Theoretical Problem No. 2 (10 points)

Compton scattering

An electron storage ring contains circulating electrons with high energy produced by an accelerator and rotating in an appropriate magnetic field. X - ray photons are directed so as to collide with electrons stored in storage ring. Phenomenon that occurs is known as Inverse Compton Scattering.

A photon of wavelength λ_i is scattered by a moving, free electron. As a result the electron stops and the resulting photon of wavelength λ_0 is scattered at an angle $\theta = 60^\circ$ with respect to the direction of the incident photon; this photon is again scattered by a second free electron at rest. In this second scattering process a photon with wavelength of $\lambda_f = 1{,}25 \times 10^{-10} m$ emerges at an angle $\theta = 60^\circ$ with respect to the direction of the photon of wavelength λ_0.

To characterize the photons and the electrons during the processes use the following notation:

	initial photon	photon after the first scattering	final photon		first electron before collision	first electron after collision	second electron before collision	Second electron after collision
moment	$\vec{p}_i$	$\vec{p}_0$	$\vec{p}_f$	moment	$\vec{p}_{1e}$	0	0	$\vec{p}_{2e}$
energy	E_i	E_0	E_f	energy	E_{1e}	E_{0e}	E_{0e}	E_{2e}
wavelength	λ_i	λ_0	λ_f	speed	$\vec{v}_{1e}$	0	0	$\vec{v}_{2e}$

The following constants are known:

$h = 6{,}6 \times 10^{-34} J \cdot s$ - Planck's constant

$m_0 = 9{,}1 \times 10^{-31} kg$ - rest mass of the electron

$c = 3{,}0 \times 10^{8} m/s$ - speed of light in vacuum

Task 1 - First collision

1.a. Draw simple sketches marking the moments of the electron and photon before and after the first collision. Clearly specify the coordinate system used.

1.b. Express the energy and moment of the electron implied in the first collision as a function of the initial speed of the electron $\vec{v}_{1e}$ and its rest mass m_0.

1.c. Express the energy and wavelength of the photon after the first collision as a function of the wavelength of the initial photon λ_i, the scattering angle θ and $\Lambda = h/(m_0 \cdot c)$.

Task 2 - Second collision

2.a. Draw simple sketches marking moment of the electron and photon before and after the second collision. Clearly specify the coordinate system used.

2.b. Express the energy and wavelength of the photon after the second collision as a function of the wavelength of the photon before the collision λ_0, the scattering angle θ and $\Lambda = h/(m_0 \cdot c)$.

2.c. Express the kinetic energy $\left(T_2 = E_{2e} - E_{0e}\right)$ and momentum p_{2e} of the electron after the second collision as a function of the photon wavelengths after the collision λ_f, m_0, c and h.

Task 3 - Quantitative description of processes

Using the values of the given physical constants and numerical values of λ_f and θ determine the expressions and numerical values for:

3.a. the De Broglie wavelength of the initial electron;

3.b. the energy and frequency of the initial photon;

3.c. the speed of the second electron after collision;

3.d. the variation of the photon wavelength after each collision process.

Compton scattering - Solution

Task 1 - First collision

1.a. Figure 1 presents the situation before the first scattering of the photon.

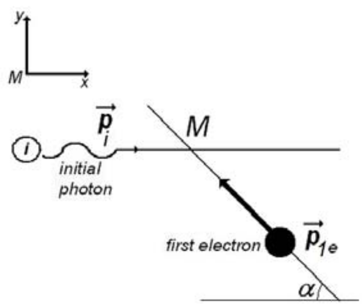

Figure 1 Photon and electron before the first collision. The collision occurs in the point M

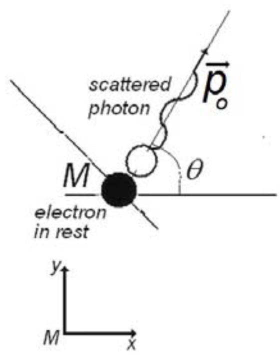

Figure 2. Photon and electron after the first collision. The collision took place in point M

1.b. The angle between the directions of the electron and photon moments before collision is denoted by α

To characterize the initial photon we will use its moment $\vec{p}_i$ and its energy E_i

$$\begin{cases} \left|\vec{p}_i\right| = \dfrac{h}{\lambda_i} = \dfrac{h \cdot f_i}{c} \\ E_i = h \cdot f_i \end{cases} \qquad (1)$$

$$f_i = \frac{c}{\lambda_i} \qquad (2)$$

is the frequency of the initial photon.
For the initial free electron in motion scenario, the moment $\vec{p}_{1e}$ and the energy E_{oe} are

$$\begin{cases} \vec{p}_{1e} = m \cdot \vec{v}_{1e} = \dfrac{m_0 \cdot \vec{v}_{1e}}{\sqrt{1-\beta^2}} \\ E_{1e} = m \cdot c^2 = \dfrac{m_0 \cdot c^2}{\sqrt{1-\beta^2}} \end{cases} \qquad (3)$$

where m_0 is the rest mass of the electron and m is the mass of the moving electron. As usual $\beta = \frac{v_{1e}}{c}$. De Broglie wavelength of the first electron is

$$\lambda_{1e} = \frac{h}{p_{1e}} = \frac{h \cdot}{m_0 \cdot v_{1e}} \sqrt{1-\beta^2} \qquad (4)$$

1.c. The situation after the scattering of photon is described in the figure 2.
To characterize the scattered photon we will use its moment $\vec{p}_0$ and its energy E_0

$$\begin{cases} |\vec{p}_o| = \dfrac{h}{\lambda_o} = \dfrac{h \cdot f_o}{c} \\ E_o = h \cdot f_o \end{cases} \qquad (5)$$

where

$$f_o = \frac{c}{\lambda_0} \qquad (6)$$

is the frequency of the scattered photon.
The magnitude of the moment of the electron (that remains at rest) after the scattering is zero; its energy is E_{0e}. The mass of the electron after collision is m_0 - the mass of electron at rest state. So,

$$E_{0e} = m_0 \cdot c^2 \qquad (7)$$

To determine the moment of the first moving electron, one can write the principles of conservation of moments and energy. That is

$$\vec{p}_i + \vec{p}_{1e} = \vec{p}_0 \qquad (8)$$

and

$$E_i + E_{1e} = E_0 + E_{0e} \qquad (9)$$

Using the referential in figures 1 and 2, the conservation of the moment in collision along the Ox direction is written as

$$\frac{h \cdot f_i}{c} + m \cdot v_{1e} \cdot \cos\alpha = \frac{h \cdot f_0}{c}\cos\theta \qquad (10)$$

and the conservation of moment along Oy direction is

$$m \cdot v_{1e} \cdot \sin\alpha = \frac{h \cdot f_0}{c}\sin\theta \qquad (11)$$

To eliminate α, the last two equations must be written again as

$$\begin{cases} (m \cdot v_{1e} \cdot \cos\alpha)^2 = \dfrac{h^2 \cdot}{c^2}(f_0 \cdot \cos\theta - f_i)^2 \\ (m \cdot v_{1e} \cdot \sin\alpha)^2 = \left(\dfrac{h \cdot f_0}{c}\sin\theta\right)^2 \end{cases} \qquad (12)$$

and then added.
The result is

$$m^2 \cdot v_{1e}^2 = \frac{h^2 \cdot}{c^2}\left(f_0^2 + f_i^2 - 2f_0 \cdot f_i \cdot \cos\theta\right) \qquad (13)$$

or

$$\frac{m_0^2 \cdot c^2}{1-\left(\frac{v_{1e}}{c}\right)^2} \cdot v_{1e}^2 = h^2 \cdot \left(f_0^2 + f_i^2 - 2f_0 \cdot f_i \cdot \cos\theta\right) \qquad (14)$$

The conservation of energy (9) can be written again as

$$m \cdot c^2 + h \cdot f_i = m_0 \cdot c^2 + h \cdot f_0 \qquad (15)$$

or

$$\frac{m_0 \cdot c^2}{\sqrt{1-\left(\frac{v_{1e}}{c}\right)^2}} = m_0 \cdot c^2 + h \cdot \left(f_0 - f_i\right) \qquad (16)$$

Squaring the last relation results in the following

$$\frac{m_0^2 \cdot c^4}{1-\left(\frac{v_{1e}}{c}\right)^2} = m_0^2 \cdot c^4 + h^2 \cdot \left(f_0 - f_i\right)^2 + 2m_0 \cdot h \cdot c^2 \cdot \left(f_0 - f_i\right) \qquad (17)$$

Subtracting (14) from (17) the result is

$$2m_0 \cdot c^2 \cdot h \cdot \left(f_0 - f_i\right) + 2h^2 \cdot f_i \cdot f_0 \cdot \cos\theta - 2h^2 \cdot f_i \cdot f_0 = 0 \qquad (18)$$

or

$$\frac{h}{m_0 \cdot c}\left(1 - \cos\theta\right) = \frac{c}{f_i} - \frac{c}{f_0} \qquad (19)$$

Using

$$\Lambda = \frac{h}{m_0 \cdot c} \qquad (20)$$

the relation (19) becomes

$$\Lambda \cdot \left(1 - \cos\theta\right) = \lambda_i - \lambda_0 \qquad (21)$$

The wavelength of scattered photon is

$$\lambda_0 = \lambda_i - \Lambda \cdot \left(1 - \cos\theta\right) \qquad (22)$$

Equation (22) show that the scattered photon wavelength is shorter than the wavelength of the initial photon. Consequently the energy of the scattered photon is greater that the energy of the initial photon.

$$\begin{cases} \lambda_0 < \lambda_i \\ E_0 > E_i \end{cases} \qquad (23)$$

Task 2 - Second collision

2.a. Let's analyze now the second collision process that occurs at point N. To study that, let's consider a new referential having the Ox axis along the direction of the photon scattered after the first collision.

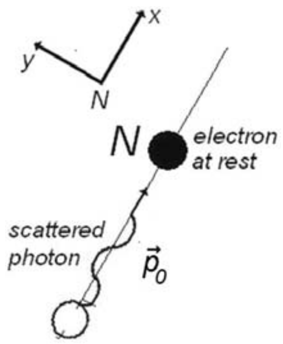

Figure 3 Photon and electron before the second collision. The collision occur at the point N

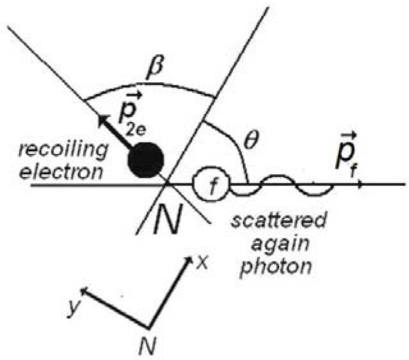

Figure 4 Photon and electron after the second collision. The collision took place at the point N*. The scattering angle for electron is denoted by* β*.*

Figure 3 presents the situation before the second collision and Figure 4 presents the situation after this scattering process.

2.b. The conservation law for moments in the scattering process gives

$$\begin{cases} \dfrac{h}{\lambda_0} = \dfrac{h}{\lambda_f} \cos\theta + m \cdot v_{2e} \cdot \cos\beta \\ \dfrac{h}{\lambda_f} \sin\theta - m \cdot v_{2e} \cdot \sin\beta = 0 \end{cases} \qquad (24)$$

To eliminate the unknown angle β, one must square and then add equations (24)

That is

$$\begin{cases} \left(\dfrac{h}{\lambda_0} - \dfrac{h}{\lambda_f}\cos\theta\right)^2 = (m \cdot v_{2e} \cdot \cos\beta)^2 \\ \left(\dfrac{h}{\lambda_f}\sin\theta\right)^2 = (m \cdot v_{2e} \cdot \sin\beta)^2 \end{cases} \qquad (25)$$

and

$$\left(\frac{h}{\lambda_f}\right)^2 + \left(\frac{h}{\lambda_0}\right)^2 - \frac{2 \cdot h^2}{\lambda_0 \cdot \lambda_f}\cos\theta = (m \cdot v_{2e})^2 \qquad (26)$$

or

$$\frac{h^2 \cdot c^2}{\lambda_f^2} + \frac{h^2 \cdot c^2}{\lambda_0^2} - \frac{2 \cdot h^2 \cdot c^2}{\lambda_0 \cdot \lambda_f}\cos\theta = m^2 \cdot c^2 \cdot v_{2e}^2 \qquad (27)$$

The conservation law for energies in the second scattering process gives

$$\frac{h\cdot c}{\lambda_0}+m_0\cdot c^2=\frac{h\cdot c}{\lambda_f}+m\cdot c^2 \tag{28}$$

or

$$h^2\cdot c^2\cdot\left(\frac{1}{\lambda_f}-\frac{1}{\lambda_0}\right)^2+m_0^2\cdot c^4+2h\cdot c^3\cdot m_0\cdot\left(\frac{1}{\lambda_f}-\frac{1}{\lambda_0}\right)=\frac{m_0^2\cdot c^4}{1-\left(v_{2e}/c\right)^2} \tag{29}$$

Subtracting (27) from (29), one obtains

$$+\frac{2\cdot h^2\cdot c^2}{\lambda_0\cdot\lambda_f}\cos\theta-\frac{2\cdot h^2\cdot c^2}{\lambda_0\cdot\lambda_f}+2h\cdot c^3\cdot m_0\cdot\left(\frac{1}{\lambda_f}-\frac{1}{\lambda_0}\right)=0 \tag{30}$$

that is

$$\begin{cases}\dfrac{h}{m_0\cdot c}\cdot(1-\cos\theta)=\lambda_f-\lambda_0\\ \lambda_f-\lambda_0=\Lambda\cdot(1-\cos\theta)\end{cases} \tag{31}$$

Concluding

$$\begin{cases}\lambda_f>\lambda_0\\ E_f<E_0\end{cases} \tag{32}$$

Comparing (31) written as

$$\lambda_f=\lambda_0+\Lambda\cdot(1-\cos\theta) \tag{33}$$

and (22) written as

$$\lambda_i=\lambda_0+\Lambda\cdot(1-\cos\theta) \tag{34}$$

results in

$$\lambda_i=\lambda_f \tag{35}$$

2.c. Taking into account (26), the moment of the electron after the second collision is

$$p_{2e}=h\sqrt{\frac{1}{\lambda_f^2}+\frac{1}{\lambda_0^2}-\frac{2\cdot\cos\theta}{\lambda_f\cdot\lambda_0}} \tag{36}$$

Or, considering (33), the moment can be rewritten as

$$p_{2e}=h\sqrt{\frac{1}{\lambda_f^2}+\frac{1}{(\lambda_f-\Lambda(1-\cos\theta))^2}-\frac{2\cdot\cos\theta}{\lambda_f\cdot(\lambda_f-\Lambda(1-\cos\theta))}} \tag{37}$$

The expression of the de Broglie wavelength of the second electron after scattering is

$$\lambda_{2e}=1\Bigg/\left(\sqrt{\frac{1}{\lambda_f^2}+\frac{1}{(\lambda_f-\Lambda(1-\cos\theta))^2}-\frac{2\cdot\cos\theta}{\lambda_f\cdot(\lambda_f-\Lambda(1-\cos\theta))}}\right) \tag{38}$$

From (28) results

$$\frac{h\cdot c}{\lambda_0}=\frac{h\cdot c}{\lambda_f}+T_2 \tag{39}$$

that is

$$T_2=h\cdot c\cdot\left(\frac{1}{\lambda_0}-\frac{1}{\lambda_f}\right) \tag{40}$$

Task 3 - Quantitative description of processes

3.a. The relation (13) can be written as

$$p_{1e}=h\sqrt{\frac{1}{\lambda_i^2}+\frac{1}{\lambda_0^2}-\frac{2\cdot\cos\theta}{\lambda_i\cdot\lambda_0}} \tag{41}$$

or, considering (34) and (35)

$$p_{1e} = h\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{\left(\lambda_f - \Lambda(1-\cos\theta)\right)^2} - \frac{2\cdot\cos\theta}{\lambda_f\cdot\left(\lambda_f - \Lambda(1-\cos\theta)\right)}} \qquad (42)$$

so that the expression of the de Broglie wavelength of the first electron before scattering is

$$\lambda_{1e} = 1\Bigg/\left(\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{\left(\lambda_f - \Lambda(1-\cos\theta)\right)^2} - \frac{2\cdot\cos\theta}{\lambda_f\cdot\left(\lambda_f - \Lambda(1-\cos\theta)\right)}}\right) \qquad (43)$$

In the condition of the problem (43) becomes

$$\lambda_{1e} = 1\Bigg/\left(\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{\left(\lambda_f - \frac{\Lambda}{2}\right)^2} - \frac{1}{\lambda_f\cdot\left(\lambda_f - \frac{\Lambda}{2}\right)}}\right) \qquad (44)$$

Because the value of λ_f and Λ are known as

$$\begin{cases} \lambda_f = \lambda_i = 1{,}25\times10^{-10}\,m \\ \Lambda = \dfrac{6{,}6\times10^{-34}}{9{,}1\times10^{-31}\cdot3{,}0\times10^{8}}\,m = 2{,}41\times10^{-12}\,m = 0{,}02\times10^{-10}\,m \end{cases} \qquad (45)$$

From (33) results that

$$\lambda_0 = \lambda_f - \Lambda\cdot(1-\cos\theta) = \lambda_f - \frac{\Lambda}{2} \qquad (46)$$

or

$$\lambda_0 = 1{,}24\times10^{-10}\,m \qquad (47)$$

The numerical value of the de Broglie wavelength of the first electron before scattering is

$$\lambda_{1e} = 10^{-10}\Bigg/\left(\sqrt{\frac{1}{1{,}25^2} + \frac{1}{1{,}24^2} - \frac{1}{1{,}25\times1{,}24}}\right)m \cong 1{,}25\times10^{-10}\,m \qquad (48)$$

Evidently the de Broglie wavelength λ_{2e} for the second electron after the scattering is the same

$$\lambda_{1e} = \lambda_{2e} \qquad (49)$$

3.b. Because $\lambda_f = \lambda_i$ the energy of the first photon before the first scattering has the following expression

$$E_i = \frac{h\cdot c}{\lambda_i} = \frac{h\cdot c}{\lambda_f} \qquad (50)$$

Its numerical value is

$$E_i = \frac{h\cdot c}{\lambda_i} = \frac{6{,}6\times10^{-34}\cdot3{,}0\times10^{8}}{1{,}25\times10^{-10}} \cong 1{,}58\times10^{-15}\,J = 9{,}9\,KeV \qquad (51)$$

The frequency f_i of the first photon before the first scattering has the following expression

$$f_i = \frac{c}{\lambda_i} = \frac{c}{\lambda_f} \qquad (52)$$

Its numerical value is

$$f_i = \frac{3\times10^{8}}{1{,}25\times10^{-10}} \cong 2{,}4\times10^{18}\,Hz \qquad (53)$$

3.c. From (40) written as

$$T_2 = T_1 = m \cdot c^2 - m_0 \cdot c^2 = h \cdot c \cdot \left(\frac{1}{\lambda_0} - \frac{1}{\lambda_f} \right) \qquad (54)$$

results

$$m = m_0 + \frac{h}{c} \cdot \left(\frac{1}{\lambda_f - (\Lambda/2)} - \frac{1}{\lambda_f} \right) \cong m_0 + \frac{h}{c} \cdot \frac{\Lambda}{2\lambda_f^2} = m_0 \cdot \left(1 + \frac{1}{2} \left(\frac{h}{m_0 \cdot c \cdot \lambda_f} \right)^2 \right) \qquad (55)$$

Its numerical value is

$$m \cong m_0 \qquad (56)$$

Consequently, the speed of the second electron after the collision v_{2e} has the following expression

$$v_{2e} = \frac{h}{\lambda_f \cdot m} \approx \frac{h}{\lambda_f \cdot m_0} \qquad (57)$$

Its numerical value is

$$v_{2e} \approx 5{,}8 \times 10^6 \, m/s = 0{,}019 \cdot c \qquad (58)$$

3.d. As results from (46)

$$\lambda_i - \lambda_0 = \lambda_f - \lambda_0 = \frac{\Lambda}{2} \qquad (59)$$

The numerical value of the variation of wavelength is

$$\frac{\Lambda}{2} \cong 0{,}01 \times 10^{-10} m \qquad (60)$$

© Solution proposed by
Delia DAVIDESCU, PhD
Adrian DAFINEI, PhD