Theoretical Question 1:

"Ping-Pong" Resistor

A capacitor consists of two circular parallel plates both with radius R separated by distance d, where $d << R$, as shown in Fig. 1.1(a). The top plate is connected to a *constant* voltage source at a potential V while the bottom plate is grounded. Then a thin and small disk of mass m with radius r ($<< R, d$) and thickness t ($<< r$) is placed on the center of the bottom plate, as shown in Fig. 1.1(b).

Let us assume that the space between the plates is in vacuum with the dielectric constant ε_0; the plates and the disk are made of perfect conductors; and all the electrostatic edge effects may be neglected. The inductance of the whole circuit and the relativistic effects can be safely disregarded. The image charge effect can also be neglected.

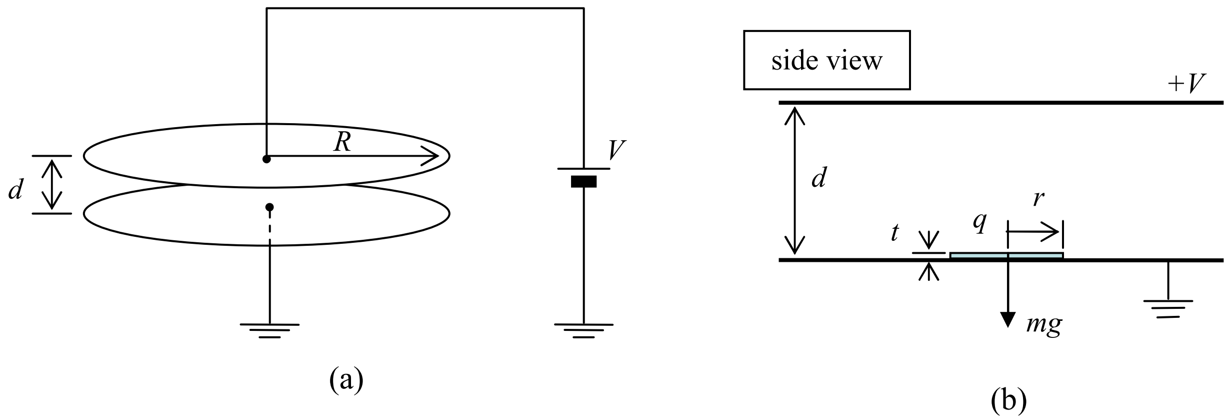

Figure 1.1 Schematic drawings of (a) a *parallel plate* capacitor connected to a constant voltage source and (b) a side view of the parallel *plates* with a small *disk* inserted inside the capacitor. (See text for details.)

(a) [1.2 points] Calculate the electrostatic force F_{p} between *the plates* separated by d before inserting the disk in-between as shown in Fig. 1.1(a).

(b) [0.8 points] When the disk is placed on the bottom plate, a charge q on *the disk* of Fig. 1.1(b) is related to the voltage V by $q = \chi V$. Find χ in terms of r, d, and ε_0.

(c) [0.5 points] The parallel plates lie perpendicular to a uniform gravitational field g. To lift up the disk at rest initially, we need to increase the applied voltage beyond a threshold voltage $V_{\rm th}$. Obtain $V_{\rm th}$ in terms of m, g, d, and χ.

(d) [2.3 points] When $V > V_{\rm th}$, the disk makes an up-and-down motion between the plates. (Assume that the disk moves only vertically *without any wobbling*.) The collisions between the disk and the plates are inelastic with the restitution coefficient $\eta \equiv ({\rm v}_{\rm after} / {\rm v}_{\rm before})$, where ${\rm v}_{\rm before}$ and ${\rm v}_{\rm after}$ are the speeds of the disk just before and after the collision respectively. The plates are stationarily fixed in position. The speed of the disk *just after* the collision at the bottom plate approaches a "steady-state speed" ${\rm v}_{\rm s}$, which depends on V as follows:

$${\rm v}_{\rm s} = \sqrt{\alpha V^2 + \beta}\,. \tag{1.1}$$

Obtain the coefficients α and β in terms of m, g, χ, d, and η. Assume that the whole surface of the disk touches the plate evenly and simultaneously so that the complete charge exchange happens instantaneously at every collision.

(e) [2.2 points] After reaching its steady state, the time-averaged current I through the capacitor plates can be approximated by $I = \gamma V^2$ when $qV >> mgd$. Express the coefficient γ in terms of m, χ, d, and η.

(f) [3 points] When the applied voltage V is decreased (extremely slowly), there exists a critical voltage $V_{\rm c}$ below which the charge will cease to flow. Find $V_{\rm c}$ and the corresponding current $I_{\rm c}$ in terms of m, g, χ, d, and η. By comparing $V_{\rm c}$ with the lift-up threshold $V_{\rm th}$ discussed in (c), make a rough sketch of the $I-V$ characteristics when V is increased and decreased in the range from $V = 0$ to $3V_{\rm th}$.