T3: Scaling laws (8 pts)

Task A: Spaghetti (2 pts)

This is section 2.2.2 (Statics) of the syllabus.

Consider only the left half of the spaghetti straw.

Torque balance at its right endpoint implies that the torque applied to its right endpoint must balance out the torque due to gravity: $\tau \propto ml \propto d^2l^2$. This torque arises from the gradient in the horizontal stress. If the typical horizontal stress is σ, then the typical force is $F \propto \sigma d^2$, so the torque is $\tau \propto Fd \propto \sigma d^3$. Hence, we obtain

$$\sigma d^3 \propto d^2 l^2 \implies l \propto \sqrt{d},$$

so

$$l' = \sqrt{\frac{d'}{d}} l = \sqrt{10} \cdot 50\,\text{cm} = 158\,\text{cm}.$$

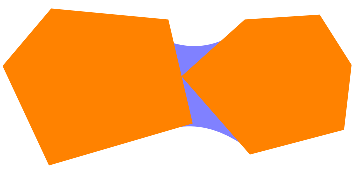
Marking scheme:

$\tau \propto d^2l^2$	0.4 pts
$F \propto \sigma d^2$	0.5 pts
$\tau \propto \sigma d^3$	0.5 pts
$l \propto \sqrt{d}$	0.4 pts
Answer: $158\,\text{cm}$	0.2 pts

Task B: Sand castle (2 pts)

This is Section 2.2.2 (statics) and 2.2.5 (hydrodynamics) of the syllabus

Due to wetting of the surfaces of the sand grains and its large surface tension water acts like a glue for sand. This means that all the grains need to be bound together by air-water interface. To achieve this there needs to be neither too little nor too much water: if there is too little water, most of the grains are dry with now surface tension binding them, and if there is too much water, almost all the grains are immersed into water, and again, there is no surface tension binding the grains. So, the overall strength of the buildings from wet sand depends on the water content; we assume that for the both types of sand, the water content is optimal, and the shape of the grains is statistically similar. Let us consider two neighbouring grains connected by a water meniscus — or "neck", as we shall be referring to it henceforth. Note that the "neck" may extend perpendicularly to the figure plane far away; so, more specifically, what the word "neck" will refer to is that part of the water-air interface for which the closest two grains are the ones under consideration.



There are two processes binding the sand grains together. The first one is the force due to the surface tension, $F_1 = \gamma l$, where γ denotes the surface tension coefficient, and l — the perimeter of the "neck"; with $l \sim r_g$, where r_g denotes the length scale of a single grain, we obtain $F_s \sim \gamma r$. The second one is the pressure force caused by the negative capillary pressure in the neck, $F_p = \Delta p A$, where A is the cross-sectional area of the "neck", and $\Delta p \sim \gamma / r$. With $A \sim r^2$ we obtain $F_2 \sim \gamma r$. Thus, the both components are of the same order of magnitude and using either of them will lead to the correct scaling law. These forces press the grains against each other, hence the normal force and friction force between the grains is also on the order of F_s and F_p.

Solution 1:

Based on what has been said above, the typical force needed to delocate a grain of sand is $F_g \propto r_g$. The force needed to delocate an entire layer of sand is then $\propto F_g N_l$, where $N_l \sim A/r_g^2$ is the number of grains in a layer. The force of cylinder destruction F thus satisfies

$$F \propto F_g N_l \propto r_g / r_g^2 = r_g^{-1} \propto V_g^{-1/3},$$

so

$$F_{fg} = (1/10)^{-1/3} \cdot F_{cg} = 21{,}5\,\text{N}.$$

Marking scheme:

a) Either $F_s \propto r_g$ or $F_p \propto r_g$	0.5 pts
b) $F_g \propto r_g$	0.5 pts
c) $F \propto F_g N_l$	0.5 pts
d) $F \propto r_g^{-1}$	0.3 pts
Answer: $21.5\,\text{N}$	0.2 pts

Notes: If the student only qualitatively explains the mechanism by which the grains of sand are held together, a maximum of 0.5 pts are given. Points b)-c) are given only if derived from a).

Solution 2:

We have seen above that grains to one side of a fictitious surface exert force per cross-sectional area on the order of magnitude as the capillary pressure $\Delta p \sim \gamma / r_g$. In order to get the grains moving, a pressure of the same order of magnitude needs to be applied externally. For the both cylinders, the surface area where the force is applied is the same, hence the force scales as the capillary pressure, $F \propto 1/r_g \propto V^{-1/3}$.

Marking scheme:

The applied pressure must be $\sim \Delta p$	0.6 pts
The curvature radius of the interface is $\sim r_g$	0.6 pts
Capillary pressure $\Delta p \sim \gamma / r_g$	0.6 pts
Answer: $21.5\,\text{N}$	0.2 pts

Solution 3:

The compression force serves to break the surface tension bonds between sand grains.

Consider the energy E required to push a single layer of sand into the layer beneath it. $E \propto F r_g$, where F is the force required and r_g is the typical height of a layer (i.e., the typical length scale of a grain).

On the other hand, $E = \gamma \Delta A$, where γ is the surface tension of water and ΔA is the total amount by which the surface of the water in the layer stretches before all the "water bonds" between the sand grains are broken.

Here, ΔA is proportional to the area A of a layer and is thus a constant between the two cylinders. Hence, $E \propto F r_g$ is a constant between the two cylinders, i.e., $F \propto r_g^{-1}$.

Marking scheme:

$E \propto Fr_g$	0.5 pts
$E \propto \gamma \Delta A$	0.5 pts
$\Delta A \propto A$	0.5 pts
$F \propto r_g^{-1}$	0.3 pts
Answer: $21.5\,\mathrm{N}$	0.2 pts

Note: If the student only qualitatively explains the mechanism by which the grains of sand are held together, a maximum of 0.5 pts are given.

Solution 4:

First of all, the force F should be proportional to the cylinder's base area A. The force required to destroy a cylinder with base area $A = nA_0$ is equal to the force required to destroy n cylinders each with base area A_0. As a result, $F \propto n \propto A$.

In addition, F depends on the grain's length scale r_g and the water's surface tension γ. Dimensional analysis thus gives

$$F \propto \frac{A\gamma}{r_g} \propto r_g^{-1}$$

for fixed A and γ.

Marking scheme:

$F \propto A$	0.6 pts
$F = F(A, r_g, \gamma)$	0.6 pts
$F \propto \frac{A\gamma}{r_g}$	0.6 pts
Answer: $21.5\,\mathrm{N}$	0.2 pts

Note: If the student only qualitatively explains the mechanism by which the grains of sand are held together, a maximum of 0.5 pts are given.

Task C: Interstellar travel (2 pts)

This is Section 2.5 (Relativity) of the syllabus

Let $T = 50\,\mathrm{yrs}$ be the astronauts' total travel time. For maximal travel distance, the spaceship accelerates at constant proper acceleration $a = g$ for proper time $T/4$, during which a distance of d is traveled. The spaceship then decelerates at $a = -g$ for proper time $T/4$ to come to a rest, during which another distance d is traveled. The spaceship then returns to Earth using the same procedure.

Solution 0: (incorrect)

If we ignore relativity, then $d \propto \frac{1}{2}gt^2 \propto g$, which gives an answer of 1.5.

Marking scheme:

$d \propto gt^2$	0.2 pts
Answer: 1.5	0.1 pts

Solution 1:

One way to approach the problem is to notice that constant acceleration in spaceship's frame means a constant force in the Earth's frame. This follows directly from the Lorentz transform for the electromagnetic field, more specifically from the fact that when going to a frame moving parallel to the x-axis, the x-directional electric field E_x remains unchanged. Hence, on the one hand, the force $F_x = eE_x$ exerted on an accelerating particle of rest mass m_0 and carrying a charge e remains constant in the lab frame. On the other hand, the acceleration of that particle in an inertial frame moving with velocity v, where v denotes the particle's velocity at a certain moment of time t, is always equal to eE_x/m_0, regardless of the value of t, i.e. constant in time.

Those who are not familiar with the Lorenz transform for electromagnetic field can derive the above described property from the Lorenz transform for momentum and coordinates. We use again (i) the lab frame, and (ii) an inertial frame moving with velocity v, where v denotes the spaceship's velocity at a certain moment of time which will be used as the origin, $t = t' = 0$; let primes denote quantities in the second frame. Assuming a very short time period t, we can neglect terms quadratic in time so that in the frame (ii), the momentum, coordinate and the relativistic mass can be expressed as $p' = F't'$, $x' = 0$, $m' = m_0$, respectively; applying the Lorenz transform yields $t = \gamma t'$ and $p = \gamma(F't' + m_0 v) = tF' + \gamma m_0 v$. On the other hand, in the frame (i), $p = \gamma m_0 v + Ft$; comparing this with the previous result yields $F = F'$.

It appears that in either case, the spaceship's speed will reach almost c much faster than the travel time. Hence, using for convenience the system of units where $c = 1$, the travel distance x equals with a very good precision the travel time t, $x = t$.

What is left to do is to relate t to the proper time τ,

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\gamma} = \mathrm{d}t\frac{m_0}{\sqrt{m_0^2 + m_0^2 g^2 t^2}};$$

upon integration we obtain

$$\tau = \mathrm{asinh}(gt)/g \Rightarrow x \approx t = \sinh(g\tau)/g \approx \exp(g\tau)/2g.$$

So we conclude that the ratio of the travel distances is

$$\frac{d_2}{d_1} = \frac{g}{1.5g}\exp(1.5g\tau - g\tau) = \frac{2}{3}\exp(gT/8) \approx 480.$$

Note that an exact relationship between x and t could have been obtained by expressing the energy of the spaceship as $m = m_0 + m_0 gx$, and the momentum as $p = m_0 gt$. Then the Lorenz invariant $(m_0 + m_0 gx)^2 - (m_0 gt)^2 = m_0^2$ yields $x(x + 2/g) = t^2 = \sinh^2(g\tau)/g^2$, hence $x = [\cosh(g\tau) - 1]/g$.

F_x is Lorentz invariant	0.4 pts
$x \approx t$	0.4 pts
$\mathrm{d}\tau = \frac{\mathrm{d}t}{\gamma}$	0.2 pts
$\gamma^{-1} = m_0/m$	0.2 pts
$m = \sqrt{m_0^2 + p^2}$	0.2 pts
$p = m_0 g t$	0.2 pts
$t = \sinh(g\tau)/g$	0.2 pts
Answer: 480	0.2 pts

Solution 2:

Let w be the rapidity of the spaceship, defined as $w \equiv \tanh^{-1}(\beta)$, where β is the spaceship's velocity. Then $\beta = \tanh w$, the Lorentz factor $\gamma = \cosh w$, and its momentum $p = m \sinh w$.

As shown by Solution 1, a spaceship experiencing a constant proper acceleration g experiences a constant three-force

$$F = mg = \frac{\mathrm{d}p}{\mathrm{d}t} = m\cosh w \frac{\mathrm{d}w}{\mathrm{d}t} \implies \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{g}{\cosh w}.$$

Meanwhile, time dilation relates t to the spaceship's proper time τ as

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \cosh w \implies \frac{\mathrm{d}w}{\mathrm{d}\tau} = \frac{\mathrm{d}w}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = g.$$

Integrating yields $w = g\tau$. Recalling that $dt = \gamma d\tau$, we get the following as the total distance traveled over a quarter of the spaceship's trip:

$$d = \int_0^{T/4} \beta\gamma\, \mathrm{d}\tau = \int_0^{T/4} \tanh w \cosh w\, \mathrm{d}\tau$$
$$= \int_0^{T/4} \sinh g\tau\, \mathrm{d}\tau = \frac{1}{g}(\cosh(gT/4) - 1).$$

The answer is thus

$$\frac{g_1}{g_2}\frac{\cosh(g_2T/4c) - 1}{\cosh(g_1T/4c) - 1} = \frac{10}{15}\frac{\cosh(19.72) - 1}{\cosh(13.15) - 1}$$
$$\approx \frac{2}{3}e^{19.72-13.15} = 480.$$

Marking scheme:

$\frac{\mathrm{d}}{\mathrm{d}t}(m\sinh w) = mg$	0.5 pts
$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{g}{\cosh w}$	0.1 pts
$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \cosh w$	0.4 pts
$\frac{dw}{d\tau} = g$	0.1 pts
$w = g\tau$	0.1 pts
$d = \int_0^{T/4} \beta\gamma\, \mathrm{d}\tau$	0.3 pts
$d = \int_0^{T/4} \tanh w \cosh w\, d\tau$	0.2 pts
$d = \frac{1}{g}(\cosh(gT/4) - 1)$	0.1 pts
Answer: 480	0.2 pts

Solution 3: The problem can be also solved by using the century-old trick of depicting things in $x-\mathrm{i}t$-diagram. The benefit of using this diagram is that the relativistic invariant $x^2 - t^2$ transforms into Euclidean squared distance $x^2 + \theta^2$ with $\theta = \mathrm{i}t$. This means that in that diagram, we can use the knowledge of Euclidean geometry. In particular, the Lorentz transform is now the rotation of the Euclidean $x - \mathrm{i}t$-space by an angle $\alpha = \arctan\frac{v}{\mathrm{i}c}$. Now, consider the trajectory of the space ship; its infinitesimal arc length is $\mathrm{i}c\mathrm{d}\tau$, where $\mathrm{d}\tau$ is the differential of the proper time, and the infintesimal rotation angle of its tangent is $\mathrm{d}\alpha = \arctan(\mathrm{d}v/\mathrm{i}c) = \mathrm{d}v/\mathrm{i}c = g\mathrm{d}\tau/\mathrm{i}c$. Therefore, the curvature radius $R = \mathrm{i}c\mathrm{d}\tau/\mathrm{d}\alpha = -c^2/g$ is constant, i.e. the trajectory is a circle of radius R. Now we can easily relate the travel distance x to the arc length $\mathrm{i}c\tau$:

$$x = R(1-\cos\alpha) = R\left(1 - \cos\frac{\mathrm{i}c\tau}{R}\right) = \frac{c^2}{g}\left(\cosh\frac{g\tau}{c} - 1\right).$$

Marking scheme:

$R =$ const in x-ict-diag.	0.5 pts
$R = -g^2/c$	0.5 pts
missing '$-$'	-0.2 pts
partial credit for $R = \frac{\mathrm{i}c\tau}{\mathrm{d}\alpha}$	0.2 pts
$x = R(1-\cos\alpha)$	0.5 pts
$\frac{c^2}{g}\left(\cosh\frac{g\tau}{c} - 1\right)$	0.3 pts
Answer: 480	0.2 pts

Solution 4: The problem can be solved by using the velocity addition formula. Let v be the speed of the spaceship in the lab frame, t be the lab time, and τ — the proper time. Also, we consider a frame which moves with constant speed v in which the spaceship accelerates from rest:

$$v + \mathrm{d}v = \frac{v + g\mathrm{d}\tau}{1 + vg\mathrm{d}\tau} = v + g\mathrm{d}\tau(1-v^2)\mathrm{d}\tau.$$

Thus,

$$\frac{\mathrm{d}v}{1-v^2} = g\mathrm{d}\tau \Rightarrow v = \tanh(g\tau).$$

From relativistic time dilation formula we obtain

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{\sqrt{1-v^2}} = \cosh(g\tau)\mathrm{d}\tau$$

so that the travel distance

$$\frac{L}{2} = \int v\mathrm{d}t = \int_0^{T/4}\sinh(g\tau)\mathrm{d}\tau = \frac{1}{g}\left[\cosh\left(\frac{gT}{4}\right) - 1\right]$$

which leads to the same answer as before.

a) $v + \mathrm{d}v = \frac{v+g\mathrm{d}\tau}{1+vg\mathrm{d}\tau}$	0.3 pts
b) $\frac{\mathrm{d}v}{1-v^2} = g\mathrm{d}\tau$	0.2 pts
c) $v = \tanh(g\tau)$	0.2 pts
d) $\mathrm{d}t = \frac{\mathrm{d}\tau}{\sqrt{1-v^2}}$	0.3 pts
e) $\mathrm{d}t = \cosh(g\tau)\mathrm{d}\tau$	0.2 pts
f) $\frac{L}{2} = \int v\mathrm{d}t$	0.2 pts
g) $\frac{L}{2} = \int_0^{T/4}\sinh(g\tau)\mathrm{d}\tau$	0.2 pts
h) $\frac{L}{2} = \frac{1}{g}\left[\cosh\left(\frac{gT}{4}\right) - 1\right]$	0.2 pts
i) Answer: 480	0.2 pts

Remark: if integration in f) is done over proper time, no points are given for f).

Task D: That sinking feeling (2 pts)

(This is Sections 2.2.5 (Hydrodynamics) and 2.4.1 (Single oscillator) of the syllabus

The oscillation of the half sink sphere is driven by the gravity. The non-damped angular frequency depends on the gravitational acceleration and a characteristic length, which is, for a sphere, its radius r, so

$$\omega_0 \propto \sqrt{g/r}$$

is the only dimensionally correct possible function.

The drag force F_d depends on the sphere's speed v [m/s], its size r [m], and viscosity of the liquid η [Pa·s]. Dimensional analysis thus gives $F_d \propto \eta r v$. The damping factor is thus

$$\beta = \frac{F_d}{2mv} \propto \frac{\eta r}{m}.$$

Since the mass scales with r^3, we have

$$\beta \propto \frac{1}{r^2}.$$

Then the relation

$$\frac{\beta^2}{\omega_0^2} = 1 - \frac{\omega^2}{\omega_0^2}$$

scales as

$$\frac{\beta^2}{\omega_0^2} \propto \frac{1}{r^3}$$

Oscillations only occur if $\beta/\omega_0 < 1$, so solve

$$\frac{r}{r_0} = \sqrt[3]{1-(0.99)^2} = 0.271$$

Notes:

1. To obtain $\omega_0 \propto 1/\sqrt{r}$ without dimensional analysis, note that a small displacement y changes the submerged volume of the ball by $\Delta V \propto r^2 y$, so the change in buoyant force $F \propto r^2 y$, which gives $\omega_0 = \sqrt{k/m} \propto \sqrt{r^2/r^3} = \sqrt{1/r}$.
2. To obtain $F_d \propto \eta r v$ without dimensional analysis, note that the typical length scale l in the variations in the velocity field of the water is proportional to r. Thus, the viscous shear $\sigma \propto \eta v/l \propto \eta v/r$. The total drag force is thus $F_d \sim A\sigma \propto \eta r v$, where A is ball's area of contact with the water.

Marking scheme:

$\omega_0 \propto \sqrt{g/r}$	0.4 pts
no justification	-0.2 pts
$F_d \propto \eta r v$	0.6 pts
no justification	-0.3 pts
$\beta \propto 1/r^2$	0.3 pts
$\frac{\beta^2}{\omega_0^2} = 1 - \frac{\omega^2}{\omega_0^2}$	0.4 pts
$\frac{\beta^2}{\omega_0^2} \propto \frac{1}{r^3}$	0.2 pts
Answer: 0.271	0.1 pts