\pi}{N+1}}\right)^{N+1} - 1}{e^{i\frac{2\pi}{N+1}} - 1} =$$

$$= \frac{\cos 2\pi + i\sin 2\pi - 1}{\cos\left(\frac{2\pi}{N+1}\right) + i\sin\left(\frac{2\pi}{N+1}\right) - 1} =$$

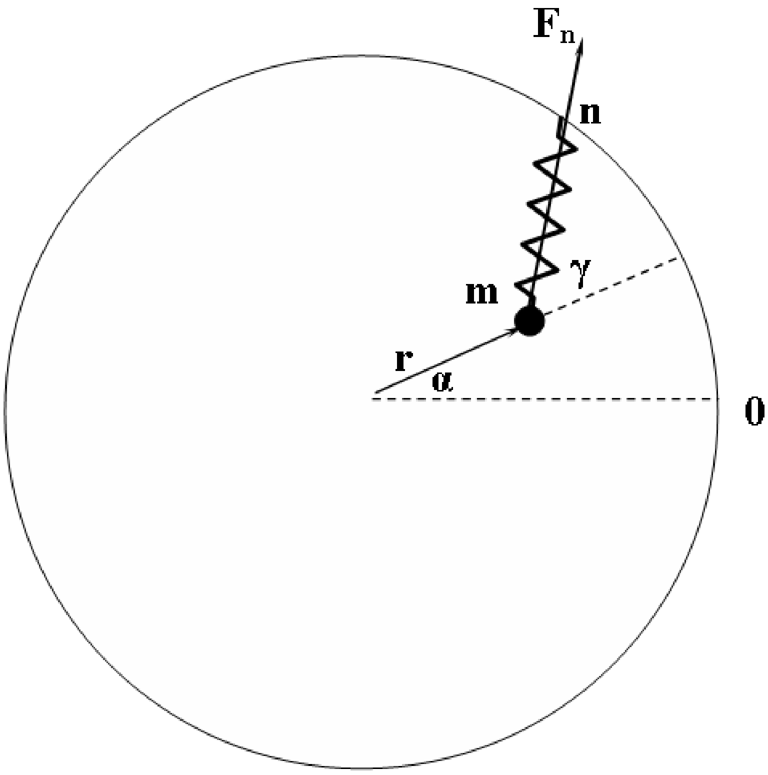
$$= 0$$

So

$$\sum_{n=0}^{N}\cos\left(\frac{2n\pi}{N+1}\right) = \sum_{n=0}^{N}\sin\left(\frac{2n\pi}{N+1}\right) = 0$$

$$E_{\text{pot}} = \frac{k}{2}\left[\sum_{n=0}^{N}\left(R^2 + r^2\right) - 2Rr\sum_{n=0}^{N}\cos\left(\frac{2n\pi}{N+1} - \alpha\right)\right] = \frac{N+1}{2}k\left(R^2 + r^2\right)$$

j. 0.5 point



The force with which the n-th spring is acting on the object is

$$F_n = kl_n$$

According to the sine rule,

$$\frac{\sin(\pi-\gamma)}{R} = \frac{\sin\left(\frac{2n\pi}{N+1}-\alpha\right)}{l_n}$$

The torque of F_n equals

$$M_n = r\cdot F_n \sin\gamma =$$

$$= r\cdot kl_n \cdot \frac{R\sin\left(\frac{2n\pi}{N+1}-\alpha\right)}{l_n} =$$

$$= kRr\sin\left(\frac{2n\pi}{N+1}-\alpha\right)$$

The total torque equals

$$M = \sum_{n=0}^{N} M_n = \sum_{n=0}^{N} kRr\sin\left(\frac{2n\pi}{N+1}-\alpha\right) = 0$$

So the angular momentum is indeed constant.

$$L = mr^2\dot\alpha$$

k. 0.5 point

$$E = E_{\text{kin}} + E_{\text{pot}} = \frac{1}{2}m\left(\dot r^2 + \frac{L^2}{m^2r^2}\right) + \frac{1}{2}k(N+1)\left(R^2+r^2\right) = \text{constant} \Rightarrow$$

$$\dot E = m\left(\dot r\ddot r - \frac{L^2}{m^2}r^{-3}\dot r\right) + k(N+1)r\dot r = 0 \Rightarrow$$

$$\ddot r - \mathcal{L}^2 r^{-3} + \omega^2 r = 0$$

l. 0.5 point

$$\dot r = \frac{2z\dot z}{2\sqrt{z^2+K}} = \frac{z\dot z}{r} \Rightarrow$$

$$\ddot r = \frac{\left(\dot z^2 + z\ddot z\right)r - z\dot z\dot r}{r^2} = \frac{\dot z^2 + z\ddot z}{r} - \frac{z\dot z}{r^2}\cdot\frac{z\dot z}{r} \Rightarrow$$

$$\frac{\dot{z}^2 + z\ddot{z}}{r} - \frac{z^2\dot{z}^2}{r^3} - \frac{\mathcal{L}^2}{r^3} + \omega^2 r = 0$$
$$\left(\dot{z}^2 + z\ddot{z}\right)\left(z^2 + K\right) - z^2\dot{z}^2 - \mathcal{L}^2 + \omega^2\left(z^2 + K\right)^2 = 0$$
$$\left(\ddot{z} + \omega^2 z\right)z^3 + K\ddot{z}z + K\dot{z}^2 + 2K\omega^2 z^2 + K^2\omega^2 - \mathcal{L}^2 = 0$$

m. **0.5 point**

The first term of the equation cancels anyway.

$$-K\omega^2 A^2\cos^2\left(\omega t + \phi_0\right) + K\omega^2 A^2\sin^2\left(\omega t + \phi_0\right) + 2K\omega^2 A^2\cos^2\left(\omega t + \phi_0\right) + K^2\omega^2 - \mathcal{L}^2 = 0$$
$$K\omega^2 A^2 + K^2\omega^2 - \mathcal{L}^2 = 0 \Rightarrow A^2 = \frac{\mathcal{L}^2}{K\omega^2} - K$$

n. **0.5 point**

$$r(t) = \sqrt{\frac{\mathcal{L}^2 - K^2\omega^2}{K\omega^2}\cos^2\left(\omega t + \phi_0\right) + K} =$$
$$= \frac{1}{\omega}\sqrt{\frac{\mathcal{L}^2\cos^2\left(\omega t + \phi_0\right) + K^2\omega^2\sin^2\left(\omega t + \phi_0\right)}{K}} =$$
$$= \frac{1}{\omega\sqrt{2K}}\sqrt{\mathcal{L}^2\left[1 + \cos\left(2\omega t + 2\phi_0\right)\right] + K^2\omega^2\left[1 - \cos\left(2\omega t + 2\phi_0\right)\right]} =$$
$$= \frac{1}{\omega\sqrt{2K}}\sqrt{\left(\mathcal{L}^2 + K^2\omega^2\right) + \left(\mathcal{L}^2 - K^2\omega^2\right)\cos\left(2\omega t + 2\phi_0\right)}$$
$$T_r = \frac{2\pi}{2\omega}$$

But since the values of r repeat after every rotation by π, it means that the period of the motion is

$$T = 2\pi\sqrt{\frac{1}{N+1}\frac{m}{k}}$$

o. **0.5 point**

$$L = 0 \Rightarrow \mathcal{L} = 0 \Rightarrow r(t) = \sqrt{K}\left|\sin\left(\omega t + \phi_0\right)\right|$$
$$L = 0 \Rightarrow \dot{\alpha} = 0 \Rightarrow \alpha = \text{constant}$$

According to the previous part, except for the moments when $r = 0$ and α is not defined, there will be two constant values for α, corresponding to two opposing directions. So in this case the object will perform simple harmonic motion with amplitude $\sqrt{K}$.

p. **0.5 point**

$$r = \text{constant} = \sqrt{K} \Rightarrow \mathcal{L}^2 - K^2\omega^2 = 0 \Rightarrow \left|\frac{L}{m}\right| = K\sqrt{\frac{k}{m}(N+1)} \Rightarrow r = \sqrt{\frac{|L|}{\sqrt{(N+1)mk}}}$$

q. **0.5 point**

We can think of this situation as having *N* + 1 springs of positive constant *k* and another (*N* + 1)/*d* springs of "negative" stiffness –k. The potential energy E_{pot} derived in **i.** holds for both ensembles (with appropriate *N* and *k*) and angular momentum is still conserved. Thus all the results from part **j.** onwards continue to hold if we replace (N + 1) by (N + 1) + [– (N + 1)/d]. Thus

$$\omega'^2 = \left[N + 1 - \frac{N+1}{d}\right]\frac{k}{m} \Rightarrow \omega' = \omega\sqrt{1 - \frac{1}{d}}$$

Barem de evaluare şi de notare

Se punctează în mod corespunzător oricare altă modalitate de rezolvare corectă a problemei

Problema teoretică nr. 2

Modelarea climei terestre

Nr. item	*Sarcina de lucru nr. 1*		Punctaj
1.a.	Pentru: $w_S \cdot \pi \cdot R_P^2 = \sigma \cdot T_P^4 \cdot 4 \cdot \pi \cdot R_P^2$ expresia temperaturii medii la suprafaţa Pământului $T_P = \sqrt[4]{\frac{w_S}{4 \cdot \sigma}}$	 0,50p 0,20p	**0,70p**
1.b.	Pentru: $T_P \cong 279\,K \qquad t_P \cong 6°C$	 0,30p	**0,30p**
Nr. item	*Sarcina de lucru nr. 2*		**Punctaj**
2.a.	Pentru: $(1-A) \cdot w_S \cdot \pi \cdot R_P^2 = \sigma \cdot T'^4_P \cdot 4 \cdot \pi \cdot R_P^2$ expresia temperaturii medii la suprafaţa Pământului $T_P' = \sqrt[4]{\frac{(1-A) \cdot w_S}{4 \cdot \sigma}}$	 0,50p 0,20p	**0,70p**
2.b.	Pentru: $\begin{cases} T_P' \cong 255K \\ t_P' \cong -18°C \end{cases}$	 0,30p	**0,30p**
Nr. item	*Sarcina de lucru nr. 3*		**Punctaj**
3.a.	Pentru: expresia energiei care ajunge la suprafaţa Pământului în unitatea de timp, venind de la Soare $E_S = \pi \cdot R_P^2 \cdot w_s \cdot (1-A) \cdot \alpha_{vis}$ bilanţul puterilor la suprafaţa Pământului $(1-A) \cdot w_s \cdot \pi \cdot R_P^2 \cdot \alpha_{vis} + E_A = \sigma \cdot T''^4_P \cdot 4 \cdot \pi \cdot R_P^2$ unde E_A este puterea radiantă, datorată atmosferei bilanţul puterilor în partea superioară a atmosferei $(1-A) \cdot w_s \cdot \pi \cdot R_P^2 = \sigma \cdot T''^4_P \cdot 4 \cdot \pi \cdot R_P^2 \cdot \alpha_{ir} + E_A$ expresia temperaturii medii la suprafaţa Pământului $T''_P = \sqrt[4]{\frac{w_s \cdot (1-A) \cdot (\alpha_{vis}+1)}{4 \cdot \sigma \cdot (\alpha_{ir}+1)}}$	 0,30p 1,00p 1,00p 0,20p	**2,50p**
3.b.	Pentru: $\begin{cases} T''_P \cong 288\,K \\ t''_P \cong 15\,°C \end{cases}$	 0,30p	**0,30p**

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3.c. Pentru: **1,20p**

Caz	I	II	III	IV
w_s	1370	1370	1370	1370
α_{vis}	1	1	1	1
α_{ir}	1	1	0	0
A	0,3	0,0	0,0	0,3
$T''_P\,(K)$	255	279	332	303
$t''_P\,(°C)$	-18	6	59	30

4x 0,30p

Nr. item	*Sarcina de lucru nr. 4*	Punctaj
4.a.	Pentru:	**1,00p**
	$w'_s = \dfrac{w_s}{(1{,}01)^2}$	0,50p
	expresia temperaturii medii la suprafaţa Pământului $T'''_P = \sqrt[4]{\dfrac{w'_s \cdot (1-A)\cdot(\alpha_{vis}+1)}{4\cdot\sigma\cdot(\alpha_{ir}+1)}}$	0,20p
	$\begin{cases} T'''_P \cong 267K \\ t'''_P \cong -6°C \end{cases}$	0,30p

Nr. item	*Sarcina de lucru nr. 5*	Punctaj
5.a.	Pentru:	**3,00p**
	expresia ariei suprafeţei „oglinzii de sticlă” din deşertul Sahara $s = x\cdot\pi\cdot R_P^2$	0,10p
	expresia ariei suprafeţei „de captură” a energiei solare $\begin{cases} S_{captura} = \pi\cdot R_P^2 - \dfrac{x}{2}\cdot\pi\cdot R_P^2 \\ S_{captura} = \pi\cdot R_P^2\cdot\left(1-\dfrac{x}{2}\right) \end{cases}$	0,20p
	expresia ariei suprafeţei emiţătoare $\begin{cases} S_{emitator,P} = 4\cdot\pi\cdot R_P^2 - x\cdot\pi\cdot R_P^2 \\ S_{emitator,P} = \pi\cdot R_P^2\cdot(4-x) \end{cases}$	0,20p
	bilanţul puterilor la suprafaţa Pământului $(1-A)\cdot w_s\cdot\pi\cdot R_P^2\cdot\left(1-\dfrac{x}{2}\right)\cdot\alpha_{vis} + E_A = \sigma\cdot\left(T'''_{P,Sahara}\right)^4\cdot\pi\cdot R_P^2\cdot(4-x)$	0,70p

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	bilanţul puterilor în partea superioară a atmosferei $$(1-A)\cdot w_s\cdot\pi\cdot R_P^2\cdot\left(1-\frac{x}{2}\right)=\sigma\cdot\left(T'''_{P,Sahara}\right)^4\cdot\pi\cdot R_P^2\cdot(4-x)\cdot\alpha_{ir}+E_A$$	0,70p	

	$$T'''_{P,Sahara}=\sqrt[4]{\frac{w_s\cdot(1-A)}{\sigma}\cdot\frac{(\alpha_{vis}+1)}{(\alpha_{ir}+1)}\cdot\frac{1-\frac{x}{2}}{4-x}}$$	0,20p	
	$$\frac{T'''_{P,Sahara}}{T''_P}=\sqrt[4]{\frac{1-\frac{x}{2}}{1-\frac{x}{4}}}$$	0,20p	
	$$\frac{1-\frac{x}{2}}{1-\frac{x}{4}}=\left(\frac{T'''_{P,Sahara}}{T''_P}\right)^4=k$$ unde $k=0{,}9862$	0,30p	
	$$\begin{cases}x=\frac{4\cdot(1-k)}{2-k}\\ x=0{,}0544\end{cases}$$	0,20p	
	aria suprafeţei „oglinzii de sticlă” din Sahara $s\cong 7\cdot 10^6 km^2$ *Observaţie: Aria suprafeţei Saharei este de* $9\cdot 10^6 km^2$.	0,20p	
TOTAL			**10p**

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Se punctează în mod corespunzător oricare altă modalitate de rezolvare corectă a problemei

PROBLEM 3: BLACK HOLES PHYSICS

a. 1.0 point

For Minkovski (flat) spacetime *f*(*r*) = *g*(*r*) = 1.

$$(ds)^2 = -c^2(dt)^2 + (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$$

b. 1.0 point

$$(ds)^2 = -c^2\left(1-\frac{r_S}{r}\right)(dt)^2 + \frac{(dr)^2}{1-\frac{r_S}{r}} = -c^2(dt')^2 \Rightarrow$$

$$\left(1-\frac{r_S}{r}\right)\left(\frac{dt}{dt'}\right)^2 - \frac{\left(\frac{dr}{dt'}\right)^2}{c^2\left(1-\frac{r_S}{r}\right)} = 1 \Rightarrow$$

$$\frac{d^2r}{dt'^2} + \frac{r_S c^2}{2r^2}\left[1+\frac{\left(\frac{dr}{dt'}\right)^2}{c^2\left(1-\frac{r_S}{r}\right)}\right] - \frac{r_S}{2r^2}\frac{1}{1-\frac{r_S}{r}}\left(\frac{dr}{dt'}\right)^2 = 0 \Rightarrow$$

$$a = \frac{d^2r}{dt'^2} = -\frac{r_S c^2}{2r^2}$$

c. 0.5 point

The observer will always experience the same acceleration as in the classical case.

$$a = -\frac{GM}{r^2} \Rightarrow r_S = \frac{2GM}{c^2}$$

d. 1.0 point

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Se punctează în mod corespunzător oricare altă modalitate de rezolvare corectă a problemei

$$a=\frac{dv}{dt'}=\frac{dv}{dr}\frac{dr}{dt'}=\frac{1}{2}\frac{d\left(v^2\right)}{dr}=-\frac{r_S c^2}{2r^2}\Rightarrow$$

$$d\left(v^2\right)=r_S c^2\frac{1}{r^2}dr\Rightarrow v^2=r_S c^2\frac{1}{r}+C$$

From the initial conditions, $C=-r_S c^2/r_0$, so

$$v=-c\sqrt{r_S\left(\frac{1}{r}-\frac{1}{r_0}\right)}=-\sqrt{\frac{2GM}{r_0}\left(\frac{r_0}{r}-1\right)}$$

Again, from the standpoint of the observer this is just the classical result.

e. 0.5 point

$$dt'=\frac{dr}{v}=-\frac{1}{c}\sqrt{\frac{r_0 r}{r_S\left(r-r_0\right)}}dr\Rightarrow$$

$$t'=\int_{r_0}^{r_S}\left(-\frac{1}{c}\right)\sqrt{\frac{r_0}{r_S}\frac{\frac{r}{r_0}}{1-\frac{r}{r_0}}}dr=$$

$$=\frac{r_0}{c}\sqrt{\frac{r_0}{r_S}}\left[-\sqrt{x(1-x)}\Big|_{\frac{r_S}{r_0}}^{1}-\arccos\left(\sqrt{x}\right)\Big|_{\frac{r_S}{r_0}}^{1}\right]=$$

$$=\frac{r_0}{c}\sqrt{\frac{r_0}{r_S}}\left[\sqrt{\frac{r_S}{r_0}\left(1-\frac{r_S}{r_0}\right)}+\arccos\left(\sqrt{\frac{r_S}{r_0}}\right)\right]=$$

$$=\frac{r_0}{c}\left[\sqrt{\left(1-\frac{2GM}{c^2 r_0}\right)}+\sqrt{\frac{c^2 r_0}{2GM}}\arccos\left(\sqrt{\frac{2GM}{c^2 r_0}}\right)\right]$$

And once again this is just the classic result for a falling body.

f. 0.5 point

$$\left(1-\frac{r_S}{r}\right)\left(\frac{dt}{dt'}\right)^2-\frac{v^2}{c^2\left(1-\frac{r_S}{r}\right)}=1\Rightarrow$$

Barem de evaluare şi de notare

Se punctează în mod corespunzător oricare altă modalitate de rezolvare corectă a problemei

$$\frac{r-r_S}{r}\left(\frac{dt}{dt'}\right)^2 = 1+\frac{c^2\dfrac{r_S\left(r_0-r\right)}{r_0 r}}{c^2\dfrac{r-r_S}{r}} \Rightarrow$$

$$\left(\frac{dt}{dt'}\right)^2 = \frac{r}{r-r_S}\cdot\frac{r_0 r - r_S r}{r_0\left(r-r_S\right)} \Rightarrow$$

$$dt = dt'\cdot\frac{r}{r-r_S}\sqrt{\frac{r_0-r_S}{r_0}} = -\frac{dr}{c}\cdot\frac{r}{r-r_S}\sqrt{\frac{r_0-r_S}{r_0}}\sqrt{\frac{r_0 r}{r_S\left(r_0-r\right)}} =$$

$$= \frac{|dr|}{c}\cdot\frac{r}{r-r_S}\sqrt{\frac{r}{r_S}\cdot\frac{r_0-r_S}{r_0-r}}$$

But

$$\left.\begin{array}{l}\dfrac{r}{r_S}>1\\[2ex]\dfrac{r_0-r_S}{r_0-r}>1\\[2ex]\dfrac{r}{r-r_S}>\dfrac{r}{r-r_S}\end{array}\right\}\Rightarrow dt > \frac{|dr|}{c}\cdot\frac{r_S}{r-r_S} \Rightarrow$$

$$\int_{r=r_0}^{r=r_S} dt > \int_{r_S}^{r_0}\frac{r_S}{c}\cdot\frac{1}{r-r_S}dr = \infty$$

So from the point of view of someone outside the black hole, the falling observer remains frozen just outside the black hole's rim for an eternity, never crossing the event horizon!! Of course, this result is just an approximation, valid in the idealized regime in which the falling observer's effect on the gravitational field (spacetime) can be ignored.

g. **1.0 point**

$$dS = \frac{dU}{T} = \frac{d\left(Mc^2\right)}{\dfrac{\hbar c^3}{8\pi G k_B M}} = \frac{8\pi G k_B}{\hbar c}MdM$$

By definition and common sense, no black hole, no entropy. So

$$S = \frac{4\pi G k_B M^2}{\hbar c} = \frac{4\pi G k_B}{\hbar c}\cdot\frac{r_S^2 c^4}{4G^2} = \frac{k_B c^3}{4G\hbar}A$$

Barem de evaluare şi de notare

Se punctează în mod corespunzător oricare altă modalitate de rezolvare corectă a problemei

h. 0.5 point

$$E_{\text{radiated}} = \left(M_1 + M_2 - M_{1+2}\right)c^2$$

$$S_{1+2} \geq S_1 + S_2 \Rightarrow A_{1+2} \geq A_1 + A_2 \Rightarrow M_{1+2}^2 \geq M_1^2 + M_2^2 \Rightarrow$$

$$E_{\max} = \left(M_1 + M_2 - \sqrt{M_1^2 + M_2^2}\right)c^2$$

i. 0.5 point

$$ds = \frac{dr}{\sqrt{G(r)}} \approx \frac{dr}{\sqrt{G'(r_h)\cdot(r - r_h)}} \Rightarrow$$

$$R = \int_{r_h}^{r_h+\varepsilon} ds = \frac{1}{\sqrt{G'(r_h)}}\cdot 2\sqrt{r - r_h}\Big|_{r_h}^{r_h+\varepsilon} = \frac{2\sqrt{\varepsilon}}{\sqrt{G'(r_h)}}$$

j. 0.5 point

$$F(r) \approx F'(r_h)\cdot(r - r_h)$$

$$ds = c\sqrt{F(r_h + \varepsilon)}d\tau \approx \sqrt{F'(r_h)\cdot\varepsilon}\cdot d(c\tau)$$

$$L = \int_0^P ds = \sqrt{F'(r_h)\cdot\varepsilon}\cdot P$$

k. 1.0 point

$$F(r_h) = G(r_h) = 0 \Rightarrow 1 - \frac{2GM}{c^2 r_h} = 0 \Rightarrow r_h = r_S$$

$$F'(r_h) = G'(r_h) = \frac{1}{r_S}$$

$$\sqrt{F'(r_h)\cdot\varepsilon}\cdot P = 2\pi\frac{2\sqrt{\varepsilon}}{\sqrt{G'(r_h)}} \Rightarrow P = \frac{4\pi}{\sqrt{F'(r_h)\cdot G'(r_h)}} = 4\pi r_S$$

$$4\pi\frac{2GM}{c^2} = \frac{\hbar c}{k_B T_H} \Rightarrow T_H = \frac{\hbar c^3}{8\pi G k_B M}$$

Barem de evaluare şi de notare

Se punctează în mod corespunzător oricare altă modalitate de rezolvare corectă a problemei

l. **1.0 point**

$$C=\frac{dU}{dT_{\mathrm{H}}}=\frac{d\left(Mc^{2}\right)}{dT_{\mathrm{H}}}=c^{2}\frac{d\left(\frac{\hbar c^{3}}{8\pi Gk_{\mathrm{B}}T_{\mathrm{H}}}\right)}{dT_{\mathrm{H}}}=-\frac{\hbar c^{5}}{8\pi Gk_{\mathrm{B}}T_{\mathrm{H}}^{2}}$$

m. **0.5 point**

$$W=\sigma T_{\mathrm{H}}^{4}A=\frac{\pi^{2}k_{\mathrm{B}}^{4}}{60\hbar^{3}c^{2}}\cdot\frac{\hbar^{4}c^{12}}{8^{4}\pi^{4}G^{4}k_{\mathrm{B}}^{4}M^{4}}\cdot 4\pi\cdot\frac{4G^{2}M^{2}}{c^{4}}=\frac{\hbar c^{6}}{15360\pi G^{2}M^{2}}$$

n. **0.5 point**

$$W=-\frac{dU}{dt}\Rightarrow\frac{\hbar c^{6}}{15360\pi G^{2}M^{2}}=-\frac{c^{2}dM}{dt}\Rightarrow$$

$$\tau=\int_{0}^{\tau}dt=-\frac{15360\pi G^{2}}{\hbar c^{4}}\int_{M}^{0}M^{2}dM=\frac{5120\pi G^{2}M^{3}}{\hbar c^{4}}$$

For the given values, τ is of the order of 10^{77} s, which is approximately 10^{60} times greater than the age of the universe!!