1. MAGNETOSTATICS

The magnetomotive force (mmf) along a curve is defined as the path integral of the projection of the magnetic induction B along the curve,

$$\int_{\text{curve}} \vec{B}\vec{dl} \, .$$

Ampere's Circuital Law states that the magnetomotive force along a closed curve (loop) is proportional to the electric current crossing **ANY** surface whose frontier is this loop. The proportionality constant is called *magnetic permeability of the vacuum* (μ_0).

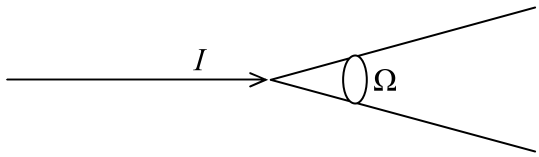
$$\oint_{\text{loop}} \vec{B}\vec{dl} = \mu_0 I_{\text{across}}$$

The positive direction of the current is associated to the path followed on the loop through *the right-handed corkscrew rule.*

a. An infinitely long straight conductor carries a steady current I. Find the magnitude and the orientation of the magnetic induction B generated by this current at a distance r from the wire. Express the result in terms of I, r, and μ_0.

b. A thin uniform rod of mass m and length L is placed parallel to the wire, at a distance d. The rod can only rotate on an axis perpendicular to the plane determined by the wire and the rod, passing through the middle of the rod. The rod carries a steady current I' in the opposite direction of I. The rod is slanted with a small angle from its equilibrium position and let to oscillate freely. Find the period of the small oscillations of the rod in terms of I, I', m, L, d, and μ_0.

c. A semi-infinite straight conductor is continued with an infinite conical conductor surface, whose axis coincides with the wire, as in the diagram alongside. The system carries a steady current I. Find the magnitude and the orientation of the magnetic induction B at a distance r from the axis, both inside and outside the conical conductor. Express the result in terms of I, r, and μ_0.



d. A semi-infinite straight conductor is connected at its end with an infinite conductor plane, placed perpendicular to the wire. The system carries a steady current I. Find the magnitude and the orientation of the magnetic induction B at a distance r from the axis of the wire, on both sides of the plane. Express the result in terms of I, r, and μ_0.

e. Define the linear current density $\vec{J}$ flowing on the plane from the previous point as:

$$J \overset{\text{def}}{=} \frac{dI}{dl},$$

where dl is an elementary length perpendicular to the line carrying an elementary current dI.

Introduce a unit vector $\vec{n}$ perpendicular to the plane, in order to indicate the positive direction of the crossing from one side of the plane to the other. The vectorial product $\vec{J} \times \vec{n}$ determines the positive direction for the component of B parallel to the plane.

Show that when crossing the plane, the difference in magnitude of the component of B parallel to the plane is proportional to the magnitude of J in the crossing point, and find the proportionality constant.

f. An infinite conductor plane is parallel to a uniform magnetic field. The magnetic induction B has the same direction on both sides of the plane, but different values B_1 and B_2. Find the pressure exerted upon the plane. Express the result in terms of B_1, B_2, and μ_0.

g. A conductor hollow sphere is connected at its poles with two semi-infinite straight conductors, oriented on the poles axis. The system carries a steady current I. Find the magnitude and the orientation of the magnetic induction B at a distance r from the axis of the poles, both inside and outside the sphere. Express the result in terms of I, r, and μ_0.

h. A conductor hollow sphere has its poles connected by an interior straight wire. A steady current I flows on the surface of the sphere from one pole to the other, and then back through the wire. Find the magnitude and the orientation of the magnetic induction B at a distance r from the axis of the poles, both inside and outside the sphere. Express the result in terms of I, r, and μ_0.

The Biot-Savart Law gives the expression of the magnetic induction generated in a point in space by an electric current flowing along an elementary path dl:

$$\overrightarrow{dB} = \frac{\mu_0 I\left(\overrightarrow{dl} \times \vec{r}\right)}{4\pi r^3},$$

where r is the position of the point relative to the elementary current.

i. A straight conductor of length L carries a steady current I. The wire is seen from a point in its mediator plane under the angle 2α. Express the magnitude of the magnetic induction in this point in terms of L, I, α, and μ_0.

j. A steady current I flows uniformly on the surface of a conductor sphere of radius R, from one pole to the other. Find the magnitude of the magnetic induction in the equatorial plane of the sphere, in a point at distance r from the axis of the poles, both inside and outside the sphere. Express the result in terms of I, R, r, and μ_0.

2. OSCILLATIONS OF ELASTIC BODIES

In this problem gravitational effects are to be neglected. Unless otherwise stated, substances are to be considered homogenous and isotropic at all times.

A. Consider a very thin elastic rod, prevented from bending. The rod has length L, density ρ, and Young elasticity modulus E. The cross section of the rod is to be taken as constant. The rod is slightly stretched from both ends and then let to oscillate freely.
a. Write the kinetic energy of the rod at some moment in terms of its mass m, length L, and change rate of the tensile strain ε.
b. Write the potential elastic energy of the rod at some moment in terms of m, ρ, E, and ε.
c. Show that the conservation of mechanical energy for the rod leads to the characteristic differential equation for an undamped harmonic oscillation. Resorting to the analogy with a point mass m acted upon by the force σS (σ being the tensile stress and S being the cross section), specify the quantity that plays here the role of coordinate.
d. Write the expression for the period of small longitudinal oscillations of the rod in terms of L, ρ, and E.

B. Consider an elastic sphere of radius R, made of the same material as the rod before. Let ε be the tensile strain $\Delta R/R$. The sphere is slightly compressed uniformly and then let to oscillate freely.
e. Write the mechanical energy of the sphere at some moment in terms of m, R, ρ, E, ε and the change rate of ε.
f. Write down the expression for the period of small radial oscillations of the sphere in terms of R, ρ, and E.

C. Consider a very thin rectangular elastic plate, prevented from bending. The plate has the linear dimensions L and l respectively. The thickness of the plate is to be taken as constant.
The effects of the tensile stresses σ_x and σ_y acting on the plate are **NOT** independent, in the sense that a stretching on one of the directions leads to a shrinking on the other direction. In the limits of Hooke's Law, this can be written as:

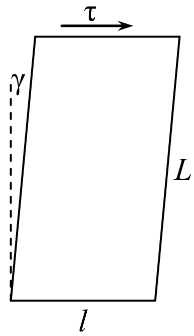
$$\varepsilon_x = -\mu \frac{\sigma_y}{E}, \qquad \varepsilon_y = -\mu \frac{\sigma_x}{E}$$

where the dimensionless factor μ (Poisson's ratio) is somewhere in the range of 0.3.
g. Express σ_x and σ_y in terms of ε_x, ε_y, E, and μ.
h. Write the system of differential equations for the movement of a point mass m on two orthogonal directions, using as coordinates the equivalent found at point **c**.
i. Find the possible values of ω for which the solutions of the above system are simple undamped harmonic oscillations (modes):

$$\varepsilon_x = A \sin \omega t, \qquad \varepsilon_y = B \sin \omega t.$$

Express the results in terms of L, l, ρ, E, and μ.

j. In general, the solution of the above system of equations is a superposition of the two modes found. For a square plate ($L = l$) and a weak Poisson ratio ($\mu^2 << 1$), express the beats period in terms of μ and the period T_{long} of longitudinal oscillations of a rod with the same length.



D. Now, instead of being squeezed, the plate is slightly slanted along one of the dimensions, by the action of a shear stress τ, as in the diagram alongside. The shear strain is defined to be $\tan\gamma \approx \gamma$, and Hooke's Law takes the form:

$$\gamma = \frac{1}{G}\tau \; ; [\tau]_{\text{SI}} = [G]_{\text{SI}} = \text{N/m}^2 \,,$$

where G is the so-called *shear modulus*.
k. Express G in terms of E and μ.
l. Find the period of the small slanting oscillations of the plate in terms of L, ρ, and G. Express the same period in terms of μ and the period of a rod with the same length undergoing longitudinal oscillations, T_{long}.

E. A cylinder of radius R and length L, made of the same material as before, is slightly twisted and let to oscillate freely. The torsion strain is defined as the angle θ with which the cylinder is twisted, under the stress of a torque. Hooke's Law takes now the form:

$$\theta = \frac{1}{C}M \; ; [C]_{\text{SI}} = [M]_{\text{SI}} = \text{Nm} \,,$$

where C is the so-called *elastic torsion constant*.
m. Find the period of the small twisting oscillations of the cylinder in terms of L, ρ, and G.
n. Express the elastic torsion constant in terms of R, L, E, and μ.

3. FUNDAMENTS OF GENERAL RELATIVITY

Einstein declared that the idea of the Equivalence Principle (1907) was "the most fortunate thought (*die glückichste Gedanken*) of my life". He recalls that "I was sitting in a chair in the patent office at Bern when all of a sudden a thought occurred to me: ***if a person falls freely, he will not feel his own weight***. I was startled. This simple thought made a deep impression on me. It impelled me toward a theory of gravitation."
So, if an observer is in free fall, he can be regarded as an inertial reference frame in which the gravitational field is abolished. Unfortunately, since different observers in space and time would be falling at different rates and/or in different directions, Einstein realized that one will have to use only ***local*** reference frames, so that inside each of them the acceleration due to gravity would be constant in both magnitude and direction.
Consequently, consider a point-like observer of negligible but nonzero mass in the vicinity of a massive object of mass M. If the massive object's gravitational field is not too strong, and in the absence of other forces, the observer will move along a gravitational field line with an acceleration equal in each point to the gravitational acceleration in that particular point. However, if mass M is sufficiently large, the notion of gravitational field loses meaning and one is forced to work instead with general relativistic concepts. What remains true, however, is that in situations with planar symmetry the elementary spacetime interval can be written as

$$ds^2 = -c^2\left(\mathrm{d}t'\right)^2 = -fc^2\left(\mathrm{d}t\right)^2 + \frac{\left(\mathrm{d}x\right)^2}{g} + \left(\mathrm{d}y\right)^2 + \left(\mathrm{d}z\right)^2,$$

where $\mathrm{d}t'$ is the infinitesimal time interval measured by the observer's clock. In general f and g are functions of the spacetime coordinates. Thus, the observed elementary displacements $\mathrm{d}x$ can be regarded as being affected by some local shrinking factor, and the corresponding times $\mathrm{d}t$ needed to accomplish the said displacements can be considered as being affected by some local stretching factor.
So the difficulty arising in General Relativity is the fact that the known expression for $\mathrm{d}s^2$ (called *the Minkowski metric*), in which there are no variable factors whatsoever, no longer holds. In what follows, we will address the simplest possible situation and we will try to describe the events in spacetime using other sets of coordinates (or, more correctly, parameters) than the usual ones, with corresponding shrinking/stretching factors. These will lead us to other more suitable expressions for the spacetime metric $\mathrm{d}s^2$.

Let us consider a point-like object of rest mass m_0, initially ($t = t' = 0$ s) at rest on the x-axis in a point $x_0 = c^2/a'$. It now starts "falling freely" in the positive direction of the axis, so that at any time the proper acceleration experienced in an inertial reference frame momentarily co-moving with the object is constant, a'.
a. Write down the expression of the acceleration of the body in the "gravitational field" reference frame, in terms of its "falling" velocity v, a' and c. Show that the net force acting upon the body is constant.
b. Find the expression of v in terms of t, a' and c.
(*Hint*: in the integral, take v to be proportional to a trigonometric function.)

For what follows, we need to define the hyperbolic functions sinh, cosh and tanh:

$$\sinh x \stackrel{\text{def}}{=} \frac{e^x - e^{-x}}{2} \; ; \; \cosh x \stackrel{\text{def}}{=} \frac{e^x + e^{-x}}{2} \; ; \; \tanh x \stackrel{\text{def}}{=} \frac{\sinh x}{\cosh x}.$$

Of course, there are also appropriate definitions for the inverse hyperbolic functions arcsinh, arccosh and arctanh.

c. Find the expression of the proper time t' of the "falling" body in terms of t, a' and c. (*Hint*: in the integral, take t to be proportional to a hyperbolic function. Denote the argument of this function by τ.)

As you can see, τ is proportional to t'. In other words, τ = constant describes events that are simultaneous from the point of view of the "falling" observer. So the next step would be to try to introduce one other parameter, say ρ, so that ρ = constant describes phenomena at rest relative to the "falling" observer.

d. Using the Minkowski metric and the transformation found for the time, find the expression of the position x of the body in terms of a', τ and c.

e. Write down the equation of the worldline (the trajectory) of the body in the two-dimensional spacetime ct-x (the coordinates y and z are of no particular interest here). Draw the graph of ct versus x, plotting also the past and future lightcones of a stationary observer found in the origin of the system. (The past lightcone is the region of spacetime from which signals can reach the origin; the future lightcone is the region of spacetime to which signals can be transmitted from origin.) On the same diagram draw the wordline of a stationary object having some coordinate $x_1 > x_0$, which the "falling" object will pass by at some time.

f. In light of what we said above, it will prove to be very convenient to choose the magnitude of the constant ρ corresponding to our body **at rest** in a reference frame deprived of gravity, say ρ_0, as being equal to the spatial constant term intervening in the equation found at the previous point. Express ρ_0 in terms of a' and c.

g. Now we will naturally extend these two new „coordinates" found for the "falling" body to an (almost) arbitrary event in spacetime. Express x and ct in terms of ρ and τ. Conversely, express ρ and τ in terms of x and ct. What is the maximal region of spacetime that can be parameterized using these coordinates?

h. Transform the Minkowski metric in terms of ρ and τ, and identify the factors f and g mentioned in the introduction to this problem.

OK, so let's sit back for a moment and get a better look at this new metric you found. It is called a *Rindler metric*, and it looks analogous to the parameterization of a plane using polar coordinates. As one would probably expect, its factors f and g **are not invariant under a Lorentz transformation**, but in this most simple case one can always return to the Minkowski metric in order to get a globally invariant metric. However in general it proves to be impossible to have an invariant metric.

You also saw that the Rindler metric cannot cover all spacetime. Even if we could extend it, one can easily see that an accelerating observer could never get information from **all** spacetime (unlike an inertial observer, whose past lightcone is bound to cover at infinity all spacetime). It is said that the events lying on the frontier of the region of spacetime from which the "falling" observer can get information make up the so-called *event horizon*.

Finally, since this new metric sees an accelerating body as being at rest, it yields that stationary objects in a gravitational field are now in motion relative to the reference frame deprived of gravity!

i. For the stationary object at x_1 mentioned earlier, express its spacetime trajectory in coordinates ρ and τ. Draw on a τ versus ρ diagram the "worldlines" of both the objects considered, and determine the limit "distance" $\Delta\rho$ of the event horizon relative to the observer in "free fall".

j. At the moment the observer starts to "fall", a beacon placed at x_0 starts emitting very short electromagnetic pulses in the positive direction of the x-axis, separated by constant time intervals T_0. How many such signals will reach the observer? Write down the expression of the wordline of the first one of them in terms of the ρ and τ coordinates. Draw on a τ versus ρ diagram the worldlines of the first three signals and of the observer.

k. Obviously the signals received will be sparser and sparser, meaning that they will have grater and greater wavelength (smaller and smaller frequency). Let ν_0 be the frequency of the emitted pulses. Express the receiving time τ in terms of the emitting time t, x_0 and c. Determine the frequency of the last received signal in terms of ν_0, x_0, T_0 and c.

l. What is the magnitude of the coordinate change rate $\mathrm{d}\rho/\mathrm{d}t'$ of the signals upon reception? Plot the graph of this "light speed along the direction of a' in the spacetime deprived of gravity" as a function of ρ.

m. One of the most important aspects when considering $\mathrm{d}\tau$ as being the time flowing in a local inertial frame, is that time at different locations on the x-axis will run differently not only as a function of that position, but also as a function of the time t elapsed from the moment when the inertial observer started to "fall freely". As the points of space pile up forming the event horizon, since the Rindler metric does not cover the entire spacetime, the time at those points seems to come to a halt. For instance, find the time $\mathrm{d}t$ elapsed at x_0 as a function of x_0, $\mathrm{d}\tau$, t and c. Considering a second point at a small distance Δx_0 to the right of x_0 (i.e. in the direction of the gravitational field), determine the relative slowing down of two clocks running in those points, $\varepsilon = \Delta(\mathrm{d}t)/\mathrm{d}t$, in terms of x_0, Δx_0, t and c.

n. Now suppose that at $t = 0$ the observer starts "falling" from rest on a short distance Δx_0, so we can approximately interpret a' as being the gravitational acceleration g of a very weak and almost uniform gravitational field, such the one in the vicinity of the surface of the Earth. Estimate the relative slowing down of a clock running at the surface of the Earth with respect to another identical clock running at the altitude of the ISS, $h = 360$ km. How much time would that mean for an astronaut spending one year on a mission on the ISS? (Neglect the fact that the station is moving around the Earth.)

Contestant code

ANSWER SHEET FOR PROBLEM No. 1

a.

$B(r) =$

Draw the magnetic field lines here:

b.

$T_{\text{slant}} =$

c.

$B_{\text{OUT}}(r) =$

$B_{\text{IN}}(r) =$

Draw the magnetic field lines here:

d.

$B_{\text{WIRE SIDE}}(r) =$

$B_{\text{OTHER SIDE}}(r) =$

Draw the magnetic field lines here:

Contestant code

e.

$\Delta B_{\parallel} =$

f.

$p =$

g.

$B_{OUT}(r) =$

$B_{IN}(r) =$

Draw the magnetic field lines here:

h.

$B_{OUT}(r) =$

$B_{IN}(r) =$

Draw the magnetic field lines here:

i.

$B(r) =$

Contestant code

j.

$B_{OUT}(r) =$

$B_{IN}(r) =$

Contestant code

ANSWER SHEET FOR PROBLEM No. 2

a.

$E_{kin} =$

b.

$E_{pot} =$

c. Write the differential equation for ε:

$\ddot{x}_{equivalent} =$

d.

$T_{long} =$

e.

$E_{mech} =$

f.

$T_{radial} =$

Contestant code

g.

$\sigma_x =$

$\sigma_y =$

h. Write the differential equations for $\ddot{x}_{\text{equivalent}}$ and $\ddot{y}_{\text{equivalent}}$:

i.

$\omega_{1;2} =$

j.

$T_{\text{beats}} =$

k.

$G =$

l.

$T_{\text{slant}} =$

m.

$T_{\text{slant}} =$

Contestant code

n.

T_{twist} =

o.

C =

Contestant code

ANSWER SHEET FOR PROBLEM No. 3

a.

$a =$
$F_{\text{net}} =$

b.

$v =$

c.

$t' =$

d.

$x =$

Contestant code

e. Write down the equation of the worldline and draw the diagram here.

f.

ρ_0 =

g.

x = ρ =

ct = τ =

Maximal region of spacetime parameterized by the Rindler metric:

Contestant code

h.

$ds^2 =$

$f =$ $g =$

i. Write down the equation of the worldline and draw the diagram here.

$\Delta\rho =$

Contestant code

j. Write down the equation of the worldline and draw the diagram here.

N =

k.

τ =

ν_N =

Contestant code

l. Write the expression of $d\rho/dt'$ and plot the graph here.

m.

$dt =$

$\varepsilon =$

n.

$\varepsilon =$

$\Delta t =$