A1 [1.00] An atom emits light with a wavelength $\lambda_0 = 300\mathrm{nm}$. Using the classical model estimate an emission time τ (that is, the period of time it takes the atom to emit the energy equal to that of a single photon). This time coincides with the characteristic time, during which the atom emits a photon, by the order of magnitude. All radiation is due to a single electron located at a distance about $a_0 = 0.1\mathrm{nm}$ from the nucleus. Express your answer in terms of the physical constants, λ_0, and a_0.

The power of radiation is

$$W = \frac{2k}{3c^3}\langle \ddot{p}(t)^2\rangle = \frac{2k}{3c^3}\langle \omega^4 p_m^2 \cos^2(\omega t+\varphi)\rangle = \frac{k\omega^4 p_m^2}{3c^3}.$$

Here p_m is the dipole moment amplitude which can be estimated as $p_m \approx ea_0$. Time of radiation is

$$\tau = \frac{\hbar\omega}{W} \approx \frac{3\hbar}{ka_0^2e^2}\left(\frac{\lambda_0}{2\pi}\right)^3 \approx 1.5\cdot 10^{-8}s.$$

Here we used the relation $\omega = 2\pi c/\lambda_0$.

Ответ:

$$\tau \approx \frac{3\hbar}{ka_0^2e^2}\left(\frac{\lambda_0}{2\pi}\right)^3 \approx 1.5\cdot 10^{-8}s.$$

A2 [0.25] Estimate the power W_s of electromagnetic radiation of all N atoms in the spontaneous emission mode, i.e. when the direction of atomic dipole and the phase of its oscillations randomly change from atom to atom. In your answer write down the formula for the power in terms of N, ω, and τ.

System of the atoms' dipole moment is

$$\vec{P}(t) = \sum_i \vec{p}_{mi}\cos(\omega t+\varphi_i).$$

The power of radiation is

$$W = \frac{2k}{3c^3}\langle \ddot{\vec{P}}(t)\rangle = \frac{k\omega^4}{3c^3}\sum_i p_{mi}^2 + \frac{k\omega^4}{3c^3}\sum_{i\neq j}(\vec{p}_{mi}\cdot\vec{p}_{mj})\cos(\varphi_i-\varphi_j).$$

We used $\langle\cos^2(\omega t+\varphi)^2\rangle = \frac{1}{2}$ and $\langle\cos(\omega t+\varphi_1)\cos(\omega t+\varphi_2)\rangle = \frac{1}{2}\cos(\varphi_i-\varphi_j)$. All dipole moments have the same magnitude, are oriented randomly and their phases are not correlated. Therefore the second sum equals zero. The first sum consists of N equal terms and te power is

$$W_s = N\frac{k\omega^4}{3c^3}p_m^2.$$

For one atom we have

$$\frac{k\omega^4 p_m^2}{3c^3} = \frac{\hbar\omega}{\tau}$$

and thus

Ответ:

$$W_s = N\frac{\hbar\omega}{\tau}.$$

A3 [0.25] Estimate the duration of the spontaneous emission pulse of this system of atoms. Express your answer in terms of the same quantities.

The energy emitted by all atoms is $E = N\hbar\omega$. Hence the time of radiation is

$$\Delta t_s = \frac{E}{W_s} \approx \tau.$$

Ответ: $\Delta t_s = \tau$.

A4 [0.50] Estimate the power W_i of electromagnetic radiation of all N atoms in the superradiance mode, i.e. when the direction of atomic dipoles and the phases of their oscillations are the same for all atoms in the excited state. Express your answer in terms of N, ω, and τ.

In this case all dipole moments are oriented in the same direction and oscillate with the same phase $\varphi_i = \varphi_j$. All terms in both sums in the expression for power in A2 are the same. The first sum has N terms and the second has $N(N-1)$ terms. The power is

$$W_i = (N + N(N-1))\frac{k\omega^4}{3c^3}p_m^2 = N^2\frac{\hbar\omega}{\tau}.$$

Ответ:

$$W_i = N^2\frac{\hbar\omega}{\tau}.$$

A5 [0.25] Estimate the duration of the radiation pulse of the system of atoms in the superradiance mode. Express your answer in terms of the same quantities.

Emitted energy is the same as in A3. The time of radiation

$$\Delta t_i = \frac{E}{W_i} \approx \frac{\tau}{N}.$$

Ответ: $\Delta t_i = \frac{\tau}{N}$.

B1 [0.50] Let the amplitudes of two wave maxima be E_{m1} and E_{m2}. Find the difference in their propagation speeds Δv. Express your answer in terms of n_0, n_2, c, E_{m1}, and E_{m2}.

Wave propagation speed is

$$v = \frac{c}{n} = \frac{c}{n_0 + n_2E_m^2} \approx \frac{c}{n_0}\left(1 - \frac{n_2E_m^2}{n_0}\right).$$

So the speed difference is

Ответ:

$$\Delta v = v_1 - v_2 = \frac{cn_2}{n_0^2}\left(E_{m2}^2 - E_{m1}^2\right).$$

B2 [2.00] A light pulse with a wavelength in vacuum of $\lambda_0 = 300\mathrm{nm}$ and a maximum intensity of $I_0 = 3\cdot 10^9\mathrm{W/cm^2}$ propagates along the axis of a quartz fiber. Assume the envelope of a time dependence of the electric field squared $E_m^2(t)$ of the wave to be a parabola. How far (find the distance s) does the pulse propagate along the fiber before its spectral width increases by the factor of $K = 200$? Express your answer in terms of K, λ_0, n_2, E_m and calculate the numerical value (in meters, rounded to an integer).

First approach.

The intensity is proportional to the square of electric field amplitude. Pulse envelope equation is

$$E_m^2(t) = E_0^2\left(1 - \frac{4(t-t_0)^2}{(\Delta t_0)^2}\right).$$

Here t_0 is a time when the center of the wavepocket crosses given point in space. Value of t_0 depends on the coordinate along the wave. Electric field in the wave is equal to $E_x(t,\, z) = E_m(t)\cos\varphi(t,\, z)$, where the coordinate z is measured along the direction of wave propagation, and the phase is

$$\varphi(t,\, z) = \omega_0 t - n_0\frac{\omega_0}{c}z - n_2E_m^2(t)\frac{\omega_0}{c}z.$$

We neglect initial frequency difference because spectral width increases by hundreds of times. In this way, nonlinear effects lead to additional phase shift, proportional to the square of $(t-t_0)$:

$$\delta\varphi = n_2E_0^2\left(1 - \frac{4(t-t_0)^2}{(\Delta t_0)^2}\right)\frac{\omega_0}{c}z.$$

Angular frequency is the the rate of phase change with time. So quadratic phase shift leads to linear frequency shift

$$\delta\omega(t,\, z) = \frac{\partial\varphi}{\partial t} = -8n_2E_0^2\frac{\omega_0}{c}z\frac{t-t_0}{(\Delta t_0)^2}.$$

Spectral width of the pulse after propagating the distance z in the medium is equal to the difference of frequencies at $t = t_0 \pm \Delta t_0/2$:

$$\Delta\omega = -\frac{8}{\Delta t_0}n_2E_0^2\frac{\omega_0}{c}z\Delta\omega_0.$$

To obtain this formula we used $\Delta\omega_0 \approx 2\pi/\Delta t_0$. The ratio of spectral width at distance $z = s$ and initial spectal width is

$$K = \frac{|\Delta\omega|}{\Delta\omega_0} = \frac{4}{\pi}n_2E_0^2\frac{\omega_0}{c}s,$$

Therefore the required distance is

$$s = \frac{\pi Kc}{4n_2E_m^2\omega_0} = \frac{K\lambda_0}{8n_2E_m^2}.$$

To calculate the numerical value of s, one should remember that the peak pulse intensity value is three times higher than the one mentioned in the problem description. Thus the value of $n_2E_m^2$ is three times higher: $n_2E_m^2 \approx 9.6\cdot 10^{-7}$.

Second approach.

Let us consider the propagation of two adjacent electric field maxima. Due to nonlinearity this maxima move with slightly different speed:

$$\delta v \approx \frac{cn_2}{n_0^2}(E_{m2}^2 - E_{m1}^2).$$

For the given pulse duration and mean frequency it consists of $N = \Delta t_0/T_0 = \omega_0/\Delta\omega_0$ oscillations. The electric field maxima can be enumerated from $-N/2$ to $N/2$. Then the amplitude of electric field maximum with number l is (the maximum with large number arrives at a given point later):

$$E_m^2(l) = E_0^2\left(1 - \frac{4l^2}{N^2}\right).$$

Consequently, the speed difference between adjacent maxima with numbers l and $l+1$ equals

$$\Delta v_l = v_{l+1} - v_l \approx \frac{cn_2}{n_0^2}E_0^2\frac{8l}{N^2}.$$

Initial frequency is equal ω_0. After propagation over distance s the frequency of the maximum with number l is:

$$\omega_l = \frac{2\pi}{T_l} = \frac{2\pi}{2\pi/\omega_0 + s/v_{l+1} - s/v_l} = \frac{\omega_0}{1 - s\Delta v_l\omega_0/(2\pi v_l^2)} = \omega_0 + \frac{\omega_0^2}{2\pi v_l^2}\frac{cn_2}{n_0^2}E_0^2\frac{8l}{N^2}s.$$

Frequency linearly depends on the maximum number. The maximal frequency difference is between the first and the last maxima and equals

$$\Delta\omega = \frac{\omega_0^2}{2\pi v_l^2}\frac{cn_2}{n_0^2}\frac{8}{N}s.$$

Hence ($v_l \approx v = c/n_0$ and $\omega_0 = N\Delta\omega_0$)

$$K = \frac{\Delta\omega}{\Delta\omega_0} = \frac{8n_2E_m^2}{\lambda_0}s,$$

and the answer is the same.

Ответ:

$$s = \frac{K\lambda_0}{8n_2E_m^2} \approx 7.8m.$$

B3 0.50 What sign should the constant β_2 have in order for the pulse chirped according to the scheme described above to be compressed in time in this medium? Please, indicate "+" or "-" in your answer. In what follows consider that β_2 has exactly this sign.

To squeeze the chirping pulse, the relatively high-frequency tail of the pulse should propagate faster than its low-frequency 'head'. This is called 'anomalous dispersion'. Speed of the wave propagation is the group velocity

$$V_g = \frac{d\omega}{dk} = \frac{1}{\beta_1 + \beta_2(\omega - \omega_0)},$$

Anomalous dispersion (larger group speed at larger frequencies) is observed for $\beta_2 < 0$.

Ответ: $\beta_2 < 0$.

Условие

B4 1.00 A pulse described in B2 has a duration $\Delta t_0 = 10\text{ps}$ and an initial spectral width $\Delta\omega_0 \approx 2\pi/\Delta t_0$ (before chirping) and propagates in the medium described above. Find the distance the pulse should travel in order to achieve the minimum possible duration after chirping with spectrum broadening by the factor of $K = 200$. Express your answer in terms of physical constants, $K, \Delta t_0, \beta_1$, and β_2 and calculate the numerical value in meters, rounded to an integer.

Since the term associated with β_2 is much less in absolute value than β_1 for the given range of frequencies, we can use approximate expression for the group velocity

$$V_g \approx \frac{1}{\beta_1} - \frac{\beta_2}{\beta_1^2}(\omega - \omega_0).$$

The length of the chirping pulse that just has entered into the considered medium is $\Delta l = V_g \Delta t_0 = \frac{1}{\beta_1}\Delta t_0$. The difference in group velocity of the 'tail' and the 'head' is

$$\Delta V_g \approx \frac{|\beta_2|}{\beta_1^2}\Delta\omega = \frac{|\beta_2|}{\beta_1^2}K\frac{2\pi}{\Delta t_0}.$$

The time in which the 'tail' will catch up with the 'head' up (which corresponds to the minimal duration of the pulse) is

$$t = \frac{\Delta l}{\Delta V_g} = \frac{\beta_1}{2\pi K|\beta_2|}(\Delta t_0)^2.$$

Since $V_g \approx \frac{1}{\beta_1}$ in this period of time the pulse will pass the distance

$$l = V_g t \approx \frac{1}{2\pi K|\beta_2|}(\Delta t_0)^2 \approx 4m.$$

Ответ:

$$l = \frac{1}{2\pi K|\beta_2|}(\Delta t_0)^2 \approx 4m.$$

B5 1.50 Nonlinearity of a medium leads to disappearance of diffraction of a light beam of sufficiently high intensity. Estimate the minimum power of a light pulse W_c at which it does not experience diffraction, i.e. propagates inside a narrow cylindrical channel of constant radius. Express your answer for W_c in terms of physical constants, frequency ω_0, n_0, and n_2. Assume the intensity distribution over the channel cross section to be approximately uniform. Find the numerical value of the power for a pulse with a wavelength in vacuum $\lambda_0 = 300\text{nm}$ propagating in quartz. Coefficient $n_0 = 1.47$.

Let be a radius of the channel. Diffraction divergence angle of the beam can be estimated as

$$\theta_d \approx \frac{\lambda}{\pi a} = \frac{2c}{n_0\omega_0 a}.$$

However, due to nonlinearity, the refractive index inside the channel $n = n_0 + n_2E_m^2$ is larger than refractive index $n = n_0$ outside of it. The beam deflecting from the channel's axis due to diffraction will not get out of it if its angle of incidence will be greater than the angle of total internal reflection α_c. So the critical power corresponds to the condition $\theta_d = \frac{\pi}{2} - \alpha_c \equiv \theta_c$.

Since

$$\cos\theta_c \approx 1 - \frac{\theta_c^2}{2} = \sin\alpha_c = \frac{n_0}{n_0 + n_2E_m^2} \approx 1 - \frac{n_2E_m^2}{n_0},$$

we have the following condition

$$E_m^2 = \frac{n_0}{2n_2}\theta_d^2 = \frac{2c^2}{n_0n_2\omega_0^2}\frac{1}{a^2}.$$

Corresponding value of the intensity is

$$I_c = \frac{\varepsilon_0 n_0 c}{2}E_m^2 = \frac{\varepsilon_0 c^3}{n_2\omega_0^2a^2}.$$

Finally, 'critical' power is

$$W_c = I\pi a^2 = \frac{\pi\varepsilon_0 c^3}{n_2\omega_0^2}$$

and does not depend on the channel diameter.

The numerical value of the power for the pulse of wavelength λ_0 in a vacuum is equal to

$$W_c = \frac{2\pi n_0\lambda_0^2 I_1}{3.2\cdot 10^{-7}} \approx 26MW.$$

Comment: The expression for the diffraction divergence of the circular beam is approximate. Therefore, it is possible to use alternative estimations, for instance $\theta_d \approx \lambda/a$, or $\theta_d \approx \lambda/2\pi a$.

Ответ:

$$W_c = \frac{\pi\varepsilon_0 c^3}{n_2\omega_0^2} = \frac{2\pi n_0 \lambda_0^2 I_1}{3.2\cdot 10^{-7}} \approx 26MW.$$

C1 1.00 Propose a method that would allow one to detect an exoplanet with a noticeable inclination of its orbital plane with respect to the line of sight by means of studying the spectrum of its star in the optical range. As an answer name the physical phenomenon underlying your method.

The most natural way is to detect the Doppler shift of some element (e.g. hydrogen) spectral line. The shift is caused by the star rotation with respect to the star-exoplanet system barycenter. This shift should be observed with the exoplanet rotation period.

Ответ: Doppler effect

C2 1.00 Suppose an exoplanet of mass m orbits a star of mass M in a circular orbit of a radius R and the period of revolution is T. The orbital plane is at an angle θ to the direction to Earth. Estimate the accuracy of the relative frequency measurement, $\Delta\omega/\omega$, required to detect such an exoplanet by your method. In your answer express $\Delta\omega/\omega$ in terms of the fundamental constants, R, T, θ, m, and M.

Planet's orbital motion velocity is $v = \frac{2\pi R}{T}$. The corresponding velocity of the star is $v_1 = \frac{vm}{M}$. The Doppler effect is sensitive to the velocity sightline component only (the nonrelativistic case is considered). Its maximum value is $v_1 \cos(\theta)$. Thus the relative frequency shift is

Ответ:

$$\frac{\Delta\omega}{\omega} = \frac{2\pi R}{Tc}\frac{m}{M}\cos\theta.$$

C3 0.25 Assume the mass of the exoplanet and its star to be equal to the mass of Earth and the Sun, respectively. Assume the radius of the circular orbit to be equal to the distance from Earth to the Sun ($R \approx 1.5\cdot 10^{11}\mathrm{m}$), the angle $\theta = 60^\circ$. The Solar mass is 330,000 times of the Earth's mass, the period of the Earth's revolution around the Sun is 1 year. Find an integer n such that 10^{-n} is the accuracy of relative frequency measurement required by your method. Usage of ultrashort (femtosecond) laser pulses makes it possible to measure frequencies in the optical range ($10^{15}\mathrm{Hz}$) with an accuracy of about 10 Hz. Is this accuracy enough to register the exoplanet?

Even for $\theta = 60^o$ the star velocity sightline component is at most half v_1. In order to estimate this component, one could assume the exoplanet orbit is circular. Since the same assumption is valid for the Earth's orbit, the maximal sightline projection is

$$v_m = \frac{2\pi R}{T}\frac{1}{330000}\cos\theta \approx 4.5\frac{sm}{s}.$$

The corresponding frequency shift is of order $\frac{\Delta\omega}{\omega} \approx \frac{v_m}{c} \approx 10^{-10}$. Thus, a frequency variation in the tenth significant figure should be detected once a year. The accuracy of an optical frequency comb is of order $\frac{10Hz}{10^{15}Hz} \approx 10^{-14}$. So, exoplanet detection is quite possible.

This method is indeed applied in practice. However a number of difficulties had to be overcome to actually use this technology. In fact, most of the stably generated femtosecond pulses have carrier frequencies outside the visible spectrum, in the microwave and infrared regions. In order to use the method in visible spectrum, a different nonlinear phenomenon should be used -- higher harmonic generation in nonlinear medium. Moreover, to increase the spectral device resolution, frequency filtering should be used which increases the narrow frequency intervals between the lines.

Ответ: $n = 10$, exoplanet detection is possible.

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