THEORETICAL COMPETITION

January 12, 2018

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country**)**,** your student code (**Student Code**), the question number (**Question Number**), the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

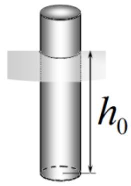
   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

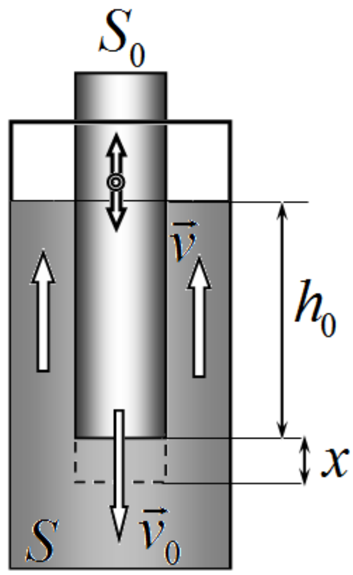
Problem 1 (10.0 points)

This problem consists of three independent parts.

Problem 1A (3.0 points)

A1 A narrow cylindrical test-tube with a displaced center of mass floats vertically in water in a very wide vessel. In equilibrium state, the test-tube is immersed into water to a depth h_0. The cross-sectional area of the tube is S_0. Determine the period of small vertical oscillations of the test-tube.



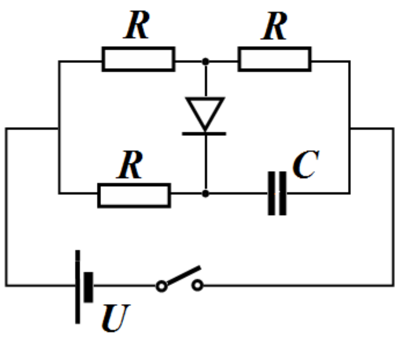


A2 The same test-tube is placed in a narrow cylindrical vessel with a cross-sectional area S filled with water. The test-tube makes small oscillations along the axis of the vessel.

A2.1 The test-tube sinks by some small value x. Express the change in the potential energy of the system through x, the initial depth of immersion h_0, cross-sectional areas S_0, S, the water density ρ and the acceleration of gravity g.

A2.2 Let the speed of the test-tube be v_0 near its equilibrium position. Express the kinetic energy of the system through the speed of the test tube v_0, the depth of immersion h_0, the cross-sectional areas S_0, S, and the water density ρ. Consider that in the gap between the test-tube and the walls of the vessel all liquid moves with the same speed v.

A2.3 Find the period of small oscillations of the test-tube in the narrow vessel.



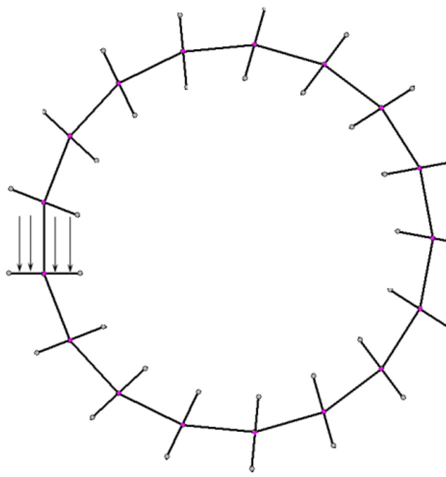
Problem 1B (4.0 points)

The circuit, shown in the figure on the left, consists of a capacitor with the capacitance C = 100 μF, an ideal diode, a constant voltage source U = 10.0 V, three identical resistors with the resistance R = 10.0 kΩ and a switch. At the initial moment, the capacitor is not charged and the switch is open. When the switch is shorted, the current through the diode goes for the time interval τ = 462 ms, and then stops.

1. Find the current through the diode immediately after shorting the switch.
2. Find the total charge that has flowed through the diode.

Problem 1C (3.0 points)

At the vertices of the regular 17-gon, there are 17 identical lenses. The optical centers of the lenses are located exactly at the vertices of the polygon, the planes of all lenses are perpendicular to one of the sides adjacent to the lens. The focal lengths of the lenses are all equal to $F = 10$ sm and coinciude with the length of the side of the 17-gon. One of the lenses is illuminated by a parallel light flux directed along its optical axis. It turns out that one of the rays has a closed trajectory. Determine the radius of the circle inscribed in this trajectory. Consider two cases: all of the lenses are collecting; all of the lenses are diverging. Consider all angles small such that $\sin\alpha \approx \tan\alpha \approx \alpha$.



Problem 2 (10.0 points)
Physics in the mountains

The atmosphere of a real planet, such as the Earth, has a rather complex structure in view of the great variety of processes and phenomena involved in its formation. In this problem we will consider two simple models of the lower layer of the atmosphere, called the troposphere, which extends to an altitude of 10-15 km above the Earth's surface. To understand the physics of some phenomena it is sufficient to consider the Earth's atmosphere consisting of a single-component diatomic gas with the molar mass $\mu_{air} = 28.9 \cdot 10^{-3}$ kg/mole.

Part 1. Isothermal atmosphere

In the atmosphere, the lowest near-surface layer has an almost constant temperature, as it is heated up by the surface of the Earth. Therefore, we assume in this part that the temperature of the atmosphere remains the same over its entire altitude and is equal to $T_0 = 293$ K, and the air pressure at the Earth's surface is $p_0 = 1.013 \cdot 10^5$ Pa. Assume that the acceleration of gravity $g = 9.81\ \mathrm{m/s^2}$ is independent of the altitude above the Earth's surface, since the total height of the atmosphere is much less than the Earth's radius $R_E = 6400$ km. The universal gas constant is $R = 8.31\ \mathrm{J/(mole \cdot K)}$.

1.1 Find and calculate the mass M of the Earth's atmosphere.

1.2 Find and calculate the air pressure p_H at the altitude of $H = 1500$ m above the Earth's surface.

From the physical point of view, the interesting question is how fast the atmosphere warms up with the change of day and night. From space observations, the so-called solar constant $\alpha = 1367\ \mathrm{W/m^2}$ is known, which is the total power of the solar radiation in the region of the Earth's orbit passing through a unit of a surface oriented perpendicular to its flow.

1.3 Estimate the amount of heat δQ needed to heat the atmosphere by $\Delta T = 1$ K.

1.4 Find and calculate the time interval τ for the Sun to shine in order to provide the Earth with the amount of heat δQ.

Part 2. Adiabatic atmosphere

The real troposphere is not isothermal at all and the air temperature decreases with altitude. Due to the constantly flowing convective processes, the troposphere can be considered practically adiabatic. Let the air temperature and pressure at the Earth's surface be $T_0 = 293$ K and $p_0 = 1.013 \cdot 10^5$ Pa, respectively. Consider the acceleration of gravity $g = 9.81\ \mathrm{m/s^2}$ be still independent of the altitude above the Earth's surface.

2.1 Find and calculate the air temperature T_H at the altitude of $H = 1500$ m above the Earth's surface.

2.2 Find and calculate the air pressure p_H at the altitude of $H = 1500$ m above the Earth's surface.

In the constructed model, the height of the Earth's troposphere is determined by reaching a certain critical temperature, at which other physical processes begin to play an important role.

2.3 Estimate the height difference ΔH_{atm} of the Earth's troposphere at day and night if the surface temperature changes within this period of time by $\Delta T_{dn} = 20$ K.

A mountain climber starts climbing on a fairly high mountain, at the foot of which the temperature and the air pressure are equal to $T_0 = 293$ K and $p_0 = 1.013 \cdot 10^5$ Pa. At the altitude of $H = 1500$ m, he decides to make a halt in order to boil some water and discovers that it boils faster than usual. He opens the handbook on physics available at the moment and finds that at temperature $T_1 = 373$ K, the saturated water vapor pressure is $p_1 = p_0 = 1.013 \cdot 10^5$ Pa, and at temperature $T_2 = 365$ K it is equal to $p_2 = 0.757 \cdot 10^5$ Pa.

2.4 Find and calculate the boiling temperature of water at the altitude of $H = 1500$ m.

After having resumed his climbing, the mountain climber discovers that snow appears at a certain altitude and special equipment is needed to proceed further.

2.5 Find and calculate the altitude h_0, at which the climber noticed the appearance of a snow cover on the mountain.

The climber recalled the conversation with the locals right before climbing, in which he was informed that the snow cover completely disappears from the mountain at temperatures $T > 310$ K at the foot of the mountain.

2.6 Find and calculate the height H_0 of the mountain on which the climber ascends.

Having risen even higher along the slope of the mountain to some altitude H', the climber notices the appearance of fog. Looking around, he realizes that there are no clouds and there is no wind. The climber knows that the molar mass of water is $\mu_{H_2O} = 18.0 \cdot 10^{-3}$ kg/mole, and according to the weather forecast, the relative humidity at the foot of the mountain was equal to $\varphi = 0,15$. In the handbook on physics, he finds a formula for the pressure of saturated water vapor in the temperature range $T \in (250,300)$K, which has the following form

$$\ln \frac{p_{vap}}{p_{vapo}} = a + b \ln \frac{T}{T_0},$$

where p_{vap} denotes the saturated vapor pressure at temperature T, p_{vap0} stands for the saturated vapor pressure at temperature T_0, $a = 3.63 \cdot 10^{-2}$, $b = 18.2$ are constants. In the calculations, consider that the water vapor is always in thermodynamic equilibrium with the surrounding air.

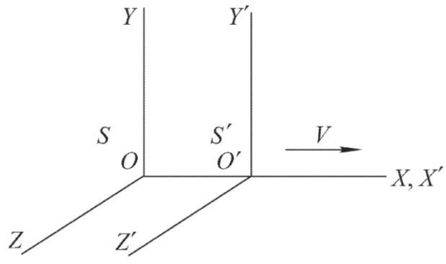
2.7 Find and calculate the altitude H'.

2.8 Find and calculate the minimum air humidity φ_{min} at the foot of the mountain such that the fog is still observed somewhere on the mountain slope.

Mathematical hint

You may need knowledge of the following integral $\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax + b|$.

Problem 3 (10.0 points)
Optics of moving media
Part 1. 4-dimensional vectors



Consider two inertial reference frames S and S' of which the second one moves at a speed V relative to the first one as shown in the figure on the right. It is assumed that the origins O and O' coincide at the initial time moment $t = t' = 0$ by the clocks of both reference frames. It is known that the Lorentz transformation of space-time coordinates (x', y', z', ct') of any event in the frame S' into the space-time coordinates (x, y, z, ct) of the same event in the frame S have the following form

$$x = \frac{x' + (V/c)ct'}{\sqrt{1 - V^2/c^2}}, \qquad y = y', \qquad z = z', \qquad ct = \frac{ct' + (V/c)x'}{\sqrt{1 - V^2/c^2}},$$

where $c = 2.9979$ m/s is the speed of light.

In the formulas of the Lorentz transformation the spatial coordinates and the time are intentionally brought to the same dimension as they together form components of the so-called 4-vector. It is known that all the components of 4-vectors are transformed in the same way at the transition from one inertial frame to the other. In particular, the momentum and the energy constitute the components of the 4-vector.

Suppose that an object moves in the reference frame S such that it has the total energy E and the momentum projections p_x, p_y, p_z on the coordinate axes OX, OY and OZ, respectively.

1.1 Write down the transformation of energy and momentum of the object from the reference frame S to the reference frame S'.

Suppose an object of the rest mass m moves in the reference frame S such that it has the total energy E and the momentum p. When converting its energy and momentum from one reference frame to another the value $E^2 - p^2c^2 = inv$ remains invariant.

1.2 Express inv in terms of m and c.

Part 2. Doppler effect and light aberration

Let a plane electromagnetic wave (EMW) propagate in XY-plane of the reference frame S so that it has the frequency ω and makes the angle φ with the axis OX.

2.1 Find the frequency ω' of EMW registered by an observer in the reference frame S'.
2.2 Find the angle φ' that EMW makes with the axis $O'X'$ in the reference frame S'.

Astronomical observations have shown that the position of a newly discovered massive star X on the celestial sphere (i.e. relative to very distant objects) does not remain constant throughout the year. It moves along an ellipse with axial ratio 0.900. Ecliptic latitude of a star is the angle between the direction to the star and the ecliptic plane, which can be assumed coincident with the plane of the Earth orbit around the Sun.

2.3 Find the ecliptic latitude δ of the star X and evaluate it in arc degrees.

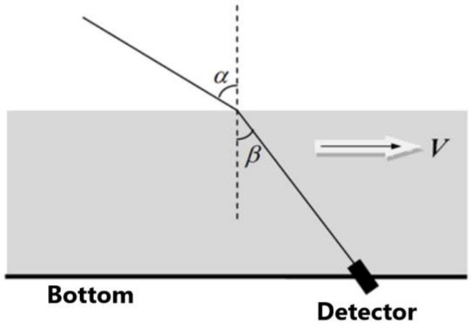
Observation of the emission spectrum of the star X has shown that all frequency in its spectrum are shifted to the red. The relative frequency deviation is $(\Delta\omega/\omega)_0 = 9.9945 \cdot 10^{-3}$. It has independently been established that the recession velocity of the star X from the Sun is equal to $v_x = \frac{1}{100}c$.

2.4 Find and evaluate the escape velocity v_{II} at the surface of the star X.

Part 3. Light in a moving medium

Consider the same two reference frames as in Part 1. Let an object move in the plane $X'Y'$ of the reference frame S' such that the projections of its velocity on the axis $O'X'$ and $O'Y'$ are equal $u_x{}'$ and $u_y{}'$, respectively.

3.1 Find the projections of the object velocity u_x on the axis OX and u_y on the axis OY in the reference frame S.



Consider a water flow moving at the speed of V relative to the bottom of the vessel. A plane EMW falls onto the water surface to make the angle α to the normal in the laboratory reference frame. A detector is fixed at the bottom of the vessel. The refractive index of water is known to be n.

If the water velocity $V \ll c$, the expression for the sine of the angle β at which the detector registers the direction of EMW propagation, takes the form:

$$\sin\beta = A_1 + B_1 V.$$

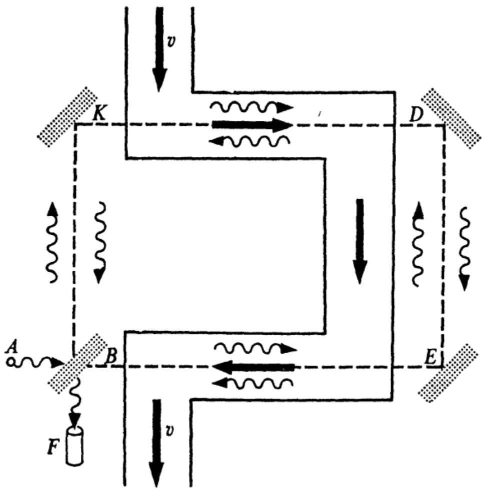
3.2 Find A_1, B_1 and express them in terms of α and n.

If the water velocity $V \ll c$, the expression for velocity v_m of the light propagation in the laboratory reference frame is found as

$$v_m = A_2 + B_2 V.$$

3.3 Find A_2, B_2 and express them in terms of β, n, c.

In 1860 A. Fizeau set up the following experiment. A monochromatic beam with the wavelength λ comes out of the source A and falls onto the semitransparent plate B at which it is divided into two coherent beams. The first beam, being reflected from the plate B, goes along the way $BKDEB$ (K, D and E are the mirrors), whereas the second beam, passing through the plate B, goes along the way $BEDKB$. The first beam, when returning to the plate B, is partially reflected from it and reaches the interferometer F. The second beam, when returning to the plate B, partially passes through it and reaches the interferometer F. Both interfering beams travels the same distance in the laboratory reference frame, including section BE and KD in which the water flows with the velocity v. The total distance covered by each beam in water in the laboratory reference frame is equal to $2L$.



3.4 Find the number of bands ΔN the interference pattern is shifted by when the liquid velocity changes from 0 to v and express it in terms of L, n, v, c, and λ.

In his own experiment A. Fizeau obtained $\Delta N = 0.230$ at $L = 1.49$ m, $v = 7.06\ \mathrm{m/s}$ and $\lambda = 536$ nm.

3.5 Evaluate refractive index of water n in Fizeau's experiment.

Mathematical hint

You may need to know the following approximate equality: $(1+x)^\alpha \approx 1 + \alpha x, \quad \text{at } x \ll 1.$