



- II.1. Consider a shell of width dr which is at the distance r from the centre of the star. Let ρ be the density of star. Gravitational Potential energy of shell is

$$dE_G = -G \frac{(4\pi r^3 \rho / 3)(4\pi r^2 dr \rho)}{r} \quad \boxed{0.5 \text{ mark}}$$

$$E_G = \int_0^R dE_G = -\frac{3}{5} \frac{GM^2}{R} \quad \boxed{0.3 \text{ mark}}$$

It is negative, i.e. gravitational force is radially inward. $\boxed{0.2 \text{ mark}}$

- II.2. Total energy of the star $E = E_G + E_e$. $\boxed{0.2 \text{ mark}}$
At equilibrium, at $R = R_{wd}$

$$\begin{aligned} \frac{dE}{dR} &= 0 & \boxed{0.8 \text{ mark}} \\ \frac{dE_G}{dR} &= -\frac{dE_e}{dR} \\ \frac{3}{5} \frac{GM^2}{R_{wd}^2} &= \frac{\hbar^2 \pi^3}{10 m_e 4^{2/3}} \left(\frac{3}{\pi} \right)^{7/3} \frac{2 N_e^{5/3}}{R_{wd}^3} \\ R_{wd} &= \frac{\hbar^2 \pi^3}{6 G m_e 4^{2/3}} \left(\frac{3}{\pi} \right)^{7/3} \frac{2 N_e^{5/3}}{M^2} & \boxed{1.0 \text{ marks}} \end{aligned}$$

- II.3. Since all the hydrogen is ionized, the number of protons (N_p) = N_e . Also $m_p \gg m_e$. Hence

$$\begin{aligned} N_e = N_p &\approx \frac{M}{m_p} & \boxed{0.5 \text{ mark}} \\ R_{wd} &= \frac{\hbar^2 \pi^3}{6 G m_e 4^{2/3}} \left(\frac{3}{\pi} \right)^{7/3} \frac{2}{M^{1/3} m_p^{5/3}} \\ R_{wd} &= 2.28 \times 10^4 \text{ km} & \boxed{1.0 \text{ mark}} \end{aligned}$$

- II.4. If r_{sep} is average separation between electrons then

$$\begin{aligned} N_e \times \frac{4}{3} \pi r_{sep}^3 &\approx \frac{4}{3} \pi R_{wd}^3 & \boxed{0.2 \text{ mark}} \\ r_{sep}^3 &= \frac{R_{wd}^3}{N_e} \approx R_{wd}^3 \frac{m_p}{M} & \boxed{0.2 \text{ mark}} \\ r_{sep} &= 2.13 \times 10^{-12} \text{ m} & \boxed{0.6 \text{ mark}} \end{aligned}$$

- II.5. For a particle confined in a box of length r_{sep} , its de Broglie wavelength λ_{dB} for the



ground state can be written as

$$\lambda_{dB} = 2r_{sep} \quad \boxed{0.2 \text{ mark}}$$

$$\text{and momentum } p = \frac{h}{\lambda_{dB}} \quad \boxed{0.3 \text{ mark}}$$

$$v \approx \frac{h}{2m_e r_{sep}} \quad \boxed{0.2 \text{ mark}}$$

$$= 1.08 \times 10^8 \text{ m.s}^{-1} \quad \boxed{0.3 \text{ mark}}$$

Correction: If one takes relativistic momentum

$$\lambda_{dB} = 2r_{sep} \quad \boxed{0.2 \text{ mark}}$$

$$p = \frac{h}{\lambda_{dB}} = \frac{h}{2r_{sep}} = \frac{m_e v}{\sqrt{1 - v^2/c^2}} \quad \boxed{0.3 \text{ mark}}$$

$$v = \frac{h}{\sqrt{4m_e^2 r^2 + \frac{h^2}{c^2}}} \quad \boxed{0.2 \text{ mark}}$$

$$= 1.06 \times 10^8 \text{ m.s}^{-1} \quad \boxed{0.3 \text{ mark}}$$

II.6. Similar to part II.2, at equilibrium

$$\frac{dE_G}{dR} = -\frac{dE_e^{rel}}{dR} \quad \boxed{0.3 \text{ mark}}$$

$$\frac{3}{5} \frac{GM^2}{R^2} = \frac{\pi^2}{4^{4/3}} \left(\frac{3}{\pi}\right)^{5/3} \frac{\hbar c}{R^2} N_e^{4/3} \quad \boxed{0.4 \text{ mark}}$$

For critical mass

$$M_c = \frac{3(5^3 \pi)^{1/2}}{16m_p^2} \left(\frac{\hbar c}{G}\right)^{3/2} \quad \boxed{0.8 \text{ mark}}$$

Alternatively: Since the total energy

$$E_G + E_e^{rel} = \left(-\frac{3}{5} GM^2 + \frac{\pi^2}{4^{2/3}} \left(\frac{3}{\pi}\right)^{5/3} \hbar c N_e^{4/3} \right) \frac{1}{R}$$

must be minimized for equilibrium, one can argue that if coefficient of $1/R$ is positive then star would collapse otherwise it would expand.

II.7. For $M > M_c$

Expand	
Contract	✓

0.5 mark

II.8.

$$M_c = 1.36 \times 10^{31} \text{ kg} \quad \boxed{1.0 \text{ mark}}$$

$$= 6.8 M_S \quad \boxed{0.5 \text{ mark}}$$