

Solutions

PART-A Product of the mass and the position of the ball ($m \times l$) **(4.0 points)**

1. Suggest and justify, by using equations, a method allowing to obtain $m \times l$. (2.0 points)

$$m \times l = (M + m) \times l_{\text{cm}}$$

(Explanation) The lever rule is applied to the Mechanical “Black Box”, shown in Fig. A-1, once the position of the center of mass of the whole system is found.



Fig. A-1 Experimental setup

2. Experimentally determine the value of $m \times l$. (2.0 points)

$$m \times l = 2.96 \times 10^{-3} \text{ kg} \cdot \text{m}$$

(Explanation) The measured quantities are

$$M + m = (1.411 \pm 0.0005) \times 10^{-1} \text{ kg}$$

and

$$l_{\text{cm}} = (2.1 \pm 0.06) \times 10^{-2} \text{ m} \quad \text{or} \quad 21 \pm 0.6 \text{ mm.}$$

Therefore

$$\begin{aligned} m \times l &= (M + m) \times l_{\text{cm}} \\ &= (1.411 \pm 0.0005) \times 10^{-1} \text{ kg} \times (2.1 \pm 0.06) \times 10^{-2} \text{ m} \\ &= (2.96 \pm 0.08) \times 10^{-3} \text{ kg} \cdot \text{m} \end{aligned}$$

PART-B The mass m of the ball (10.0 points)

1. Measure v for various values of h . Plot the data on a graph paper in a form that is suitable to find the value of m . Identify the slow rotation region and the fast rotation region on the graph. (4.0 points)
2. Show from your measurements that $h = C v^2$ in the slow rotation region, and $h = A v^2 + B$ in the fast rotation region. (1.0 points)

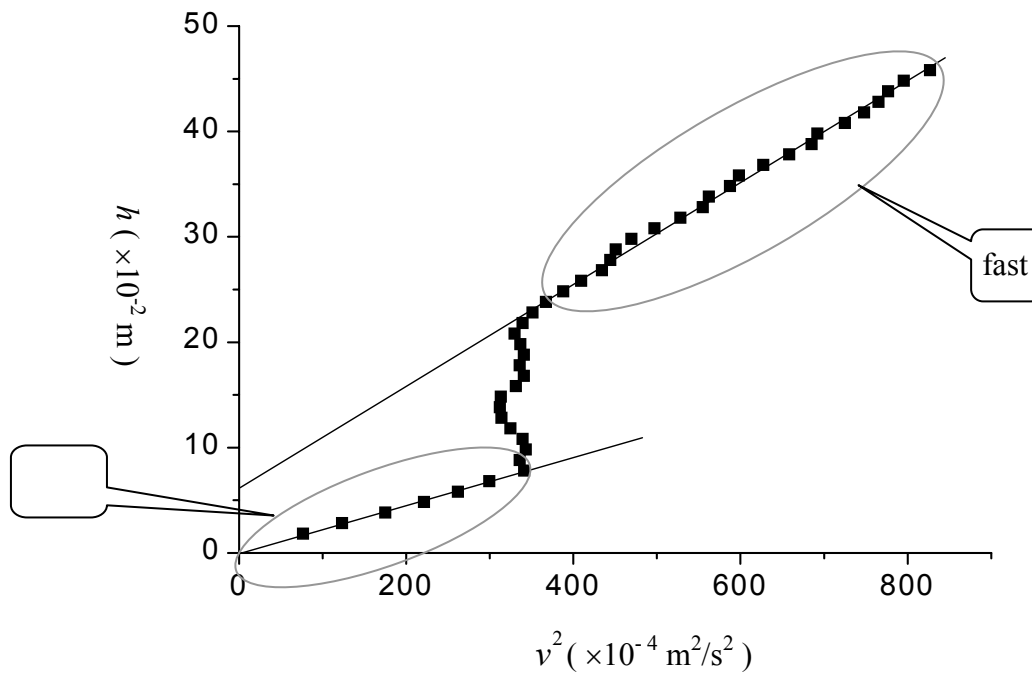


Fig. B-1 Experimental data

(Explanation) The measured data are

	$h_1 (\times 10^{-2} \text{ m})^a$	$\Delta t (\text{ms})$	$h (\times 10^{-2} \text{ m})^b$	$v (\times 10^{-2} \text{ m/s})^c$	$v^2 (\times 10^{-4} \text{ m}^2/\text{s}^2)$
1	25.5 ± 0.1	269.4 ± 0.05	1.8 ± 0.1	8.75 ± 0.02	76.6 ± 0.2
2	26.5 ± 0.1	235.7 ± 0.05	2.8 ± 0.1	11.12 ± 0.02	123.7 ± 0.3
3	27.5 ± 0.1	197.9 ± 0.05	3.8 ± 0.1	13.24 ± 0.03	175.3 ± 0.6
4	28.5 ± 0.1	176.0 ± 0.05	4.8 ± 0.1	14.89 ± 0.03	221.7 ± 0.6
5	29.5 ± 0.1	161.8 ± 0.05	5.8 ± 0.1	16.19 ± 0.03	262.1 ± 0.7
6	30.5 ± 0.1	151.4 ± 0.05	6.8 ± 0.1	17.31 ± 0.03	299.6 ± 0.7
7	31.5 ± 0.1	141.8 ± 0.05	7.8 ± 0.1	18.48 ± 0.04	342 ± 1
8	32.5 ± 0.1	142.9 ± 0.05	8.8 ± 0.1	18.33 ± 0.04	336 ± 1

9	33.5±0.1	141.4±0.05	9.8±0.1	18.53±0.04	343±1
10	34.5±0.1	142.2±0.05	10.8±0.1	18.42±0.04	339±1
11	35.5±0.1	145.4±0.05	11.8±0.1	18.02±0.04	325±1
12	36.5±0.1	147.8±0.05	12.8±0.1	17.73±0.04	314±1
13	37.5±0.1	148.3±0.05	13.8±0.1	17.67±0.04	312±1
14	38.5±0.1	148.0±0.05	14.8±0.1	17.70±0.04	313±1
15	39.5±0.1	143.9±0.05	15.8±0.1	18.21±0.04	332±1
16	40.5±0.1	141.9±0.05	16.8±0.1	18.46±0.04	341±1
17	41.5±0.1	142.9±0.05	17.8±0.1	18.33±0.04	336±1
18	42.5±0.1	141.9±0.05	18.8±0.1	18.46±0.04	341±1
19	43.5±0.1	142.8±0.05	19.8±0.1	18.35±0.04	337±1
20	44.5±0.1	144.3±0.05	20.8±0.1	18.16±0.04	330±1
21	45.5±0.1	142.2±0.05	21.8±0.1	18.42±0.04	339±1
22	46.5±0.1	139.8±0.05	22.8±0.1	18.74±0.04	351±1
23	47.5±0.1	136.7±0.05	23.8±0.1	19.17±0.04	368±1
24	48.5±0.1	133.0±0.05	24.8±0.1	19.70±0.04	388±1
25	49.5±0.1	129.5±0.05	25.8±0.1	20.23±0.04	409±1
26	50.5±0.1	125.7±0.05	26.8±0.1	20.84±0.04	434±1
27	51.5±0.1	124.3±0.05	27.8±0.1	21.08±0.04	444±1
28	52.5±0.1	123.4±0.05	28.8±0.1	21.23±0.04	451±1
29	53.5±0.1	120.9±0.05	29.8±0.1	21.67±0.04	470±1
30	54.5±0.1	117.5±0.05	30.8±0.1	22.30±0.04	497±1
31	55.5±0.1	114.0±0.05	31.8±0.1	22.98±0.04	528±1
32	56.5±0.1	111.2±0.05	32.8±0.1	23.56±0.05	555±2
33	57.5±0.1	110.5±0.05	33.8±0.1	23.71±0.05	562±2
34	58.5±0.1	108.1±0.05	34.8±0.1	24.24±0.05	588±2
35	59.5±0.1	107.1±0.05	35.8±0.1	24.46±0.05	598±2
36	60.5±0.1	104.6±0.05	36.8±0.1	25.05±0.05	628±2
37	61.5±0.1	102.1±0.05	37.8±0.1	25.66±0.05	658±2
38	62.5±0.1	100.1±0.05	38.8±0.1	26.17±0.05	685±2
39	63.5±0.1	99.6±0.05	39.8±0.1	26.31±0.05	692±2
40	64.5±0.1	97.3±0.05	40.8±0.1	26.93±0.05	725±2
41	65.5±0.1	95.8±0.05	41.8±0.1	27.35±0.05	748±2
42	66.5±0.1	94.7±0.05	42.8±0.1	27.67±0.05	766±2
43	67.5±0.1	94.0±0.05	43.8±0.1	27.87±0.06	777±2
44	68.5±0.1	92.9±0.05	44.8±0.1	28.20±0.06	795±2
45	69.5±0.1	91.1±0.05	45.8±0.1	28.76±0.06	827±2

- where
- a) h_1 is the reading of the top position of the weight before it starts to fall,
 - b) h is the distance of fall of the weight which is obtained by $h = h_1 - h_2 + d/2$,
 h_2 ($= (25 \pm 0.05) \times 10^{-2}$ m) is the top position of the weight at the start of blocking of the photogate,
 d ($= (2.62 \pm 0.005) \times 10^{-2}$ m) is the length of the weight, and
 - c) v is obtained from $v = d/\Delta t$.

3. Relate the coefficient C to the parameters of the MBB. (1.0 points)

$$h = C v^2, \text{ where } C = \{m_o + I/R^2 + m(l^2 + 2/5 r^2)/R^2\}/2m_o g$$

(Explanation) The ball is at static equilibrium ($x = l$). When the speed of the weight is v , the increase in kinetic energy of the whole system is given by

$$\begin{aligned} \Delta K &= 1/2 m_o v^2 + 1/2 I \omega^2 + 1/2 m(l^2 + 2/5 r^2) \omega^2 \\ &= 1/2 \{m_o + I/R^2 + m(l^2 + 2/5 r^2)/R^2\} v^2, \end{aligned}$$

where ω ($= v/R$) is the angular velocity of the Mechanical “Black Box” and I is the *effective* moment of inertia of the whole system except the ball. Since the decrease in gravitational potential energy of the weight is

$$\Delta U = - m_o g h,$$

the energy conservation ($\Delta K + \Delta U = 0$) gives

$$\begin{aligned} h &= 1/2 \{m_o + I/R^2 + m(l^2 + 2/5 r^2)/R^2\} v^2 / m_o g \\ &= C v^2, \text{ where } C = \{m_o + I/R^2 + m(l^2 + 2/5 r^2)/R^2\}/2m_o g \end{aligned}$$

4. Relate the coefficients A and B to the parameters of the MBB. (1.0 points)

$$\begin{aligned} h &= A v^2 + B, \text{ where } A = [m_o + I/R^2 + m\{(L/2 - \delta - r)^2 + 2/5 r^2\}/R^2]/2m_o g \\ &\text{and } B = [-k_1(L/2 - l - \delta - r)^2 \\ &\quad + k_2\{(L - 2\delta - 2r)^2 - (L/2 + l - \delta - r)^2\}]/2m_o g \end{aligned}$$

(Explanation) The ball stays at the end cap of the tube ($x = L/2 - \delta - r$). When the speed of the weight is v , the increase in kinetic energy of the whole system is given by

$$K = 1/2 [m_o + I/R^2 + m\{(L/2 - \delta - r)^2 + 2/5 r^2\}/R^2]v^2.$$

Since the increase in elastic potential energy of the springs is

$$\Delta U_e = 1/2 [-k_1 (L/2 - l - \delta - r)^2 + k_2 \{(L - 2\delta - 2r)^2 - (L/2 + l - \delta - r)^2\}],$$

the energy conservation ($K + \Delta U + \Delta U_e = 0$) gives

$$h = 1/2 [m_o + I/R^2 + m\{(L/2 - \delta - r)^2 + 2/5 r^2\}/R^2]v^2/m_o g + \Delta U_e/m_o g = A v^2 + B,$$

where

$$A = [m_o + I/R^2 + m\{(L/2 - \delta - r)^2 + 2/5 r^2\}/R^2]/2m_o g$$

and

$$B = [-k_1 (L/2 - l - \delta - r)^2 + k_2 \{(L - 2\delta - 2r)^2 - (L/2 + l - \delta - r)^2\}] / 2m_o g.$$

5. Determine the value of m from your measurements and the results obtained in **PART-A**. (3.0 points)

$$m = 6.2 \times 10^{-2} \text{ kg}$$

(Explanation) From the results obtained in **PART-B** 3 and 4 we get

$$A - C = \frac{m}{2gm_o R^2} \left\{ \left(\frac{L}{2} - \delta - r \right)^2 - l^2 \right\}.$$

The measured values are

$$L = (40.0 \pm 0.05) \times 10^{-2} \text{ m}$$

$$m_o = (100.4 \pm 0.05) \times 10^{-3} \text{ kg}$$

$$2R = (3.91 \pm 0.005) \times 10^{-2} \text{ m}$$

Therefore,

$$(L/2 - \delta - r)^2 = \{(20.0 \pm 0.03) - 0.5 - 1.1\}^2 \times 10^{-4} \text{ m}^2 = (338.6 \pm 0.8) \times 10^{-4} \text{ m}^2$$

and

$$2gm_o R^2 = 2 \times 980 \times (100.4 \pm 0.05) \times (1.955 \pm 0.003)^2 \times 10^{-6} \text{ kg} \cdot \text{m}^3/\text{s}^2 = (752 \pm 2) \times 10^{-6} \text{ kg} \cdot \text{m}^3/\text{s}^2.$$

The slopes of the two straight lines in the graph (Fig. B-1) of **PART-B** 1 are

$$A = 5.0 \pm 0.1 \text{ s}^2/\text{m} \quad \text{and} \quad C = 2.4 \pm 0.1 \text{ s}^2/\text{m},$$

respectively, and

$$A - C = 2.6 \pm 0.1 \text{ s}^2/\text{m}.$$

Since we already obtained $m \times l = (M + m) \times l_{\text{cm}} = 2.96 \times 10^{-3} \text{ kg} \cdot \text{m}$ from **PART-A**, the equation

$$(338.6 \pm 0.8)m^2 - (752 \pm 2) \times 10^3 \times (0.026 \pm 0.001)m - (296 \pm 8)^2 = 0$$

or

$$(338.6 \pm 0.8)m^2 - (19600 \pm 800)m - (88000 \pm 3000) = 0$$

is resulted, where m is expressed in the unit of g.

The roots of this equation are

$$m = \frac{(9800 \pm 400) \pm \sqrt{(9800 \pm 400)^2 + (338.6 \pm 0.8) \times (88000 \pm 3000)}}{(338.6 \pm 0.8)}.$$

The physically meaningful positive root is

$$m = \frac{(9800 \pm 400) + \sqrt{(9800 \pm 400)^2 + (338.6 \pm 0.8) \times (88000 \pm 3000)}}{(338.6 \pm 0.8)} = (62 \pm 2) \text{ g} = (6.2 \pm 0.2) \times 10^{-2} \text{ kg}.$$

PART-C The spring constants k_1 and k_2 (6.0 points)

1. Measure the periods T_1 and T_2 of small oscillation shown in Figs. 3 (1) and (2) and write down their values, respectively. (1.0 points)

$$T_1 = 1.1090 \text{ s} \quad \text{and} \quad T_2 = 1.0193 \text{ s}$$

(Explanation)

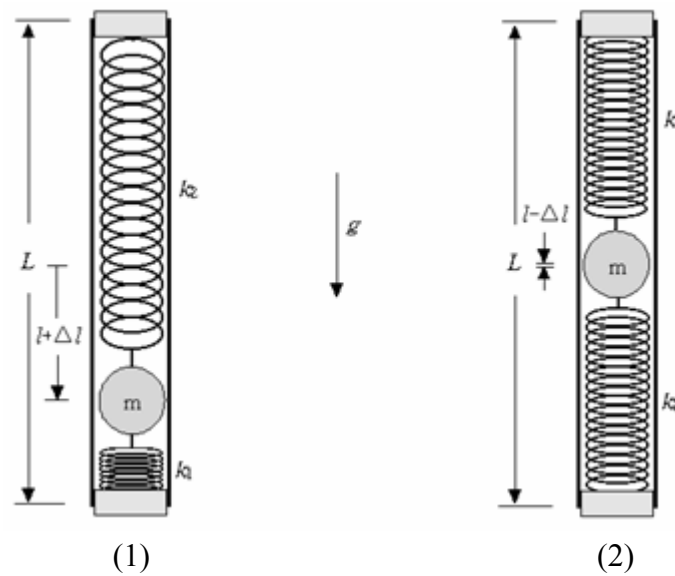


Fig. C-1 Small oscillation experimental set up

The measured periods are

	T_1 (s)		T_2 (s)
1	1.1085 ± 0.00005	1	1.0194 ± 0.00005
2	1.1092 ± 0.00005	2	1.0194 ± 0.00005
3	1.1089 ± 0.00005	3	1.0193 ± 0.00005
4	1.1085 ± 0.00005	4	1.0191 ± 0.00005
5	1.1094 ± 0.00005	5	1.0192 ± 0.00005
6	1.1090 ± 0.00005	6	1.0194 ± 0.00005
7	1.1088 ± 0.00005	7	1.0194 ± 0.00005
8	1.1090 ± 0.00005	8	1.0191 ± 0.00005
9	1.1092 ± 0.00005	9	1.0192 ± 0.00005
10	1.1094 ± 0.00005	10	1.0193 ± 0.00005

By averaging the 10 measurements for each configuration, respectively, we get

$$T_1 = 1.1090 \pm 0.0003 \text{ s} \quad \text{and} \quad T_2 = 1.0193 \pm 0.0001 \text{ s}.$$

2. Explain, by using equations, why the angular frequencies ω_1 and ω_2 of small oscillation of the configurations are different. (1.0 points)

$$\omega_1 = \sqrt{\frac{Mg \frac{L}{2} + mg \left(\frac{L}{2} + l + \Delta l \right)}{I_o + m \left\{ \left(\frac{L}{2} + l + \Delta l \right)^2 + \frac{2}{5} r^2 \right\}}}$$

$$\omega_2 = \sqrt{\frac{Mg \frac{L}{2} + mg \left(\frac{L}{2} - l + \Delta l \right)}{I_o + m \left\{ \left(\frac{L}{2} - l + \Delta l \right)^2 + \frac{2}{5} r^2 \right\}}}$$

(Explanation) The moment of inertia of the Mechanical “Black Box” with respect to the pivot at the top of the tube is

$$I_1 = I_o + m \left\{ \left(\frac{L}{2} + l + \Delta l \right)^2 + \frac{2}{5} r^2 \right\} \quad \text{or} \quad I_2 = I_o + m \left\{ \left(\frac{L}{2} - l + \Delta l \right)^2 + \frac{2}{5} r^2 \right\}$$

depending on the orientation of the MBB as shown in Figs. C-1(1) and (2), respectively.

When the MBB is slightly tilted by an angle θ from vertical, the torque applied by the gravity is

$$\tau_1 = Mg \left(\frac{L}{2} \right) \sin \theta + mg \left(\frac{L}{2} + l + \Delta l \right) \sin \theta \approx \left\{ Mg \left(\frac{L}{2} \right) + mg \left(\frac{L}{2} + l + \Delta l \right) \right\} \theta$$

or

$$\tau_2 = Mg \left(\frac{L}{2} \right) \sin \theta + mg \left(\frac{L}{2} - l + \Delta l \right) \sin \theta \approx \left\{ Mg \left(\frac{L}{2} \right) + mg \left(\frac{L}{2} - l + \Delta l \right) \right\} \theta$$

depending on the orientation.

Therefore, the angular frequencies of oscillation become

$$\omega_1 = \sqrt{\frac{\tau_1 / \theta}{I_1}} = \sqrt{\frac{Mg \frac{L}{2} + mg \left(\frac{L}{2} + l + \Delta l \right)}{I_o + m \left\{ \left(\frac{L}{2} + l + \Delta l \right)^2 + \frac{2}{5} r^2 \right\}}}$$

and

$$\omega_2 = \sqrt{\frac{\tau_2 / \theta}{I_2}} = \sqrt{\frac{Mg \frac{L}{2} + mg \left(\frac{L}{2} - l + \Delta l \right)}{I_o + m \left\{ \left(\frac{L}{2} - l + \Delta l \right)^2 + \frac{2}{5} r^2 \right\}}}.$$

3. Evaluate Δl by eliminating I_o from the previous results. (1.0 points)

$$\Delta l = (7.2 \pm 0.9) \text{ cm} = (7.2 \pm 0.9) \times 10^{-2} \text{ m}$$

(Explanation) By rewriting the two expressions for the angular frequencies ω_1 and ω_2 as

$$Mg \frac{L}{2} + mg \left(\frac{L}{2} + l + \Delta l \right) = I_o \omega_1^2 + m \omega_1^2 \left\{ \left(\frac{L}{2} + l + \Delta l \right)^2 + \frac{2}{5} r^2 \right\}$$

and

$$Mg \frac{L}{2} + mg \left(\frac{L}{2} - l + \Delta l \right) = I_o \omega_2^2 + m \omega_2^2 \left\{ \left(\frac{L}{2} - l + \Delta l \right)^2 + \frac{2}{5} r^2 \right\}$$

one can eliminate the unknown moment of inertia I_o of the MBB without the ball.

By eliminating the I_o one gets the equation for Δl

$$(\omega_2^2 - \omega_1^2) \left\{ \frac{(M+m)gL}{2} + mg\Delta l \right\} + (\omega_1^2 + \omega_2^2) mgl = \omega_1^2 \omega_2^2 m(L + 2\Delta l)(2l).$$

From the measured or given values we get,

$$\begin{aligned} (\omega_2^2 - \omega_1^2) &= \left\{ \left(\frac{2\pi}{T_2} \right)^2 - \left(\frac{2\pi}{T_1} \right)^2 \right\} = \left(\frac{6.2832}{1.0193 \pm 0.0001} \right)^2 - \left(\frac{6.2832}{1.1090 \pm 0.0003} \right)^2 \\ &= 5.90 \pm 0.01 \text{ s}^{-2} \end{aligned}$$

$$\frac{(M+m)gL}{2} = \frac{(141.1 \pm 0.05) \times 980 \times (40.0 \pm 0.05)}{2} = (27.66 \pm 0.04) \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}^2$$

$$\begin{aligned} (\omega_1^2 + \omega_2^2) mgl &= \left\{ \left(\frac{2\pi}{T_1} \right)^2 + \left(\frac{2\pi}{T_2} \right)^2 \right\} (M+m) l_{cm} g \\ &= \left\{ \left(\frac{6.2832}{1.1090 \pm 0.0003} \right)^2 + \left(\frac{6.2832}{1.0193 \pm 0.0001} \right)^2 \right\} \times (296 \pm 8) \times 980 \end{aligned}$$

$$= (203 \pm 5) \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}^4$$

$$\begin{aligned} \omega_1^2 \omega_2^2 m l &= \left(\frac{2\pi}{T_1} \right)^2 \left(\frac{2\pi}{T_2} \right)^2 (M + m) l_{cm} \\ &= \left(\frac{6.2832}{1.1090 \pm 0.0003} \right)^2 \left(\frac{6.2832}{1.0193 \pm 0.0001} \right)^2 \times (296 \pm 8) \\ &= (3.6 \pm 0.1) \text{ kg} \cdot \text{m}/\text{s}^4. \end{aligned}$$

Therefore, the equation we obtained in **PART-C** 3 becomes

$$\begin{aligned} (5.90 \pm 0.01) \{ (27.66 \pm 0.04) \times 10^5 + (62 \pm 2) \times 980 \times \Delta l \} + (203 \pm 5) \times 10^5 \\ = (7.2 \pm 0.2) \times 10^5 \times \{ (40.0 \pm 0.05) + 2\Delta l \}, \end{aligned}$$

where Δl is expressed in the unit of cm. By solving the equation we get

$$\Delta l = (7.2 \pm 0.9) \text{ cm} = (7.2 \pm 0.9) \times 10^{-2} \text{ m}$$

4. Write down the value of the effective total spring constant k of the two-spring system. (2.0 points)

$$k = 9 \text{ N/m}$$

(Explanation) The effective total spring constant is

$$k \equiv \frac{mg}{\Delta l} = \frac{(62 \pm 2) \times 980}{7.2 \pm 0.9} = 9000 \pm 1000 \text{ dyne/cm} \quad \text{or} \quad 9 \pm 1 \text{ N/m}.$$

5. Obtain the respective values of k_1 and k_2 . Write down their values. (1.0 point)

$$k_1 = 5.7 \text{ N/m}$$

$$k_2 = 3 \text{ N/m}$$

(Explanation) When the MBB is in equilibrium on a horizontal plane the force balance condition for the ball is that

$$\frac{L/2 - l - \delta - r}{L/2 + l - \delta - r} = \frac{N_1}{N_2} = \frac{k_2}{k_1}.$$

Since $k = k_1 + k_2$, we get

$$k_1 = \frac{k}{\frac{L/2 - l - \delta - r}{L/2 + l - \delta - r} + 1} = \frac{L/2 + l - \delta - r}{L - 2\delta - 2r} k$$

and

$$k_2 = k - k_1 = \frac{L/2 - l - \delta - r}{L - 2\delta - 2r} k.$$

From the measured or given values

$$\frac{L/2 + l - \delta - r}{L - 2\delta - 2r} = \frac{(20.0 \pm 0.03) + \left(\frac{296 \pm 8}{62 \pm 2} \right) - 0.5 - 1.1}{(40.0 \pm 0.05) - 1.0 - 2.2} = 0.63 \pm 0.005.$$

Therefore,

$$k_1 = (0.63 \pm 0.005) \times (9000 \pm 1000) = 5700 \pm 600 \text{ dyne/cm} \quad \text{or} \quad 5.7 \pm 0.6 \text{ N/m},$$

and

$$k_2 = (9000 \pm 1000) - (5700 \pm 600) = 3000 \pm 1000 \text{ dyne/cm} \quad \text{or} \quad 3 \pm 1 \text{ N/m}.$$