Theoretical Question 1

Gravitational Red Shift and the Measurement of Stellar Mass

(a) (3 marks)
A photon of frequency f possesses an effective inertial mass m determined by its energy. Assume that it has a gravitational mass equal to this inertial mass. Accordingly, a photon emitted at the surface of a star will lose energy when it escapes from the star's gravitational field. Show that the frequency shift Δf of the photon when it escapes from the surface of the star to infinity is given by

$$\frac{\Delta f}{f} \simeq -\frac{GM}{Rc^2}$$

for $\Delta f \ll f$ where:

- $G =$ gravitational constant
- $R =$ radius of the star
- $c =$ velocity of light
- $M =$ mass of the star.

Thus, the red-shift of a known spectral line measured a long way from the star can be used to measure the ratio M/R. Knowledge of R will allow the mass of the star to be determined.

(b) (12 marks)
An unmanned spacecraft is launched in an experiment to measure both the mass M and radius R of a star in our galaxy. Photons are emitted from He^+ ions on the surface of the star. These photons can be monitored through resonant absorption by He^+ ions contained in a test chamber in the spacecraft. Resonant absorption accors only if the He^+ ions are given a velocity towards the star to allow exactly for the red shifts.

As the spacecraft approaches the star radially, the velocity relative to the star ($v = \beta c$) of the He^+ ions in the test chamber at absorption resonance is measured as a function of the distance d from the (nearest) surface of the star. The experimental data are displayed in the accompanying table.

Fully utilize the data to determine graphically the mass M and radius R of the star. There is no need to estimate the uncertainties in your answer.

Data for Resonance Condition

Velocity parameter	$\beta = v/c$ $(\times 10^{-5})$	3.352	3.279	3.195	3.077	2.955
Distance from surface of star	d $(\times 10^8 \text{m})$	38.90	19.98	13.32	8.99	6.67

(c) (5 marks)
In order to determine R and M in such an experiment, it is usual to consider the frequency correction due to the recoil of the emitting atom. [Thermal motion causes emission lines to be broadened without displacing emission maxima, and we may therefore assume that all thermal effects have been taken into account.]

(i) (4 marks)
Assume that the atom decays at rest, producing a photon and a recoiling atom. Obtain the relativistic expression for the energy hf of a photon emitted in terms of ΔE (the difference in rest energy between the two atomic levels) and the initial rest mass m_0 of the atom.

(ii) (1 mark)
Hence make a numerical estimate of the relativistic frequency shift $\left(\frac{\Delta f}{f}\right)_{\text{recoil}}$ for the case of He^+ ions.

Your answer should turn out to be much smaller than the gravitational red shift obtained in part (b).

Data:

Velocity of light	c	$=$	$3.0 \times 10^8 \mathrm{ms}^{-1}$
Rest energy of He	m_0c^2	$=$	$4 \times 938 (\mathrm{MeV})$
Bohr energy	E_n	$=$	$-\frac{13.6Z^2}{n^2} (\mathrm{eV})$
Gravitational constant	G	$=$	$6.7 \times 10^{-11} \mathrm{Nm}^2\mathrm{kg}^{-2}$

Theoretical Question 2

Sound Propagation

Introduction

The speed of propagation of sound in the ocean varies with depth, temperature and salinity. Figure 1(a) below shows the variation of sound speed c with depth z for a case where a minimum speed value c_0 occurs midway betweeen the ocean surface and the sea bed. Note that for convenience $z = 0$ at the depth of this sound speed minimum, $z = z_S$ at the surface and $z = -z_b$ at the sea bed. Above $z = 0$, c is given by

$$c = c_0 + bz \quad .$$

Below $z = 0$, c is given by

$$c = c_0 - bz \quad .$$

In each case $b = \left|\frac{dc}{dz}\right|$, that is, b is the magnitude of the sound speed gradient with depth; b is assumed constant.

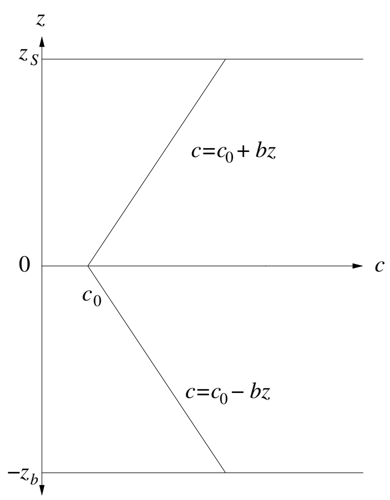


Figure 1 (a)

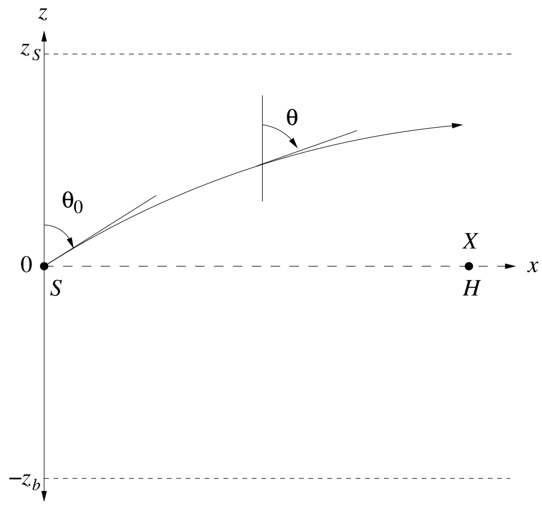


Figure 1 (b)

Figure 1(b) shows a section of the z–x plane through the ocean, where x is a horizontal direction. The variation of c with respect to z is shown in figure 1(a). At the position $z = 0$, $x = 0$, a sound source S is located. A 'sound ray' is emitted from S at an angle θ_0 as shown. Because of the variation of c with z, the ray will be refracted.

(a) (6 marks)
Show that the trajectory of the ray, leaving the source S and constrained to the z–x plane forms an arc of a circle with radius R where

$$R = \frac{c_0}{b \sin\theta_0} \quad \text{for} \quad 0 \le \theta_0 < \frac{\pi}{2} \quad .$$

(b) (3 marks)
Derive an expression involving z_S, c_0 and b to give the smallest value of the angle θ_0 for upwardly directed rays which can be transmitted without the sound wave reflecting from the sea surface.

(c) (4 marks)
Figure 1(b) shows the position of a sound receiver H which is located at the position $z = 0$, $x = X$. Derive an expression involving b, X and c_0 to give the series of angles θ_0 required for the sound ray emerging from S to reach the receiver H. Assume that z_S and z_b are sufficiently large to remove the possibility of reflection from sea surface or sea bed.

(d) (2 marks)
Calculate the smallest four values of θ_0 for refracted rays from S to reach H when

- $X = 10000$ m
- $c_0 = 1500$ ms^{-1}
- $b = 0.02000$ s^{-1}

(e) (5 marks)
Derive an expression to give the time taken for sound to travel from S to H following the ray path associated with the **smallest** value of angle θ_0, as determined in part (c). Calculate the value of this transit time for the conditions given in part (d). The following result may be of assistance:

$$\int \frac{dx}{\sin x} = \ln \tan\left(\frac{x}{2}\right)$$

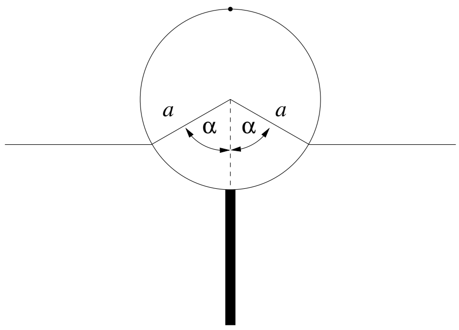
Calculate the time taken for the direct ray to travel from S to H along $z = 0$. Which of the two rys will arrive first, the ray for which $\theta_0 = \pi/2$, or the ray with the smallest value of θ_0 as calculated for part (d)?

Theoretical Question 3

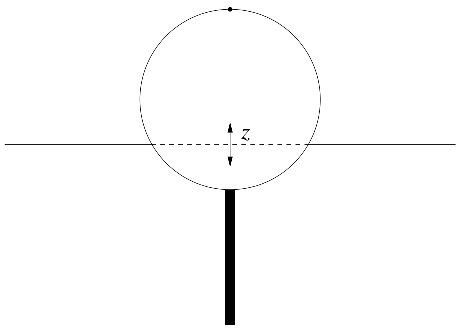
Cylindrical Buoy

(a) (3 marks)
A buoy consists of a solid cylinder, radius a, length l, made of lightweight material of uniform density d with a uniform rigid rod protruding directly outwards from the bottom halfway along the length. The mass of the rod is equal to that of the cylinder, its length is the same as the diameter of the cylinder and the density of the rod is greater than that of seawater. This buoy is floating in sea-water of density ρ.

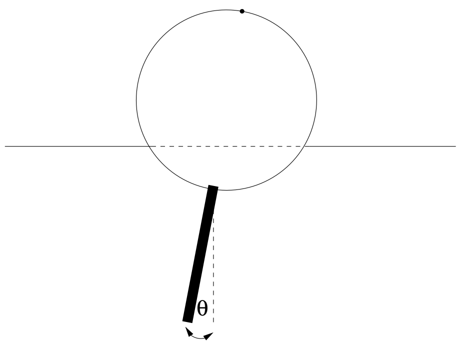
In equilibrium derive an expression relating the floating angle α, as drawn, to d/ρ. Neglect the volume of the rod.



(b) (4 marks)
If the buoy, due to some perturbation, is depressed vertically by a small amount z, it will experience a nett force, which will cause it to begin oscillating vertically about the equilibtium floating position. Determine the frequencty of this vertical mode of vibration in terms of α, g and a, where g is the acceleration due to gravity. Assume the influence of water motion on the dynamics of the buoy is such as to increase the effective mass of the buoy by a factor of one third. You may assume that α is not small.



(c) (8 marks)
In the approximation that the cylinder swings about its horizontal central axis, determine the frequency of swing again in terms of g and a. Neglect the dynamics and viscosity of the water in this case. The angle of swing is assumed to be small.



(d) (5 marks)
The buoy contains sensitive acelerometers which can measure the vertical and swinging motions and can relay this information by radio to shore. In relatively calm waters it is recorded that the vertical oscillation period is about 1 second and the swinging oscillation period is about 1.5 seconds. From this information, show that the floating angle α is about 90° and thereby estimate the radius of the buoy and its total mass, given that the cylinder length l equals a.

[You may take it that $\rho \simeq 1000$ kgm^{-3} and $g \simeq 9.8$ ms^{-2}.]

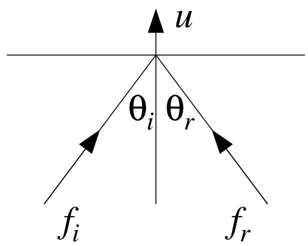
Original Theoretical Question 3

The following question was not used in the XXVI IPhO examination.

Laser and Mirror

(a)

Light of frequency f_i and speed c is directed at an angle of incidence θ_i to the normal of a mirror, which is receding at speed u in the direction of the normal. Assuming the photons in the light beam undergo an elastic collision *in the rest frame of the mirror*, determine in terms of θ_i and u/c the angle of reflection θ_r of the light and the reflected frequency f_r, with respect to the original frame.



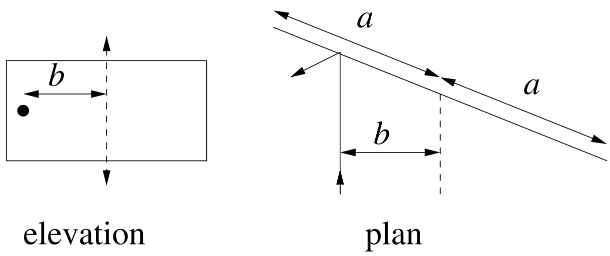
[You may assume the following Lorentz transformation rules apply to a particle with energy E and momentum $\mathbf{p}$:

$$p_\perp = p_\perp \ , \quad p_\| = \frac{p_\| - vE/c^2}{\sqrt{1 - v^2/c^2}} \ , \quad E = \frac{E - vp_\|}{\sqrt{1 - v^2/c^2}} \ ,$$

where $\mathbf{v}$ is the relative velocity between the two inertial frames; p stands for the component of momentum perpendicular to $\mathbf{v}$ and p represents the component of momentum parallel to $\mathbf{v}$.]

(b)

A thin rectangular light mirror, perfectly reflecting on each side, of width $2a$ and mass m, is mounted in a vacuum (to eliminate air resistance), on essentially frictionless needle bearings, so that it can rotate about a vertical axis. A narrow laser beam operating continuously with power P is incident on the mirror, at a distance b from the axis, as drawn.



Suppose the mirror is originally at rest. The impact of the light causes the mirror to acquire a very small but not constant angular acceleration. To analyse the siuation approximately, assume that at any given stage in the acceleration process the angular velocity ω of the mirror is constant throughout any one complete revolution, but takes on a slightly larger value in the next revolution due to the angular momentum imparted to the mirror by the light during the preceding revolution. Ignoring second order terms in the ratio (mirror velocity / c), calculate this increment of angular momentum per revolution at any given value of ω. [HINT: You may find it useful to know that $\int \sec^2\theta \ d\theta = \tan\theta$.]

(c)

Using the fact that the velocity of recoil of the mirror remains small compared with c, derive an approximate expression for ω as a function of time.

(d)

As the mirror rotates, there will be instants when the light is reflected from its edge, giving the reflected ray an angle of somewhat more than 90° with respect to the incident beam.. A screen 10 km away, with its normal perpendicular to the incident beam, intercepts the beam reflected from near the mirror's edge. Find the deviation ξ of that extreme spot from its initial position (as shown by the dashed line, when the mirror was almost at rest), after the laser has operated for 24 hours. You may suppose the laser power is $P = 100$ W, that the mirror has mass $m = 1$ gram and that the geometry of the apparatus corresponds to $a = b\sqrt{2}$. Neglect diffraction effects at the edge.