SOLUTION
COMPUTER EXPERIMENT:
A mathematical pendulum or what angle can be considered rather small ...

1. Constructing a theoretical model.

1.1 The formula for the period of small oscillations of a mathematical pendulum has the form

$$T = 2\pi\sqrt{\frac{l}{g}}. \quad (1)$$

1.2 From the law of conservation of mechanical energy for the ball of the pendulum (the zero level of the potential energy is taken at the suspension point)

$$\frac{ml^2\omega^2}{2} = mgl(\cos\varphi - \cos\varphi_0), \quad (2)$$

the following formula is derived for the angular velocity of the pendulum

$$\omega = \sqrt{\frac{2g}{l}(\cos\varphi - \cos\varphi_0)}. \quad (3)$$

1.3 Let us divide the entire section of motion from φ_0 to zero into infinitely small intervals $d\varphi$. The time dt it takes for the pendulum to pass this interval is found as

$$dt = \frac{d\varphi}{\omega} = \frac{d\varphi}{\sqrt{\frac{2g}{l}(\cos\varphi - \cos\varphi_0)}}. \quad (4)$$

Then the time of motion t_1 is obtained as the sum of small intervals, which finally reduces to a simple integration

$$t_1 = \int_0^{\phi_0} dt = \int_0^{\phi_0} \frac{d\varphi}{\sqrt{\frac{2g}{l}(\cos\varphi - \cos\varphi_0)}}. \quad (5)$$

1.4 The oscillation period is 4 times longer than the found time t_1, viz.

$$T = 4t_1. \quad (6)$$

1.5 The angular velocity in dimensionless units is expressed as follows

$$\tilde{\omega} = \frac{d\varphi}{d\tau} = \frac{d\varphi}{\sqrt{\frac{g}{l}}dt} = \sqrt{\frac{l}{g}}\,\omega. \quad (7)$$

1.6 The period of small oscillations in dimensionless units reads as

$$\tilde{T} = \sqrt{\frac{g}{l}}\,T = 2\pi. \quad (8)$$

1.7 The dependence of the angular velocity $\tilde{\omega}$ on the deflection angle φ has the form

$$\tilde{\omega}(\varphi) = \sqrt{2(\cos\varphi - \cos\varphi_0)}. \quad (9)$$

2. Designing an experimental setup, planning an experiment.

2.1 The partition interval is equal to

$$\Delta\varphi = \frac{\varphi_0}{N}. \quad (10)$$

2.2 First, you should set the "zeroth" angle of deflection, and the coordinates of the rest of the splitting points are given by the formula

$$\varphi_k = \varphi_{k-1} - \Delta\varphi. \quad (11)$$

2.3 The angular velocity ω_k at the point φ_k is described by the formula

$$\omega_k = \sqrt{2(\cos\varphi_k - \cos\varphi_0)}. \quad (12)$$

In the particular case of $\varphi_0 = \frac{\pi}{2}$ this formula further simplifies to

$$\omega_k = \sqrt{2\cos\varphi_k}\ . \quad (12a)$$

2.4 In the recommended approximation of uniformly accelerated motion, the average speed at the selected interval is equal to the arithmetic mean of the angular velocities at the ends of the interval, i.e.

$$\langle\omega\rangle = \frac{1}{2}\left(\omega_{k-1} + \omega_k\right),$$

Therefore, the travel time Δt_k of the k 'th interval from φ_{k-1} to φ_k is obtained as

$$\Delta t_k = \frac{2\Delta\varphi}{\omega_{k-1} + \omega_k}\ . \quad (13)$$

2.5 The time t_k to reach the angle φ_k is found via

$$t_k = t_{k-1} + \Delta t_k \quad (14)$$

at the initial condition $t_0 = 0$.

2.6 The oscillation period T_N when dividing into N intervals is eventually written as

$$T_N = 4t_N\ . \quad (15)$$

3. Trial experiment, estimation of errors.

3.1 The results of calculating the angular velocities, times and periods of oscillations for the indicated values of the number of partition intervals are shown in Table 1.

Table 1. Calculation of periods of oscillations with different numbers of partition intervals.

***N*=**	**32**				***N*=**	**16**			
$\Delta\varphi$	0,0491				$\Delta\varphi$	0,0982			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	1,5708	0,0000		0,0000	0	1,5708	0,0000		0,0000
1	1,5217	0,3133	0,3134	0,3134	1	1,4726	0,4428	0,4435	0,4435
2	1,4726	0,4428	0,1299	0,4432	2	1,3744	0,6246	0,1840	0,6274
3	1,4235	0,5417	0,0997	0,5430	3	1,2763	0,7620	0,1416	0,7690
4	1,3744	0,6246	0,0842	0,6271	4	1,1781	0,8749	0,1200	0,8890
5	1,3254	0,6971	0,0743	0,7014	5	1,0799	0,9710	0,1064	0,9954
6	1,2763	0,7620	0,0673	0,7687	6	0,9817	1,0541	0,0970	1,0923
7	1,2272	0,8208	0,0620	0,8307	7	0,8836	1,1264	0,0900	1,1824
8	1,1781	0,8749	0,0579	0,8886	8	0,7854	1,1892	0,0848	1,2672
9	1,1290	0,9247	0,0546	0,9432	9	0,6872	1,2434	0,0807	1,3479
10	1,0799	0,9710	0,0518	0,9950	10	0,5890	1,2896	0,0775	1,4254
11	1,0308	1,0140	0,0495	1,0444	11	0,4909	1,3281	0,0750	1,5004
12	0,9817	1,0541	0,0475	1,0919	12	0,3927	1,3593	0,0731	1,5735
13	0,9327	1,0915	0,0458	1,1377	13	0,2945	1,3834	0,0716	1,6451
14	0,8836	1,1264	0,0443	1,1819	14	0,1963	1,4006	0,0705	1,7156
15	0,8345	1,1589	0,0430	1,2249	15	0,0982	1,4108	0,0698	1,7854
16	0,7854	1,1892	0,0418	1,2667	16	0,0000	1,4142	0,0695	1,8549
17	0,7363	1,2173	0,0408	1,3075					
18	0,6872	1,2434	0,0399	1,3474					
19	0,6381	1,2674	0,0391	1,3865					
20	0,5890	1,2896	0,0384	1,4249					
21	0,5400	1,3098	0,0378	1,4626					
22	0,4909	1,3281	0,0372	1,4999					
23	0,4418	1,3446	0,0367	1,5366					

24	0,3927	1,3593	0,0363	1,5729					
25	0,3436	1,3723	0,0359	1,6088					
26	0,2945	1,3834	0,0356	1,6445					
27	0,2454	1,3929	0,0354	1,6798					
28	0,1963	1,4006	0,0351	1,7150					
29	0,1473	1,4065	0,0350	1,7500					
30	0,0982	1,4108	0,0348	1,7848					
31	0,0491	1,4134	0,0348	1,8196					
32	0,0000	1,4142	0,0347	1,8543					
			T_N=	**7,4171**				**T_N=**	**7,4197**

***N*=**	**8**				***N*=**	**4**			
$\Delta\varphi$	0,1963				$\Delta\varphi$	0,3927			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	1,5708	0,0000		0,0000	0	1,5708	0,0000		0,0000
1	1,3744	0,6246	0,6287	0,6287	1	1,1781	0,8749	0,8977	0,8977
2	1,1781	0,8749	0,2619	0,8906	2	0,7854	1,1892	0,3805	1,2783
3	0,9817	1,0541	0,2036	1,0941	3	0,3927	1,3593	0,3082	1,5864
4	0,7854	1,1892	0,1751	1,2692	4	0,0000	1,4142	0,2832	1,8696
5	0,5890	1,2896	0,1584	1,4276					
6	0,3927	1,3593	0,1483	1,5759					
7	0,1963	1,4006	0,1423	1,7182					
8	0,0000	1,4142	0,1395	1,8577					
			T_N=	**7,4307**				**T_N=**	**7,4785**

***N*=**	**2**				***N*=**	**1**			
$\Delta\varphi$	0,7854				$\Delta\varphi$	1,5708			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	1,5708	0,0000		0,0000	0	1,5708	0,0000		0,0000
1	0,7854	1,1892	1,3209	1,3209	1	0,0000	1,4142	2,2214	2,2214
2	0,0000	1,4142	0,6034	1,9242					
			T_N=	**7,6969**				**T_N=**	**8,8858**

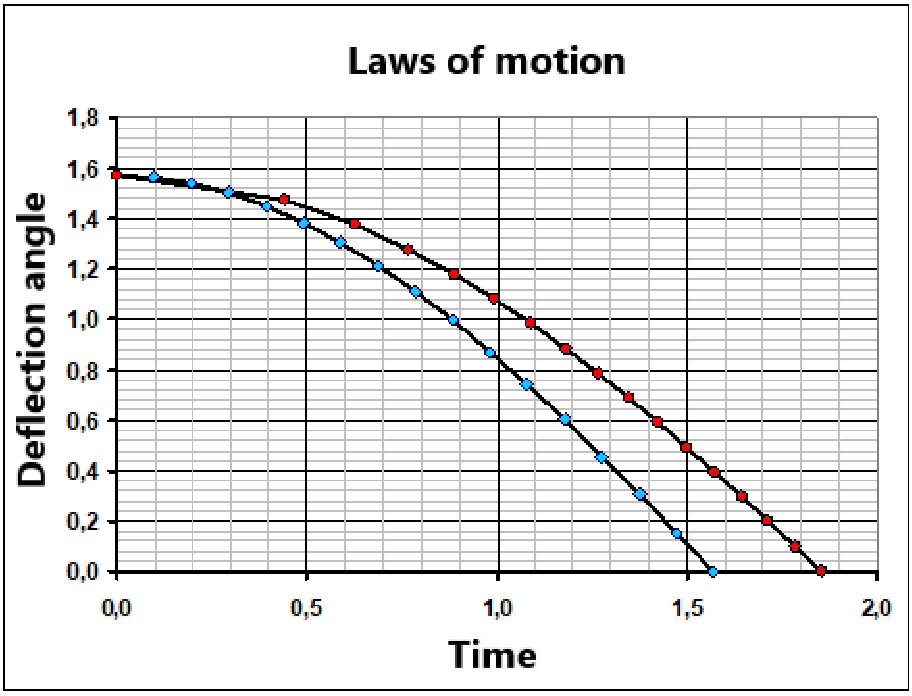
3.2 – 3.3 To calculate the graph points of the law of motion in the approximation of small oscillations, it is necessary to use the formula

$$\varphi(t) = \frac{\pi}{2}\cos t\,. \qquad (16)$$

The calculation results for this law are presented in Table 2 and in the graph. There is also a line in the graph representing the calculated law of motion (16). It is interesting to note that in the first case, the time values are set, and the corresponding deflection angles are calculated; and in the second one, on the contrary, the deflection angles are set and corresponding times are calculated.

Table 2.

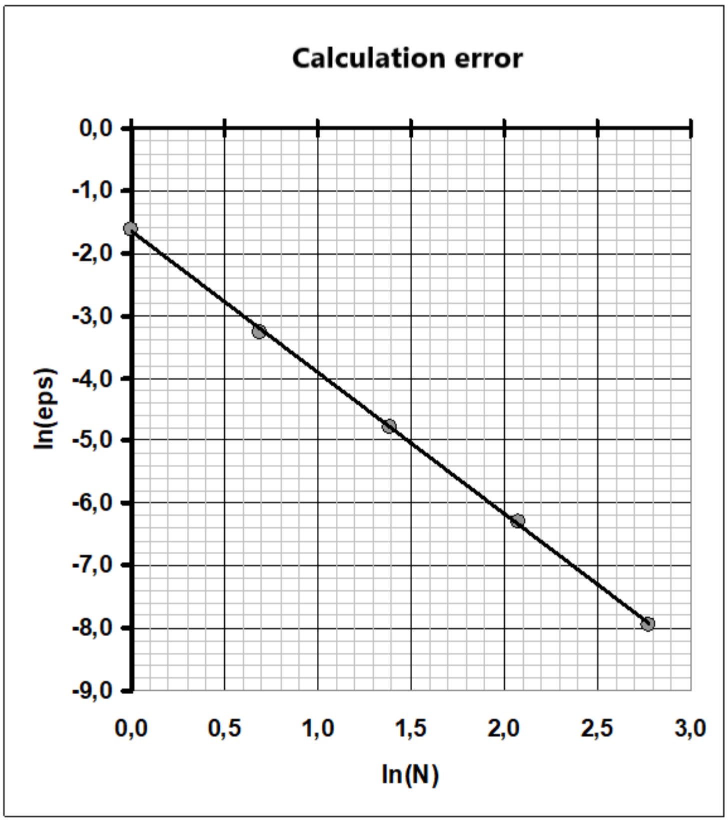
k	t	φ
0	0,0000	1,5708
1	0,0982	1,5632
2	0,1963	1,5406
3	0,2945	1,5032
4	0,3927	1,4512
5	0,4909	1,3853
6	0,5890	1,3061
7	0,6872	1,2142
8	0,7854	1,1107
9	0,8836	0,9965
10	0,9817	0,8727
11	1,0799	0,7405
12	1,1781	0,6011
13	1,2763	0,4560
14	1,3744	0,3064
15	1,4726	0,1540
16	1,5708	0,0000



3.4 - 3.5 The results of calculating the errors ε_N of the oscillation periods for different numbers N of partition intervals are shown in Table 3.

Table 3. Calculation errors.

N	T	ε_N	$\ln N$	$\ln \varepsilon_N$
1	8,8858	1,98E-01	0,0000	-1,6195
2	7,6969	3,77E-02	0,6931	-3,2774
4	7,4785	8,27E-03	1,3863	-4,7952
8	7,4307	1,83E-03	2,0794	-6,3026
16	7,4197	3,50E-04	2,7726	-7,9584
32	7,4171	0,00E+00		



To determine the parameters of the dependence $\varepsilon_N = \dfrac{C}{N^\gamma}$, it must be represented on a double logarithmic scale as

$$\ln \varepsilon_N = \ln C - \gamma \ln N . \qquad (17)$$

The figure on the right shows a graph of this dependence, drawn according to the data in Table 3. The linearity of the obtained dependence clearly proves the applicability of the formula for the dependence of the calculation error on the number N .

The parameters of this linear relationship, calculated using the least squares method, are equal: the slope coefficient $a \approx -2{,}3$ and the shift $b \approx -1{,}65$. Therefore, the sought parameters of the dependence are found as

$$\begin{aligned} \gamma &\approx -a = 2{,}3 \\ C &= \exp(b) \approx 0{,}19 \end{aligned} . \qquad (18)$$

3.6 It is easy to find from formula (17) that the number of partition intervals required to achieve the error $\varepsilon = 0{,}002$ is expressed as

$$N_{\min} = \left(\frac{C}{\varepsilon}\right)^{1/\gamma} \approx 8. \qquad (19)$$

So, all further calculations should be carried out at $N = N_{\min} = 8$.

4. Experiment: the dependence of the period on the amplitude.

4.1 The results of calculating the periods of oscillations for various amplitudes of oscillations are shown in Table 4.

Table 4. Calculation of periods of oscillations.

φ_0	0,2618		15°		φ_0	0,5236		30°	
N=	8				N=	8			
$\Delta\varphi$	0,0327				$\Delta\varphi$	0,0654			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	0,2618	0,0000		0,0000	0	0,5236	0,0000		0,0000
1	0,2291	0,1261	0,5190	0,5190	1	0,4581	0,2484	0,5270	0,5270
2	0,1963	0,1724	0,2193	0,7383	2	0,3927	0,3402	0,2224	0,7494
3	0,1636	0,2036	0,1741	0,9124	3	0,3272	0,4023	0,1763	0,9257
4	0,1309	0,2259	0,1524	1,0648	4	0,2618	0,4470	0,1541	1,0799
5	0,0982	0,2419	0,1399	1,2047	5	0,1963	0,4791	0,1413	1,2212
6	0,0654	0,2527	0,1323	1,3370	6	0,1309	0,5008	0,1336	1,3548
7	0,0327	0,2590	0,1279	1,4649	7	0,0654	0,5135	0,1291	1,4839
8	0,0000	0,2611	0,1259	1,5908	8	0,0000	0,5176	0,1269	1,6108
			T=	6,3630				T=	6,4432
			T/To=	1,0127				T/To=	1,0255

φ_0	0,7854		45°		φ_0	1,0472		60°	
N=	8				N=	8			
$\Delta\varphi$	0,0982				$\Delta\varphi$	0,1309			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	0,7854	0,0000		0,0000	0	1,0472	0,0000		0,0000
1	0,6872	0,3631	0,5408	0,5408	1	0,9163	0,4664	0,5613	0,5613
2	0,5890	0,4987	0,2278	0,7687	2	0,7854	0,6436	0,2359	0,7972
3	0,4909	0,5913	0,1801	0,9488	3	0,6545	0,7660	0,1857	0,9829
4	0,3927	0,6584	0,1571	1,1059	4	0,5236	0,8556	0,1614	1,1444
5	0,2945	0,7069	0,1438	1,2497	5	0,3927	0,9207	0,1474	1,2917
6	0,1963	0,7398	0,1357	1,3855	6	0,2618	0,9653	0,1388	1,4306
7	0,0982	0,7590	0,1310	1,5165	7	0,1309	0,9914	0,1338	1,5643
8	0,0000	0,7654	0,1288	1,6453	8	0,0000	1,0000	0,1315	1,6958
			T=	6,5810				T=	6,7832
			T/To=	1,0474				T/To=	1,0796

φ_0	1,3090		75°		φ_0	1,5708		90°	
N=	8				N=	8			
$\Delta\varphi$	0,1636				$\Delta\varphi$	0,1963			
k	φ	ω	Δt_k	t_k	***k***	φ	ω	Δt_k	t_k
0	1,3090	0,0000		0,0000	0	1,5708	0,0000		0,0000
1	1,1454	0,5548	0,5899	0,5899	1	1,3744	0,6246	0,6287	0,6287
2	0,9817	0,7704	0,2469	0,8368	2	1,1781	0,8749	0,2619	0,8906
3	0,8181	0,9217	0,1934	1,0302	3	0,9817	1,0541	0,2036	1,0941
4	0,6545	1,0340	0,1673	1,1976	4	0,7854	1,1892	0,1751	1,2692

5	0,4909	1,1163	0,1522	1,3497	5	0,5890	1,2896	0,1584	1,4276
6	0,3272	1,1731	0,1429	1,4927	6	0,3927	1,3593	0,1483	1,5759
7	0,1636	1,2065	0,1375	1,6302	7	0,1963	1,4006	0,1423	1,7182
8	0,0000	1,2175	0,1350	1,7652	8	0,0000	1,4142	0,1395	1,8577
			T=	7,0608				T=	7,4307
			T/To=	1,1238				T/To=	1,1826

Below is a summary table for calculations of the oscillation period for different amplitudes.

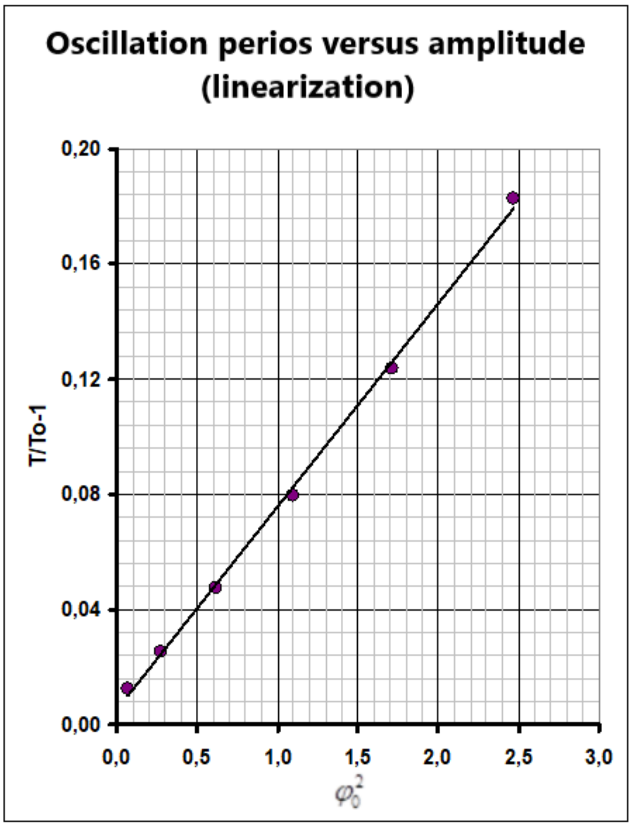
Table 5.

$\varphi_0,°$	φ_0	T	$\frac{T}{T_0}$	φ_0^2	$\frac{T}{T_0}-1$
15	0,2618	6,3630	1,0127	0,0685	0,0127
30	0,5236	6,4432	1,0255	0,2742	0,0255
45	0,7854	6,5810	1,0474	0,6169	0,0474
60	1,0472	7,0608	1,0796	1,0966	0,0796
75	1,3090	7,0608	1,1238	1,7135	0,1238
90	1,5708	7,4307	1,1826	2,4674	0,1826

4.2 – 4.3 For small oscillations, the formula for the oscillation period

$$T(\varphi_0) = T_0\left(a + \frac{\varphi_0^2}{b}\right) \qquad (20)$$

must coincide with formula (1), whence it follows that the parameter $a = 1$. To check the applicability of formula (20) to the description of the calculation results, it is necessary to draw a graph of the dependence of the value $\left(\frac{T}{T_0} - 1\right)$ on the square of the amplitude φ_0^2, which is shown in the figure on the right.



The slope coefficient of the obtained dependence is 0,0706, therefore, the parameter b appearing in formula (20) is approximately equal to

$$b \approx 14. \qquad (21)$$

4.4 With an acceptable error of a real experiment is 5%, the deviation of the period from the period of small oscillations would not be noticeable if the inequality holds

$$\frac{\varphi_0^2}{b} < 0.05, \qquad (22)$$

whence it follows that the angles can be considered rather small for $\varphi_0 < 45°$.

	Content	**Total for each part**	**Points**
	1. Constructing a theoretical model.	**1.5**	
1.1	Formula (1): $T = 2\pi\sqrt{\frac{l}{g}}$	**0.1**	0.1

1.2	Law of motion - conservation of energy (2); - formula (3): $\omega = \sqrt{\frac{2g}{l}\left(\cos\varphi - \cos\varphi_0\right)}$	**0.4**	 0.1 0.3
1.3	Integration of the law of motion - formula (4): $dt = \frac{d\varphi}{\omega}$ - formula (5): $t_1 = \int_0^{\phi_0} \frac{d\varphi}{\sqrt{\frac{2g}{l}\left(\cos\varphi - \cos\varphi_0\right)}}$	**0.3**	 0.1 0.2
1.4	Period (6): $T = 4t_1$	**0.1**	0.1
1.5	Formula (7): $\widetilde{\omega} = \sqrt{\frac{l}{g}}\omega$	**0.2**	0.2
1.6	Formula (8): $\widetilde{T} = 2\pi$	**0.2**	0.2
1.7	Formula (9): $\tilde{\omega}\left(\varphi\right) = \sqrt{2(\cos\varphi - \cos\varphi_0)}$	**0.2**	0.2
2. Designing an experimental setup, planning an experiment.		**1.5**	
2.1	Formula (10): $\Delta\varphi = \frac{\varphi_0}{N}$	**0.1**	0.1
2.2	Formula (11): $\varphi_k = \varphi_{k-1} - \Delta\varphi$	**0.1**	0.1
2.3	Formula (12): $\omega_k = \sqrt{\cos\varphi_k - \cos\varphi_0}$	**0.1**	0.1
2.4	Uniformly accelerated motion Main idea $\langle\omega\rangle = \frac{1}{2}\left(\omega_{k-1} + \omega_k\right)$ Formula (13): $\Delta t_k = \frac{2\Delta\varphi}{\omega_{k-1} + \omega_k}$	**1.0**	 0.5 0.5
2.5	Formula (14): $t_k = t_{k-1} + \Delta t_k$	**0.1**	0.1
2.6	Formula (15): $T_N = 4t_N$	**0.1**	0.1
3. Trial experiment, estimation of errors.		**9.0**	
3.1	*Graded only if the periods obtained differ from those in this official solution lees than.* Periods are correctly calculated for N=32 N=16 N=8 N=4 N=2 N=1	**4.2**	 1.2 1.0 0.8 0.6 0.4 0.2
3.2	Graph: All points are plotted in accordance with the table; Smooth line is drawn;	**0.5**	 0.3 0.2
3.3	Graph Law of motion is obtained, table 2 All points are plotted in accordance with the table Smooth line is drawn;	**1.0**	 0.5 0.3 0.2
3.4	Errors are correctly evaluated, table 3	**0.5**	0.5
3.5	Graph Double logarithm scale is used Linearized dependence is drawn	**2.3**	 0.5 0.5

	Linear dependence is obtained The power is found as $\gamma \approx 2{,}3 \pm 0{,}1$ The coefficient is found as $C \approx 0{,}19 \pm 0{,}1$		0.3 0.5 0.5
3.6	Concluded that $N = 8 \pm 1$	**0.5**	0.5
4. Experiment: the dependence of the period on the amplitude.		**8.0**	
4.1	*Graded only if the periods obtained differ from those in this official solution lees than 0,02* For each period obtained One period is added from part 3	**4.2**	 0.8 0.2
4.2	Graph of the linearized dependence Correct linearization is used $T(\varphi_0^2)$ Graph is plotted Linearized dependence is obtained	**1.5**	 0.5 0.5 0.5
4.3	Parameters of the linearized dependence Parameter $a = 1$ (only exact value is accepted) Parameter b in the range 13-16	**1.3**	 0.3 1.0
4.4	Small angle is estimated as $\varphi < 45°$	**1.0**	1.0
	TOTAL	**20.0**	