T2: James Webb Space Telescope (10 pts)

Part A: Imaging a Star (1.8 pt)

1. Diameter of image

   The ratio of diameter d_o for an object at a distance $D_o \gg f$ and an image diameter d_i is given by

$$\frac{d_i}{d_o} = \frac{f}{D_o}, \tag{1}$$

   so the diameter of the image is

$$\begin{aligned} d_i &= \frac{(1.7\times 10^{11}\ \text{m})(130\text{m})}{(89\ \text{ly})(3\times 10^8\ \text{m/s})(365\ \text{d/y})(86,400\ \text{s/d})} = \\ &= 2.6\times 10^{-5}\ \text{m} = 26\,\mu\text{m}. \end{aligned}$$

   Marking scheme:

correct formula Eq 1	0.2 pts
$d_i = (26 \pm 1)\,\mu\text{m}$	0.2 pts
sum	**0.4pts**

   Must show units for numerical result!

2. Diameter of central maximum

   The angular radius of the central maximum is

$$\theta_{\text{min}} = 1.22\frac{\lambda}{D}$$

   λ is a typical wavelength; here assumed to be the longest visible, or $\lambda \approx 700\,\text{nm}$.

   D is the aperture size, which is the primary mirror, or $\frac{\pi}{4}D^2 = 25\ \text{m}^2$.

   The diameter of the central maximum is then

$$d_d = 2\theta_{\text{min}} f = 2.44\frac{\lambda}{D} f = 1.22\frac{\lambda f}{\sqrt{A/\pi}} \tag{2}$$

   The numerical value is

$$\begin{aligned} d_d &= 2(1.22)\frac{(7\times 10^{-7}\ \text{m})}{(5.6\ \text{m})}(130\ \text{m}) = \\ &= 4.0\times 10^{-5}\ \text{m} = 40\,\mu\text{m}. \end{aligned}$$

   Marking scheme:

correct formula Eq 2	0.1 pts
$800\text{nm} \geq \lambda \geq 600\text{nm}$	0.1 pts
numerical result	0.2 pts
sum	**0.4pts**

   No penalty for ignoring factor of 1.22, so check their math. There is a penalty for using a short wavelength; this is a red giant, not a blue star!

3. Equilibrium temperature of the detector at the location of the image?

   This is section 2.7.1 and 2.7.3 from the syllabus

   The radiant power from the star is

$$P_g = 4\pi {r_o}^2 \sigma {T_g}^4 \tag{3}$$

   The intensity at the location of the scope is

$$I_g = \frac{P_g}{4\pi {D_o}^2} = \left(\frac{r_o}{D_o}\right)^2 \sigma {T_g}^4 \tag{4}$$

   This is collected onto the mirror with area A and focused on a single spot of radius r_i, so that the power incident is

$$P_i = A\left(\frac{r_o}{D_o}\right)^2 \sigma {T_g}^4 = A\left(\frac{r_i}{f}\right)^2 \sigma {T_g}^4 \tag{5}$$

   But at the image we have an equilibrium temperature of

$$P_i = a\sigma {T_p}^4,$$

   where $a = \pi r_i^2$, so

$$a\sigma {T_p}^4 = \left(\frac{r_i}{f}\right)^2 A\sigma {T_g}^4$$

   or

$$T_p = \left(\frac{A}{\pi f^2}\right)^{\frac{1}{4}} T_g \tag{6}$$

   This shows $T_p \approx 530\,\text{K}$ which seems really impressive for a star 90 light years away.

   Perhaps more perplexing is that the answer doesn't depend on the size or distance of the star!

   Marking scheme:

power of source, Eq 3	0.2 pts
intensity at mirror, Eq 4	0.2 pts
power of image, Eq 5	0.2 pts
simplify, Eq. 6	0.2 pts
numerical result	0.2 pts
sum	**1.0 pt**

   An equation which is dimensionally correct, but missing a multiplicative factor or having a single transcription error from a previous equation, will receive only +0.1 pts instead of +0.2 pts.

   An equation which is dimensionally incorrect or one which has more than two transcription errors will receive no points.

   Follow on errors are not transcription errors; the only penalty will be in the first occurrence of a mistake, except in the case of a dimensionally incorrect equation, which still receives no points, even if a follow on error.

   If an equation can be implied to have been used, then the assumption is that it did exist and would get points. For example, writing Eq. 4 without explicitly writing Eq. 3 would get points for both equations, subject to error rules above.

Part B: Counting Photons (1.8 pt)

1. Temperature of source

   We are interested in the slope of the graph, which is

   $$|\frac{\Delta E_g}{k_B}| = -\frac{(3)-(-1)}{(0.11)-(0.15)} = 100\,\mathrm{K}$$

   Since this is a characteristic temperature, it is at least a partial answer to the problem.

   However, the $E_g/6$ is the gap between the bottom of the Fermi level and the top of the impurity conduction band, since the detection of infrared photons is more complex, and when considering the bias of the semiconductor, as well as the unique construction of the CCD pixel, the cutoff energy in this case will be higher.

   As such, the characteristic temperature of 600 K is the characteristic temperature of the detectable source.

   Marking scheme:

slope of graph = 0.01	0.2 pts
$T_{graph} = 100\mathrm{K}$	0.1 pts
$T_{source} = 600\mathrm{K}$	0.1 pts
sum	**0.4 pt**

   Writing either temperature correctly implies they found the slope of graph, and would get the +0.2 pts. Just writing $T_{source} = 600\mathrm{K}$ gets full marks, as it really is possible to solve this in one's head.

2. Write an expression for the total count uncertainty σ_t

   The three uncertainties are

   $$\sigma_r$$

   and

   $$\sigma_d = \sqrt{i_d \tau}$$

   and

   $$\sigma_p = \sqrt{p\tau}$$

   and then

   $$\sigma_t{}^2 = \sigma_r{}^2 + (i_d + p)\,\tau$$

   Marking scheme:

correct error for dark current	0.1 pts
correct read photon	0.1 pts
added in quadrature	0.2 pts
sum	**0.4 pt**

   Writing

   $$\sigma_t = \sigma_r + \sqrt{i_d\tau} + \sqrt{p\tau}$$

   only gets +0.1, instead of the quadrature +0.2

   Correct dark current and photon count errors in final answer are acceptable evidence for those points; it is not necessary for the student to explicitly state what is what.

3. Determine the photon count for $S/N = 10$.

   At a temperature of $T = 7.5\mathrm{K}$, the dark current is $i_d = 5$ electrons/second. This gives a total dark current count of

   $$i_d\tau = 5 \times 10^4$$

   Let P be the photon count. Then

   $$P = 10\sigma_t$$

   so

   $$P^2 = 100\left(\sigma_r^2 + i_d\tau + P\right)$$

   with solution $P \approx 2290$, and a rate of $p = 0.229$ photons per second.

4. What is intensity of source?

   The near-infrared photons have an energy of $E_g = 6k_BT$, so

   $$E_\lambda = (600\,\mathrm{K})(1.38 \times 10^{-23}\,\mathrm{J/K}) = 8.3 \times 10^{-21}\,\mathrm{J}$$

   The energy received every second is

   $$E = (0.23)(8.3 \times 10^{-21}\,\mathrm{J}) = 1.9 \times 10^{-21}\,\mathrm{J}$$

   and the incident intensity on the primary mirror is then

   $$I = \frac{E/t}{A} = \frac{(1.8 \times 10^{-21}\,\mathrm{J/s})}{(25\,\mathrm{m}^2)} = 7.6 \times 10^{-23}\,\mathrm{W}$$

   Marking scheme:

energy estimate for photon	0.2 pts
substitute into error equation	0.1 pts
solve for count	0.2 pts
solve for rate	0.1 pts
sum	**0.5 pt**

Part C: The Passive Cooling

1. Find expressions for the temperatures of first and fifth sheet

   Start with a statement of net energy flow q_{01} into the first sheet from the sun:

$$q_{01} = \epsilon A \left(I_0 - \sigma T_1^4 \right) \tag{7}$$

where A is the area of the sheet, ϵ is the emissivity, σ is the Stefan-Boltzman constant, and T_1 is the temperature of the first sheet.

Now consider the space between two sheets i and j. Each sheet radiates an energy flow

$$\epsilon A \sigma T^4$$

toward the other sheet, but a fraction β is ejected into space out the gap.

We have defined α as the fraction emitted from one sheet that is absorbed by the other sheet, so the net energy flow from sheet i into sheet j is

$$q_{ij} = \alpha \epsilon A \sigma \left({T_i}^4 - {T_j}^4 \right) \tag{8}$$

There is also a lost fraction emitted into space from between the sheets, given by

$$q'_{ij} = \beta \epsilon A \sigma \left({T_i}^4 - {T_j}^4 \right) = \frac{\beta}{\alpha} q_{ij} \tag{9}$$

Don't make the mistake of assuming that $\alpha + \beta = 1$, as some of the energy emitted from a sheet could be reabsorbed by that sheet.

Finally, write an expression for the net thermal radiant energy flow into space, with an ambient temperature of $T_{space} = 0$, from the far side of the fifth sheet.

$$q_{5s} = \epsilon A \left(\sigma T_5^4 - \sigma T_s^4 \right) = A \epsilon \sigma T_5^4 \tag{10}$$

Write each of the Eq. 8, above in the form

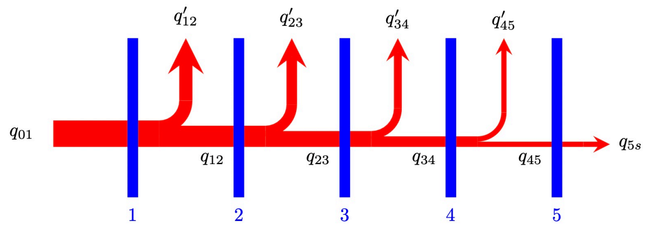
$$\frac{1}{\alpha} q_{ij} = A \epsilon \sigma (T_i^4 - T_j^4), \tag{11}$$

and then sum up the terms from Eq. 7, the four from Eqs. 11, and Eq. 10:

$$q_{01} + \frac{1}{\alpha} \left(q_{12} + q_{23} + q_{34} + q_{45} \right) + q_{5s} = \epsilon A I_0 \tag{12}$$

as all of the T_i terms cancel out on the right!

Now consider a schematic of the energy flow below



From energy conservation, the net flow into sheet one from the sun and the net flow out of sheet one toward sheet two or ejected from gap is

$$q_{01} = q_{12} + q'_{12}, \tag{13}$$

where q'_{12} is the part emitted into space from the gap.

Combine with Eq. 9 and

$$q_{01} = \left(1 + \frac{\beta}{\alpha} \right) q_{12} = \frac{\alpha + \beta}{\alpha} q_{12} \tag{14}$$

Similarly, for the remaining pairs of sheets,

$$q_{23} = \frac{\alpha}{\alpha + \beta} q_{12} = \left(\frac{\alpha}{\alpha + \beta} \right)^2 q_{01},$$

and

$$q_{34} = \frac{\alpha}{\alpha + \beta} q_{23} = \left(\frac{\alpha}{\alpha + \beta} \right)^3 q_{01},$$

and

$$q_{45} = \frac{\alpha}{\alpha + \beta} q_{34} = \left(\frac{\alpha}{\alpha + \beta} \right)^4 q_{01},$$

Finally, for the fifth (last) sheet all of the net energy flow in from the fourth sheet must be completely ejected into space on the dark side.

$$q_{5s} = q_{45} = \left(\frac{\alpha}{\alpha + \beta} \right)^4 q_{01}. \tag{15}$$

The sum on the left side of Eq. 12 can then be written as

$$k q_{01} = \epsilon A I_0 \tag{16}$$

where

$$k = 1 + \frac{1}{\alpha + \beta} + \frac{\alpha}{(\alpha + \beta)^2} + \frac{\alpha^2}{(\alpha + \beta)^3} + \frac{\alpha^3}{(\alpha + \beta)^4} + \frac{\alpha^4}{(\alpha + \beta)^4}$$

is a convenient constant.

Combining Eq. 7 with Eq. 16,

$$\frac{\epsilon A I_0}{k} = \epsilon A \left(I_0 - \sigma T_1^4 \right)$$

so

$$T_1 = \sqrt[4]{\frac{I_0}{\sigma} \left(1 - \frac{1}{k} \right)} = \sqrt[4]{\frac{I_0}{k \sigma} (k - 1)} \tag{17}$$

From above,

$$q_{5s} = \left(\frac{\alpha}{\alpha + \beta} \right)^4 q_{01}.$$

so

$$A \epsilon \sigma T_5^4 = \left(\frac{\alpha}{\alpha + \beta} \right)^4 \frac{\epsilon A I_0}{k}$$

or

$$T_5 = \frac{\alpha}{\alpha + \beta} \sqrt[4]{\frac{I_0}{k \sigma}} \tag{18}$$

which can also be written elegantly as

$$T_5 = \frac{\alpha}{\alpha + \beta} \sqrt[4]{\frac{1}{k - 1}} \, T_1.$$

As this part of the question is complex, with multiple ways to go wrong, and many opportunities for approximations, the marking scheme will be necessarily convoluted.

Marking scheme:

Net flow into sheet 1, Eq 7	0.3 pts
Net flow sheet $i \to j$, Eq 8	0.3 pts
Net flow out of sheet 5, Eq 10	0.3 pts
Sum to eliminate sheet temps, Eq 12	0.3 pts
Energy flow conservation, Eq 13	0.2 pts
Flow ratio, Eq 14	0.3 pts
Special case of sheet 5, Eq 15	0.2 pts
Simplify sum, Eq 16	0.2 pts
Final Expression for T_1, Eq 17	0.2 pts
Final Expression for T_5, Eq 18	0.1 pts
sum	**2.4 pt**

Some expected mistakes:

(a) Failing to account for the back flux of energy. This would be

$$I_0 = 2\sigma T_1^4$$

and then

$$\alpha\sigma T_1^4 = 2\sigma T_2^4,$$

and so on, concluding with

$$I_0 = \sigma \left(\frac{2}{\alpha}\right)^4 T_5$$

or

$$T_5 = \frac{\alpha}{2} T_1$$

(b) Inconsistent treatment of emissivity

The most likely error is of the form

$$\epsilon I_0 = \sigma T_1^4$$

(c) Incorrectly resolving β and α.

2. Find α and β

The process that happens is complex: for photons radiated from a sheet, they will bounce around several times before finally getting absorbed or getting ejected into space. Students will need to make some simplifying assumptions!

If the sheets were parallel and infinite the heat flux per area would be given by

$$\frac{q_{ij}}{A} = \sigma \frac{T_i^4 - T_j^4}{\frac{2}{\epsilon} - 1}$$

The expression is exact, but you don't need to know it or even derive it to solve the problem.

In our case $\epsilon \ll 1$, so this becomes

$$q_{ij} \approx \frac{\epsilon}{2} A \left(T_i^4 - T_j^4\right)$$

The factor of two is because only half of the photons from any sheet are absorbed by the other sheet; the rest are reabsorbed by the first sheet.

We don't have infinite sheets, so there will be a net flux of heat out of the system. There are at least two ways to approach this

Choice A: Assume absorptive effective areas given by ϵA

If the sheets are square with an area A, then the perimeter has length $4\sqrt{A}$. This means that the space between the sheets encloses some volume, and that volume is bounded by a surface with three parts: the area of one sheet A, the area of the other sheet A, and the area of the gap at the perimeter, $4h\sqrt{A}$, where $h = 0.25\,\mathrm{m}$ is the gap size.

The sheets each have an emissivity $\epsilon = 0.05$, while the perimeter gap has an effective emissivity of 1.

If a photon in the space is absorbed by a bounding surface, the probability that it will be absorbed by the perimeter gap instead of the sheets is the ratio of the effective areas:

$$\beta = \frac{A_{\text{gap}}}{A_{\text{effective}}} = \frac{4h\sqrt{A}}{2\epsilon A + 4h\sqrt{A}} = \frac{1}{1 + \epsilon\sqrt{A}/2h} = 0.414 \tag{19}$$

A photon can be absorbed by the original sheet, the transfer sheet, or the gap. Note that α in this case would then be given by $\beta + 2\alpha = 1$, so $\alpha = 0.293$

Marking Scheme:

Estimating fractional areas	0.4 pts
Weighting areas by emissivity	0.3 pts
Correct estimate of gap area	0.2 pts
Assuming $2\alpha + \beta \approx 1$	0.2 pts
Finding α	0.2 pts
$0.2 \le \alpha \le 0.4$	0.1 pts
Finding β	0.1 pts
$0.2 \le \beta \le 0.6$	0.1 pts
sum	**1.6 pt**

Find α and β means that it is consistent with own work.

Choice B: Estimate the number of bounces before a photon is ejected

If the sheets are circular, then the radius is about 8 meters, so the angle between two sheets is about

$$\frac{0.2\,\mathrm{m}}{8\,\mathrm{m}} \approx 0.025\,\text{radians} = \phi$$

Assuming a photon starts orthogonal to one sheet close to the center, then it will bounce no more than

$$\frac{\pi/2}{0.025} \approx 60$$

times before being ejected; a similar photon emitted halfway between the center and the edge will bounce no more than 40 times.

For a fractional distance r from the center,

$$\cos N\phi = r$$

gives the number N of bounces before it escape into space, assuming it is not absorbed first.

In any case, the probability of being absorbed by a sheet after N bounces is

$$1-(1-\epsilon)^N$$

In the case of large N, half of this probability goes to each sheet; if N is small, then the receiving sheet is more likely to collect the photon than the transmitting sheet.

We just need to compute the average number of bounces:

$$\bar{N} \approx \frac{2\pi}{\pi}\int_0^1 N(r)r\,dr = 2\frac{\pi}{8\phi} \approx 30$$

and then

$$\alpha = \frac{1}{2}\left(1-(1-\epsilon)^{30}\right) \approx 0.4$$

and

$$\beta \approx 0.2$$

This is an overestimate of $\bar{N}$, as the photon could be ejected into any initial direction, not necessarily straight up. Being ejected toward to edge is a lower N, while being ejected toward the center is a larger N, but these effects are not equal because there is a cone shape to the sheets. The result is that $\bar{N}$ is closer to 20, so

$$\alpha = \frac{1}{2}\left(1-(1-\epsilon)^{20}\right) \approx 0.3$$

and

$$\beta \approx 0.4$$

which is fairly consistent with the first approach.

Marking Scheme:

Bounces from center	0.3 pts
Bounces from halfway to edge	0.2 pts
Averaging of bounces	0.1 pts
Fractional transfer per bounce	0.2 pts
Transfer after N bounces	0.1 pts
Showing $2\alpha+\beta\approx 1$	0.2 pts
Finding α	0.2 pts
$0.2\le\alpha\le 0.4$	0.1 pts
Finding β	0.1 pts
$0.2\le\beta\le 0.6$	0.1 pts
sum	**1.6 pt**

Choice C: Estimate the radiant flux from the gap

Assuming that the enclosed volume is a black body in equilibrium, which it isn't, at a temperature equal to a quartic averaging of the two temperatures: $\frac{1}{2}(T_i^4+T_j^4)$. Then the energy is radiated out of the area according to

$$q_{lost} = \sigma A_g \frac{1}{2}(T_i^4+T_j^4)$$

where A_g is the area of the gap, given by

$$A_g = 4h\sqrt{A}$$

But energy was entering the region at the rate

$$q_{in} = \epsilon\sigma A(T_i^4+T_j^4),$$

so the fraction lost is

$$\beta = \frac{A_g}{2A} = \frac{2h}{\epsilon\sqrt{A}} = 0.7$$

Finally, this means that $\alpha\approx 0.15$.

Marking Scheme:

Estimating flux out of gap	0.2 pts
Exact flux into volume	0.2 pts
Correct estimate of gap area	0.2 pts
Assuming $2\alpha+\beta\approx 1$	0.2 pts
Finding α	0.2 pts
$0.2\le\alpha\le 0.4$	0.1 pts
Finding β	0.1 pts
$0.2\le\beta\le 0.6$	0.1 pts
sum	**1.3 pt**

Note that this approach has fewer possible points, as the expression for the flux out of gap makes an assumption that is based on unchecked physics.

Choice D: Another Approach?

Surely there will be some creative students who show other approaches. We will try and expand the marking scheme to recognise these approaches as soon as they occur. A rough guide for an incomplete approach is

Tentative Marking Scheme:

Relevant correct physics equation, each	0.2 pts
Reasonable approximation, each	0.1 pts
Assuming $2\alpha+\beta\approx 1$	0.2 pts
Finding α	0.2 pts
$0.2\le\alpha\le 0.4$	0.1 pts
Finding β	0.1 pts
$0.2\le\beta\le 0.6$	0.1 pts

The maximum possible is still 1.6 pts.

An equation is only relevant if it can be argued that it would lead to an answer to the question within the bounds of the approach that they are following. For example, don't award points for both counting bounces and effective surfaces, unless each equation contributes to a unfied approach that would lead to the answer. Find the most rewarding approach, and award points for that line of reasoning.

If a student only finds one of α or β, then they get 0.2 pts for the first. The marking scheme assumed they would look for α first, but they might have looked for β, and only found that.

Be very careful with mixing and matching approaches!

A student will not get half the points for approach A plus half the points for approach B if they attempt, but don't succeed, with both approaches. They will be awarded the higher of the two scores, not the sum.

3. Numerically determine the temperature of sheet 1 and the temperature of sheet 5.

   The solar intensity is $I_0 = 1360\,\mathrm{W/m^2}$, the background temperature of space is $T_b = 20\,\mathrm{K}$ and is negligible.

   Assuming a student does C.1 correctly, and uses $2\alpha + \beta = 1$, then

β	α	T_1 (K)	T_2 (K)
0.3	0.35	383	120
0.4	0.3	380	102
0.5	0.25	376	83
0.6	0.2	373	65
0.7	0.15	369	48

   The most common expected mistake is $\alpha + \beta = 1$, in that case:

β	α	T_1 (K)	T_2 (K)
0.3	0.7	370	189
0.4	0.6	368	165
0.5	0.5	365	140
0.6	0.4	363	114
0.7	0.3	361	87

   The numbers agree well with the theoretical performance of 320 K and 90 K. Some of the major differences are explained by different coatings on different surfaces, a temperature and wavelength dependence on emissivity that is designed to reflect visible light from the sun while radiating infrared on the sunside of sheet 1, and the sheets are not uniform temperature.

   Marking Scheme:

T_1 consistent with own formula	0.1 pts
$250\mathrm{K} \le T_1 \le 400\mathrm{K}$	0.1 pts
T_5 consistent with own formula	0.1 pts
$45\mathrm{K} \le T_5 \le 200\mathrm{K}$	0.1 pts
sum	**0.4 pt**

   The grade depends on self consistency with the previous work, so the numbers must be checked!

   Note that here is a case where follow on errors could be penalized twice; students should recognize that an answer is not reasonable, as T_1 should be on the order of the temperature of the Earth, and that T_5 ought to have shown significant, but not incredible, cooling.

Part D: The Cryo-Cooler

1. What state variables change?

   (a) In order to force the gas through the plug, which offers up considerable viscous friction, $P_1 > P_2$; it is this pressure difference that is the source of the force.

   (b) Viscous friction is dissipative, so the internal energy of the gas must decrease as it moves through the plug, and then $U_1 > U_2$.

   (c) Though no heat is gained or lost, this is not a constant entropy process; that can be seen because it is an irreversible process. As such, $S_1 < S_2$

   (d) Since the process of moving across a pressure gradient imparts kinetic energy to an object, it is expected that the fluid velocity on the right will be higher than the left. Since mass is conserved, the volume of a mole of gas on the right must also be higher than the volume of a mole on the left, and $V_1 < V_2$.

   (e) The correct answer is $T_1 ? T_2$. If this were an ideal gas, $T_1 > T_2$ since $U \propto T$. But this is not an ideal gas, and U will be a function of temperature and density. As such, it is not possible to know the comparative relation between T_1 and T_2. That's the whole point of this problem, and the challenge of trying to make liquid helium.

   Marking scheme:

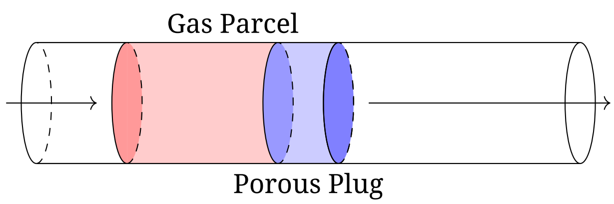
For each correct response	+0.2 pts
For each incorrect response	-0.05 pts
For a blank answer	0 pts
sum	**1.0 pt**

   Explanations by the students are not needed, and the final score for this task cannot be less than zero.

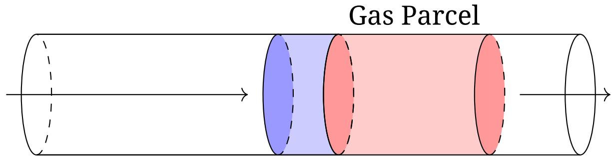
2. A mole of gas at P_1, V_1, T_1, U_1 enters the porous plug from the left, and that mole of gas exits the porous plug on the other side at P_2, V_2, T_2, U_2.

   Consider first a control volume approach

   The figure below shows the motion of a mole of gas through the plug; the mole is shown in pink. Gas to the left of the mole pushes the mole through the plug with a constant force $P_1 A$ through a volume V_1.



   The mole of gas moves through the plug to the right hand side, in the process pushing on the air to the right of the mole with a constant force $P_2 A$, through a volume V_2.



The work that the surrounding gas in region 1 does on the gas pushing it into the plug is

$$W_1 = P_1 V_1$$

because the pressure is constant, and the effective change of volume is V_1. Similarly, when the gas enters region 2 it must displace a volume V_2 of gas that was already there, so

$$W_2 = -P_2 V_2$$

The net work is then

$$W_{net} = P_1 V_1 - P_2 V_2 \tag{20}$$

Since there is no heat exchanged,

$$U_2 - U_1 = \Delta U = Q + W_{net} = P_1 V_1 - P_2 V_2 \tag{21}$$

which implies

$$\Delta U = U_2 - U_1 = P_1 V_1 - P_2 V_2.$$

Upon rearranging

$$U_2 + P_2 V_2 = U_1 + P_1 V_1$$

and therefore

$$U + PV$$

is a conserved quantity.

Marking scheme:

Compute correct W_1	0.1 pts
Compute correct W_2	0.1 pts
Write energy law, Eq 21	0.2 pts
Show $U + PV$ conserved	0.2 pts
sum	**0.6 pt**

Consider instead a differential approach

Another way to look at this problem is to focus on a differential sample of gas as it moves through the plug.

The figure below illustrates this

The total energy of parcel of molar size δm has two relevant energy terms: the internal energy δU and the bulk kinetic energy δK. It has a volume δV. These four quantities are extrinsic, but to simplify notation, we will drop the δ. It's still there, just invisible.

For simplicity's sake, assume a cylindrical shape to the parcel, with an end cap area δA and a length dx. Once again, we will drop the δ. There are three forces that act on the shape, one associated with pressure on the left end, one associated with pressure on the right end, and frictional force associated with viscosity against the walls of the container.

Since this is a parcel of differential length dx, the net force associated with the pressure difference between the ends is

$$F_{ends} = -V\frac{dP}{dx}$$

where V is again the volume of the cylinder.

But this force is (mostly) balanced by the viscous frictional force F_{walls} with the walls of the sponge; these two forces effectively add to zero. In fact, it is the viscous forces with the wall that cause the pressure gradient across the sponge.

The bulk kinetic energy of the parcel does not change significantly as it moves through the sponge. This is seen in that the bulk speed of the gas doesn't change significantly as it moves through the sponge.

The problem with this approach is that the system is not in thermodynamic equilibrium; the process is not reversible, so it is not possible to attach well defined state variables. This means that

$$dU = TdS - PdV \tag{22}$$

is not a function that can be integrated; in fact, $dS \neq 0$ from the previous part of the problem. Arguing that $VdP = -TdS$ is rather handwavy, and resolving this actually requires considering a control volume approach.

Still, the energy conservation ideas still hold true, even if thermodynamically poorly defined, so

$$dU = -PdV - VdP$$

since the part associated with $-VdP$ doesn't change the bulk kinetic energy, and instead dissipates into internal energy of the gas.

The result is that

$$dU = -d(PV)$$

or

$$U + PV$$

is a constant

Marking scheme:

Traditional $\delta W = -PdV$	0.1 pts
Bulk kinetic $\delta K = -VdP$	0.1 pts
Explain where δK goes	0.1 pts
Differential Eq 22	0.1 pts
integrate $U + PV$ constant	0.1 pts
sum	**0.5/0.6 pt**

Because of the many subtle traps, this approach will not get the same number of points as the control volume approach.

3. One can find pressure on this graph by applying

$$dU = TdS - PdV$$

and then requiring constant entropy so that $dS = 0$, and then

$$P = -\left(\frac{\partial U}{\partial V}\right)_S \tag{23}$$

which are the negative slopes of the constant entropy curves on a $U - V$ graph.

Then

$$U + PV = U - \left(\frac{\partial U}{\partial V}\right)_S V$$

is the conserved quantity.

Now $-(\partial U/\partial V)_S$ is measured only at the point V_1, U_1, and is the slope of the tangent line to the constant entropy curve. Following that tangent line back a distance V takes it to an intercept with the U axis, and that intercept is then the conserved quantity.

More mathematically, define a function H

$$H = U + PV$$

then

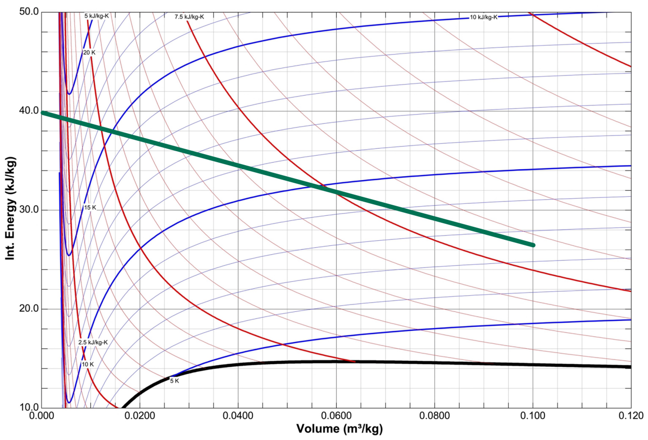
$$U = H_2 - P_2 V$$

is the equation of a line,

$$U = H_2 + \left(\frac{\partial U}{\partial V}\right)_S\bigg|_2 V \tag{24}$$

with the U intercept equal to the conserved H_2.

An estimate can be made visually, but it is difficult to be accurate. Try constructing a line from the point $V_2 = 0.120$, $T_2 = 7.5$ that is tangent to the local isentrope, and the result will intercept the U axis. This result is somewhere around 40. This is shown in green below.

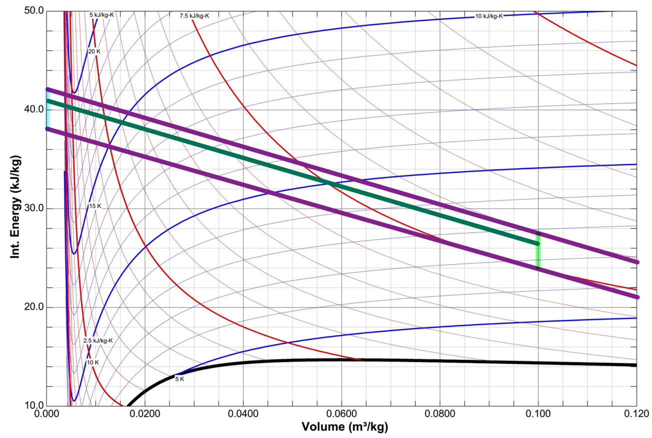


Now to improve the result.

Draw a line out from 39 that is tangent to the nearest isentrope to $V_2 = 0.120$, $T_2 = 7.5$; draw another line out from 41 that is also tangent to the nearest isentrope to $V_2 = 0.120$, $T_2 = 7.5$. These are shown in purple below.

Measuring the distance with a ruler, find the fractional distance between the two purple lines to the point $V_2 = 0.120$, $T_2 = 7.5$ along the highlighted green line. It is about 75% the way from the bottom purple line. This means that the conserved quantity ought be 75% the way up on the highlighted blue section on the graph. A line connecting the two is shown in green.

This point is about 41 kJ/kg. The actual value for the conserved quantity is $U + PV = 40.7$ kJ/kg.



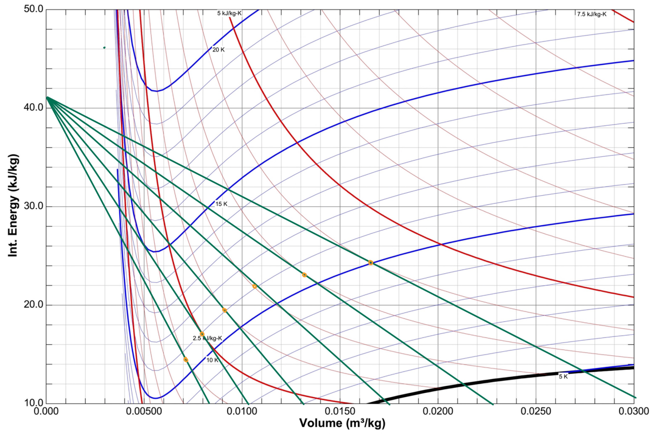
Marking scheme:

Pressure formula stated, Eq 23	0.2 pts
Tangent intercept concept	0.4 pts
A first estimate for H	0.2 pts
Upper bound for estimate set	0.2 pts
Upper bound for estimate set	0.2 pts
Interpolated estimate set	0.2 pts
$40.5 < H < 41.0$	0.2/0.2 pts
$40.2 < H < 41.2$	0.1/0.2 pts
sum	**1.4 pt**

As the task asks for a graphical construction, and it is not possible to construct an accurate tangent to the isentrope at $T_2 = 7.5$K based on a single line, students *must* do something to improve or verify the result, even if it is correct on the first guess. Hence the upper and lower bound approach and interpolation, or something equivalent.

4. Draw a series radial lines out from the conserved point that are tangent to lines of constant entropy. Mark the tangent point. Connect with a smooth curve; this curve is the set of points U_1 as a function of V_1 that has the conserved quantity. Look for the maximum temperature intercept.

   This happens at about $T_1 = 11$K. If T_1 is higher than this, it would not be possible to cool down to $T_2 = 7.5$K.



Students don't need to draw every line, as with a straight edge one can find the tangent that maximizes the temperature T_1 by shifting it around visually.

Line starts from student's H	0.2 pts
Line intercepts an isentrope	0.2 pts
The isentrope matches max T_1	0.2 pts
Stated T_1 within 0.5K of student's contruction	0.1 pts
10K $\leq T_1 \leq 12$K	0.1 pts
sum	**0.8 pt**

5. Using the slope of the line from the conserved quantity to the maximum temperature point, compute the pressure.

   Using the results from above,

$$P_1 = -\frac{(41) - (10)}{(0) - (0.0170)} = 1.8\,\text{MPa}$$

   If they didn't know to use slope by this point, they can't generate an answer. As such, they would already have received points for the pressure formula, and we only consider the numerical result

P agrees with the slope of the graph	0.1 pts
$1.6\,\text{MPa} \leq P_1 \leq 2.4\,\text{MPa}$	0.1 pts
sum	**0.2 pt**