A. Earth as a blackbody

A-1. All the energy emitted from the surface of the Sun, will reach a sphere of radius d , therefore:

$$\sigma T_S^4.(4\pi R_S^2) = (4\pi d^2).S_0$$

$$S_0 = \sigma T_S^4.\left(\frac{R_S}{d}\right)^2 = 5.67\times 10^{-8}\frac{\mathrm{W}}{\mathrm{m^2K^4}}\times(5.77\times 10^3\ \mathrm{K})^4\times\left(\frac{6.96\times 10^8\ \mathrm{m}}{1.5\times 10^{11}\ \mathrm{m}}\right)^2 = 1.35\times 10^3\frac{\mathrm{W}}{\mathrm{m^2}}$$

A-1 (0.6 pt)	
$S_0 = \sigma T_S^4.\left(\frac{R_S}{d}\right)^2$	, Numerical value of $S_0 = 1.35\times 10^3\ \mathrm{W/m^2}$

A-2. It is assumed that the Earth is in thermal equilibrium. Therefore, the energy it receives per unit time should be equal to the energy it radiates per unit time. The Earth's cross-section intercepting the solar radiation at this distance has an area of πR_E^2 , but the Earth radiates heat from all points on its surface with an area of $4\pi R_E^2$, so:

$$\pi R_E^2.S_0 = 4\pi R_E^2\sigma T_E^4 \ \to T_E = \left(\frac{S_0}{4\sigma}\right)^{\frac{1}{4}} = 278\ \mathrm{K}$$

A-2 (0.6 pt)	
$T_E = \left(\frac{S_0}{4\sigma}\right)^{\frac{1}{4}} = \sqrt{\frac{R_S}{2d}}T_S$	, Numerical value of $T_E = 278\ \mathrm{K}$

A-3. The radiation is maximum at the wavelength for which the derivative of u with respect to λ is zero:

$$\frac{du}{d\lambda} = \frac{2\pi hc^2}{\lambda^6}.\frac{1}{exp(\frac{hc}{\lambda k_B T})-1}.\left[-5+\frac{hc}{\lambda k_B T}\frac{exp(\frac{hc}{\lambda k_B T})}{exp(\frac{hc}{\lambda k_B T})-1}\right]$$

$$\frac{du}{d\lambda}|_{\lambda=\lambda_m} = 0 \quad \Rightarrow \quad \frac{hc}{\lambda_m k_B T}\frac{exp\left(\frac{hc}{\lambda_m k_B T}\right)}{exp\left(\frac{hc}{\lambda_m K_B T}\right)-1} = 5$$

Defining $x_m \equiv \frac{hc}{\lambda_m k_B T}$ we obtain the following transcendental equation:

$$5(1-e^{-x_m}) - x_m = 0$$

A-3 (0.4 pt)

$f(x) = 5(1 - e^{-x}) - x$

A-4. The first guess is $x_m^{(1)} = 5$. Substituting repeatedly for x_m we can continue as follows:

$$x_m^{(2)} = 5(1 - e^{-5}) = 4.97$$
$$x_m^{(3)} = 5(1 - e^{-4.97}) = 4.97$$

Further iterations do not change the value of x_m to three significant figures, so:

$$\lambda_m T = \frac{hc}{x_m k_B} = b = 1240 \text{ eV} \cdot \text{nm} \times \frac{1}{4.97 \times 8.62 \times 10^{-5} \text{ eVK}^{-1}} = 2.89 \times 10^6 \text{ nm} \cdot \text{K}$$

A-4 (0.4 pt)

$x_m = \{4.96, 4.97\}$, Numerical value of $b = [2.89, 2.90] \times 10^6 \text{ nm} \cdot \text{K}$

A-5. Using Wien's displacement law and the constant b obtained in the previous part, we can calculate the wavelength at which the radiation from the Sun and the Earth reaches its maximum:

$$\lambda_{max}^{Sun} = \frac{b}{T_S} = \frac{2.89 \times 10^6 \text{ nm} \cdot \text{K}}{5.77 \times 10^3 \text{ K}} = [5.01, 5.02] \times 10^2 \text{ nm}$$

$$\lambda_{max}^{Earth} = \frac{b}{T_E} = \frac{2.89 \times 10^6 \text{ nm} \cdot \text{K}}{278 \text{ K}} = 1.04 \times 10^4 \text{ nm}$$

A-5 (0.2 pt)

$\lambda_{max}^{Sun} = [5.01, 5.02] \times 10^2 \text{ nm}$, $\lambda_{max}^{Earth} = 1.04 \times 10^4 \text{ nm}$

A-6. From the diagram, it can clearly be seen that $\gamma \tilde{u}_S(\lambda_{max}^S) = u(\lambda_{max}^{Earth}, T_E)$, so we have:

$$\tilde{u}_S(\lambda_{max}^{Sun}) = \left(\frac{R_S}{d}\right)^2 \frac{2\pi hc^2}{(\lambda_{max}^{Sun})^5} \frac{1}{exp\left(\frac{hc}{\lambda_{max}^{Sun} k_B T_S}\right) - 1} = \left(\frac{R_S}{d}\right)^2 \frac{2\pi hc^2}{(\lambda_{max}^{Sun})^5} \frac{1}{exp\left(\frac{hc}{k_B b}\right) - 1}$$

$$u(\lambda_{max}^{Earth}, T_E) = \frac{2\pi hc^2}{(\lambda_{max}^{Earth})^5} \frac{1}{exp\left(\frac{hc}{\lambda_{max}^{Earth} k_B T_E}\right) - 1} = \frac{2\pi hc^2}{(\lambda_{max}^{Earth})^5} \frac{1}{exp\left(\frac{hc}{k_B b}\right) - 1}$$

Dividing these two quantities we'll find:

$$\gamma = \left(\frac{d}{R_\mathrm{S}}\right)^2 \times \left(\frac{T_\mathrm{E}}{T_\mathrm{S}}\right)^5 = [1.20, 1.21] \times 10^{-2}$$

A-6 (0.8 pt)

$\gamma = \left(\frac{d}{R_\mathrm{S}}\right)^2 \times \left(\frac{T_\mathrm{E}}{T_\mathrm{S}}\right)^5 = \left(\frac{d}{R_\mathrm{S}}\right)^2 \times \left(\frac{\lambda_\mathrm{max}^\mathrm{Sun}}{\lambda_\mathrm{max}^\mathrm{Earth}}\right)^5$, Numerical value of $\gamma = [1.20, 1.21] \times 10^{-2}$

B. The Greenhouse Effect

B-1. Both the Earth and its atmosphere are in thermal equilibrium, so one can write an equation that balances the input and output powers. For the Earth we have:

$$(\pi R_\mathrm{E}^2)(1 - r_\mathrm{A})S_0 + (4\pi R_\mathrm{E}^2)\sigma T_\mathrm{A}^4 = (4\pi R_\mathrm{E}^2)\sigma T_\mathrm{E}^4,$$

and for the atmosphere:

$$(4\pi R_\mathrm{E}^2)\sigma T_\mathrm{E}^4 = 2(4\pi R_\mathrm{E}^2)\sigma T_\mathrm{A}^4.$$

Note that the coefficient 2 on the right-hand side of the equation is due to the atmosphere radiating heat on both sides (above and below). Eliminating T_E from the two relations we obtain:

$$T_\mathrm{A} = \left(\frac{(1 - r_\mathrm{A})\frac{S_0}{4}}{\sigma}\right)^{\frac{1}{4}} = 2.58 \times 10^2\ \mathrm{K} \qquad \Rightarrow \qquad T_\mathrm{E} = (2T_\mathrm{A}^4)^{\frac{1}{4}} = 3.07 \times 10^2\ \mathrm{K}$$

B-1 (1.0 pt)

$T_\mathrm{A} = \left(\frac{(1-r_\mathrm{A})\frac{S_0}{4}}{\sigma}\right)^{\frac{1}{4}}$, Numerical value of $T_A = 2.58 \times 10^2$ K

$T_\mathrm{E} = \left(\frac{(1-r_\mathrm{A})\frac{S_0}{2}}{\sigma}\right)^{\frac{1}{4}}$, Numerical value of $T_\mathrm{E} = 3.07 \times 10^2$ K

B-2. As can be seen in the figure, a fraction $(1 - r_\mathrm{A})$ of the solar radiation reaches the Earth's surface after traversing the atmosphere. A fraction r_E of this light is reflected back and reaches the atmosphere, where a fraction r_A is reflected and returns to the Earth's surface. This process repeats *ad infinitum* and the sum of the powers transmitted at all these instances, determines the albedo. Denoting the power returned to space after n reflections by $\tilde{S}_n$, we'll have $\tilde{S}_0 = r_\mathrm{A} S_0$ and the remaining power i.e. $(1-r_\mathrm{A})S_0$, reaches the Earth's surface. From this power, $(1-r_\mathrm{A})r_\mathrm{E}S_0$ is reflected, and a fraction $1-r_\mathrm{A}$ of it is transmitted through the atmosphere to the space, hence:

$$\tilde{S}_1 = (1-r_\mathrm{A})^2 r_\mathrm{E} S_0 = \frac{(1-r_\mathrm{A})^2}{r_\mathrm{A}}\, r_\mathrm{E}\tilde{S}_0$$

The power that is reflected back to the Earth by the atmosphere after $(n-1)$ reflections is $\tilde{S}_{n-1}\left(\frac{r_\mathrm{A}}{1-r_\mathrm{A}}\right)$, of which a fraction r_E is again sent back towards the atmosphere on the n'th reflection, and the atmosphere allows a fraction $1-r_\mathrm{A}$ of this reflected power to escape into the space, thus:

$$\tilde{S}_n = \frac{\tilde{S}_{n-1}}{1-r_\mathrm{A}} r_\mathrm{A} r_\mathrm{E} \times (1-r_\mathrm{A}) = r_\mathrm{A} r_\mathrm{E} \tilde{S}_{n-1} = (r_\mathrm{A} r_\mathrm{E})^{n-1}\tilde{S}_1$$

By adding all these terms, one obtains the power returned per unit area from the Earth-atmosphere system:

$$\tilde{S} = \sum_{n=0}^{\infty}\tilde{S}_n = \tilde{S}_0 + \tilde{S}_1\sum_{n=1}^{\infty}(r_\mathrm{A} r_\mathrm{E})^{n-1} = r_\mathrm{A} S_0 + (1-r_\mathrm{A})^2 r_\mathrm{E} S_0 \times \frac{1}{1-r_\mathrm{A} r_\mathrm{E}}$$
$$= \left[r_\mathrm{A} + \frac{(1-r_\mathrm{A})^2 r_\mathrm{E}}{1-r_\mathrm{A} r_\mathrm{E}}\right] \times S_0$$

Dividing by the solar constant we get the value for albedo:

$$\alpha = \frac{\tilde{S}}{S_0} = r_\mathrm{A} + \frac{(1-r_\mathrm{A})^2 r_\mathrm{E}}{1-r_\mathrm{A} r_\mathrm{E}} = 3.13\times 10^{-1}$$

B-2 (1.6 pt)	
$\alpha = r_\mathrm{A} + \frac{(1-r_\mathrm{A})^2 r_\mathrm{E}}{1-r_\mathrm{A} r_\mathrm{E}}$	, Numerical value of $\alpha = 3.13\times 10^{-1}$

B-3. Again, thermal equilibrium requires the input and output powers to be equal both for the Earth and for the atmosphere, the only difference being that the Earth absorbs now a fraction $1-\alpha$ of the Sun's radiation. Thus, for Earth we have:

$$(4\pi R_\mathrm{E}^2)\epsilon\sigma T_\mathrm{A}^4 + (\pi R_\mathrm{E}^2)(1-\alpha)S_0 = (4\pi R_\mathrm{E}^2)\sigma T_\mathrm{E}^4,$$

and for the atmosphere:

$$(4\pi R_\mathrm{E}^2)\epsilon\sigma T_\mathrm{E}^4 = 2(4\pi R_\mathrm{E}^2)\epsilon\sigma T_\mathrm{A}^4$$

$$T_\mathrm{E} = \left[\frac{(1-\alpha)}{2\sigma(2-\epsilon)}S_0\right]^{\frac{1}{4}} \quad , \quad T_\mathrm{A} = \left(\frac{T_\mathrm{E}^4}{2}\right)^{\frac{1}{4}}$$

$$\epsilon = \frac{\left[\sigma T_E^4 - \frac{(1-\alpha)}{4} S_0\right]}{\sigma T_A^4} = 2\frac{\left[\sigma T_E^4 - \frac{(1-\alpha)}{4} S_0\right]}{\sigma T_E^4} = [8.07, 8.11] \times 10^{-1}$$

B-3 (1.0 pt)

$T_E = \left[\frac{(1-\alpha)}{2\sigma(2-\epsilon)} S_0\right]^{\frac{1}{4}}$, Numerical value of $\epsilon = [8.07\,, 8.11] \times 10^{-1}$

B-4.

$$\frac{dT_E}{d\epsilon} = \frac{1}{4}\left[\frac{(1-\alpha)S_0}{2\sigma(2-\epsilon)}\right]^{\frac{1}{4}} \frac{1}{(2-\epsilon)}$$

$$dT_E = \frac{dT_E}{d\epsilon}\epsilon\frac{d\epsilon}{\epsilon} = \left[\frac{4\sigma T_E^4}{(1-\alpha)S_0} - 1\right]\frac{T_E}{4} \times 0.01 \;=\; [4.87, 4.92] \times 10^{-1}$$

B-4 (0.8pt)

$\frac{dT_E}{d\epsilon} = \frac{1}{4}\left[\frac{(1-\alpha)S_0}{2\sigma(2-\epsilon)}\right]^{\frac{1}{4}} \frac{1}{(2-\epsilon)}$, Numerical value of $\delta T_E = [4.87, 4.92] \times 10^{-1}$ K

B-5. The equations for thermal equilibrium are similar to those for Part B.3, only a non-radiative thermal current needs to be added. For the Earth:

$$(\pi R_E^2)(1-\alpha)S_0 + (4\pi R_E^2)\epsilon\sigma T_A^4 = (4\pi R_E^2)\sigma T_E^4 + (4\pi R_E^2)k(T_E - T_A),$$

and for the atmosphere:

$$(4\pi R_E^2)\epsilon\sigma T_E^4 + (4\pi R_E^2)k(T_E - T_A) = 2(4\pi R_E^2)\epsilon\sigma T_A^4.$$

After completing the calculations, we will have:

$$\epsilon = \frac{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}}{\sigma(T_E^4 - T_A^4)} = [8.47, 8.52] \times 10^{-1}$$

$$k = \frac{\epsilon\sigma(2T_A^4 - T_E^4)}{T_E - T_A} = \frac{(2T_A^4 - T_E^4) \times \left[\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}\right]}{(T_E^4 - T_A^4) \times (T_E - T_A)} = [3.57, 3.66] \times 10^{-1}\ \mathrm{W/m^2K}$$

B-5 (1.6pt)

$\epsilon = \frac{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}}{\sigma(T_E^4 - T_A^4)}$, Numerical value of $\epsilon = [8.47, 8.52] \times 10^{-1}$

$k = \frac{(2T_A^4 - T_E^4) \times \left[\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}\right]}{(T_E^4 - T_A^4) \times (T_E - T_A)}$, Numerical value of $k = [3.57, 3.66] \times 10^{-1}$ $\mathrm{W/m^2K}$

B-6. In order to find the change in the temperatures of the Earth and the atmosphere in terms of ϵ and k, we take the logarithm of both sides of the relations before taking the derivative:

$$ln\, \epsilon = ln\left[\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}\right] - ln\, \sigma - ln\,(T_E^4 - T_A^4)$$

$$ln\, k = ln\, \epsilon + ln\, \sigma + ln(2T_A^4 - T_E^4) - ln(T_E - T_A)$$

$$\frac{1}{\epsilon} = \frac{4\sigma T_E^3 \frac{dT_E}{d\epsilon}}{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}} - \frac{4T_E^3 \frac{dT_E}{d\epsilon} - 4T_A^3 \frac{dT_A}{d\epsilon}}{T_E^4 - T_A^4}$$

$$0 = \frac{1}{\epsilon} + \frac{8T_A^3 \frac{dT_A}{d\epsilon} - 4T_E^3 \frac{dT_E}{d\epsilon}}{2T_A^4 - T_E^4} - \frac{\frac{dT_E}{d\epsilon} - \frac{dT_A}{d\epsilon}}{T_E - T_A}$$

$$\epsilon\left[\frac{1}{T_E - T_A} + \frac{4T_E^3}{2T_A^4 - T_E^4}\right]\frac{dT_E}{d\epsilon} = 1 + \epsilon\left[\frac{8T_A^3}{2T_A^4 - T_E^4} + \frac{1}{T_E - T_A}\right]\frac{dT_A}{d\epsilon}$$

$$1 + \epsilon\left[\frac{4T_E^3}{T_E^4 - T_A^4} - \frac{4\sigma T_E^3}{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}}\right]\frac{dT_E}{d\epsilon} = \frac{4T_A^3}{T_E^4 - T_A^4}\epsilon\frac{dT_A}{d\epsilon}$$

Solving this set of linear equations and substituting ϵ in B-5, we find:

$$\frac{dT_\mathrm{E}}{d\epsilon}=\frac{\left[\frac{\sigma(T_\mathrm{E}^4-T_\mathrm{A}^4)}{\sigma T_\mathrm{E}^4-(1-\alpha)\frac{S_0}{4}}\right]\left[1+\left(\frac{T_\mathrm{E}^4-T_\mathrm{A}^4}{4T_\mathrm{A}^3}\right)\left[\frac{8T_\mathrm{A}^3}{2T_\mathrm{A}^4-T_\mathrm{E}^4}+\frac{1}{T_\mathrm{E}-T_\mathrm{A}}\right]\right]}{\left[\frac{1}{T_\mathrm{E}-T_\mathrm{A}}+\frac{4T_\mathrm{E}^3}{2T_\mathrm{A}^4-T_\mathrm{E}^4}\right]-\left(\frac{\sigma T_\mathrm{A}^4-(1-\alpha)\frac{S_0}{4}}{\sigma T_\mathrm{E}^4-(1-\alpha)\frac{S_0}{4}}\right)\left(\frac{T_\mathrm{E}}{T_\mathrm{A}}\right)^3\left[\frac{8T_\mathrm{A}^3}{2T_\mathrm{A}^4-T_\mathrm{E}^4}+\frac{1}{T_\mathrm{E}-T_\mathrm{A}}\right]}$$

$$\epsilon\frac{dT_\mathrm{E}}{d\epsilon}=\frac{1+\left(\frac{T_\mathrm{E}^4-T_\mathrm{A}^4}{4T_\mathrm{A}^3}\right)\left[\frac{8T_\mathrm{A}^3}{2T_\mathrm{A}^4-T_\mathrm{E}^4}+\frac{1}{T_\mathrm{E}-T_A}\right]}{\left[\frac{1}{T_\mathrm{E}-T_\mathrm{A}}+\frac{4T_\mathrm{E}^3}{2T_\mathrm{A}^4-T_\mathrm{E}^4}\right]-\left(\frac{\sigma T_\mathrm{A}^4-(1-\alpha)\frac{S_0}{4}}{\sigma T_\mathrm{E}^4-(1-\alpha)\frac{S_0}{4}}\right)\left(\frac{T_\mathrm{E}}{T_\mathrm{A}}\right)^3\left[\frac{8T_\mathrm{A}^3}{2T_\mathrm{A}^4-T_\mathrm{E}^4}+\frac{1}{T_\mathrm{E}-T_\mathrm{A}}\right]}$$

$$dT_\mathrm{E}=\epsilon\frac{dT_\mathrm{E}}{d\epsilon}\frac{d\epsilon}{\epsilon}=[5.21\,,5.28]\times10^{-1}\ \mathrm{K}$$

B-6 (1.0pt)

(a)
$$\begin{cases}\epsilon\left[\frac{1}{T_\mathrm{E}-T_\mathrm{A}}+\frac{4T_\mathrm{E}^3}{2T_\mathrm{A}^4-T_\mathrm{E}^4}\right]\frac{dT_\mathrm{E}}{d\epsilon}=1+\epsilon\left[\frac{8T_\mathrm{A}^3}{2T_\mathrm{A}^4-T_\mathrm{E}^4}+\frac{1}{T_\mathrm{E}-T_\mathrm{A}}\right]\frac{dT_\mathrm{A}}{d\epsilon}\\ 1+\epsilon\left[\frac{4T_\mathrm{E}^3}{T_\mathrm{E}^4-T_\mathrm{A}^4}-\frac{4\sigma T_\mathrm{E}^3}{\sigma T_\mathrm{E}^4-(1-\alpha)\frac{S_0}{4}}\right]\frac{dT_\mathrm{E}}{d\epsilon}=\frac{4T_\mathrm{A}^3}{T_\mathrm{E}^4-T_\mathrm{A}^4}\epsilon\frac{dT_\mathrm{A}}{d\epsilon}\end{cases}$$

(b) $\delta T_\mathrm{E}=[5.21\,,5.28]\times10^{-1}\ \mathrm{K}$