Theoretical Question 3: Birthday Balloon

SOLUTION

a. Solution using forces:

Let the balloon's radius be r, and let P be the pressure of the inside air. Consider the balloon's rear half, and write down the equilibrium of forces on it along the cylinder's axis:

$$\pi r^2 (P - P_0) = 2\pi r \sigma_L$$

On the other hand, let us cut the balloon in half with a plane that runs along its axis, and consider a half-cylindrical section of length x. The equilibrium of forces in perpendicular to the cutting plane reads:

$$2rx(P - P_0) = 2x\sigma_t$$

from which we derive $\sigma_L/\sigma_t = 1/2$.

Solution using energies:

If we stretch the balloon longitudinally by length dL, the energy cost is:

$$E_1 = 2\pi r \sigma_L \cdot dL$$

If we inflate the balloon radially with an increment dr, the energy cost is:

$$E_2 = L\sigma_t \cdot 2\pi dr$$

The two deformations can be combined while keeping the volume fixed, if we take $\pi r^2 dL = -Ld(\pi r^2) = -2\pi L r dr$, i.e. $rdL = -2Ldr$. The equilibrium state is the one where the combined energy cost $E_1 + E_2$ of such a deformation is zero. This gives again the result $\sigma_L/\sigma_t = 1/2$.

b. From part (a), we are reminded of the relation between surface tension and pressure:

$$P = P_0 + \frac{\sigma_t}{r} = P_0 + \frac{k(r - r_0)}{r_0 r} = P_0 + k\left(\frac{1}{r_0} - \frac{1}{r}\right)$$

The volume is related to the radius by:

$$V = \pi r^2 L_0$$

So we get:

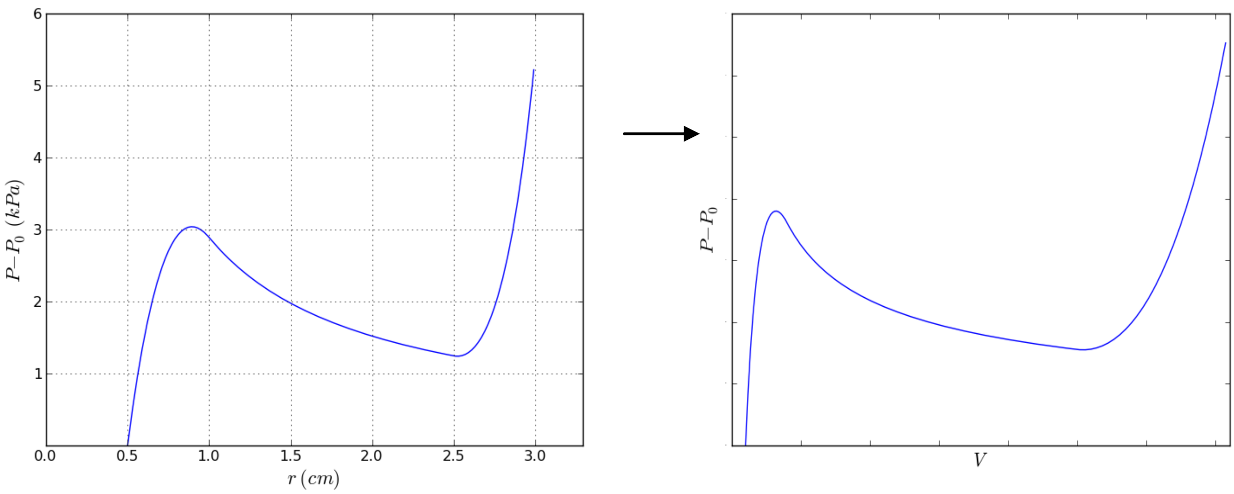
$$P(V) = P_0 + k\left(\frac{1}{r_0} - \sqrt{\frac{\pi L_0}{V}}\right)$$

The graph of $P - P_0$ is a hyperbola-like function increasing from 0 at $V = \pi r_0^2 L_0$ to an asymptotic value of k/r_0 at $V \to \infty$.

The maximal pressure is obtained at $V \to \infty$:

$$P_{max} = P_0 + \frac{k}{r_0}$$

c. The graph of $P - P_0$ as a function of V has the same qualitative form as $P - P_0 = \sigma_t / r$ as a function of r, shown below. The graph rises from zero, then decreases, and then increases again. The points $r = 1\text{cm}$ and $r = 2.5\text{cm}$ lie in the decreasing portion (and not on the local extrema).



The pressures at the two requested points are approximately given by:

$$P - P_0(r = 1\text{cm}) = \frac{\sigma}{r} = \frac{30}{0.01} = 3000\text{Pa}; \qquad P - P_0(r = 2.5\text{cm}) = \frac{30}{0.025} = 1200\text{Pa}$$

d. The work done on the pressure-controlling mechanism during continuous inflation from volume V_i to volume V_f is:

$$W_{mech} = -P(V_f - V_i)$$

The work done on the atmosphere is:

$$W_{surr} = P_0(V_f - V_i)$$

The condition for the jump is:

$$W_{rubber} + W_{surr} + W_{mech} = 0$$

This translates into Maxwell's equal-areas condition:

$$\int_{V_i}^{V_f} (P - P_0) dV = (P - P_0)\left(V_f - V_i\right)$$

Or, equivalently:

$$\int_{V_i}^{V_f} PdV = P(V_f - V_i)$$

The cubic function $P(V)$ is symmetric around the point $V = u$, $P - P_0 = ac$.

The equal-areas condition is therefore satisfied at:

$$P_c = P_0 + ac$$

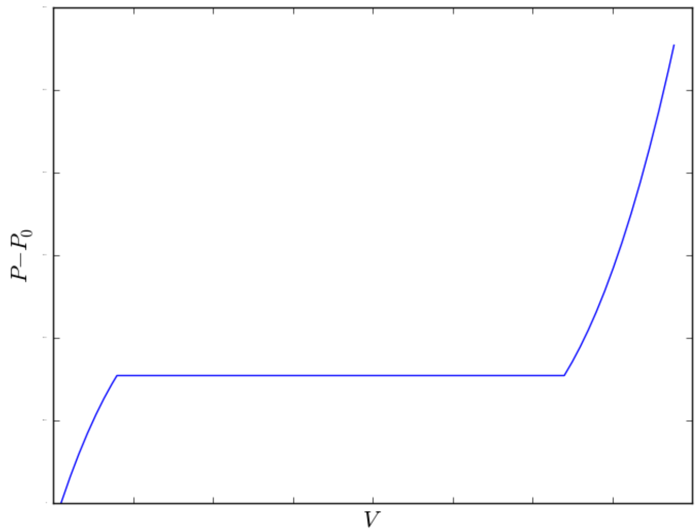
The volumes V_1 and V_2 are given by the points where:

$$(V-u)^3 - b(V-u) = 0$$

This gives:

$$V_{1,2} = u \pm \sqrt{b}$$

e. The range of volumes where a phase separation will occur is $V_1 < V < V_2$. The pressure is constant throughout this range, and equals the transition pressure P_c. The graph of $P - P_0$ as a function of V is monotonous, with a rising piece, a horizontal plateau at $V_1 < V < V_2$, $P = P_c$, followed by another rising piece. At the start and end of the plateau, the slope has a discontinuity, i.e. the graph has a kink.



f. The radii of the two domains correspond to the volumes V_1 and V_2. As the total volume increases from V_1 to V_2, the volume of the thin domain changes linearly from V_1 to 0. We get:

$$V_{thin} = \frac{V_1}{V_2 - V_1}(V_2 - V)$$

Converting this into length, we have:

$$L_{thin} = \frac{V_{thin}}{\pi r_1^2} = \frac{V_1(V_2 - V)}{\pi r_1^2(V_2 - V_1)}$$

g. The increase in the balloon's volume as a result of converting a length L_{thin} into the thick phase is:

$$\Delta V = \frac{V_2 - V_1}{V_1}\Delta V_{thin} = \frac{\pi r_1^2(V_2 - V_1)}{V_1}\Delta L_{thin}$$

The corresponding work is:

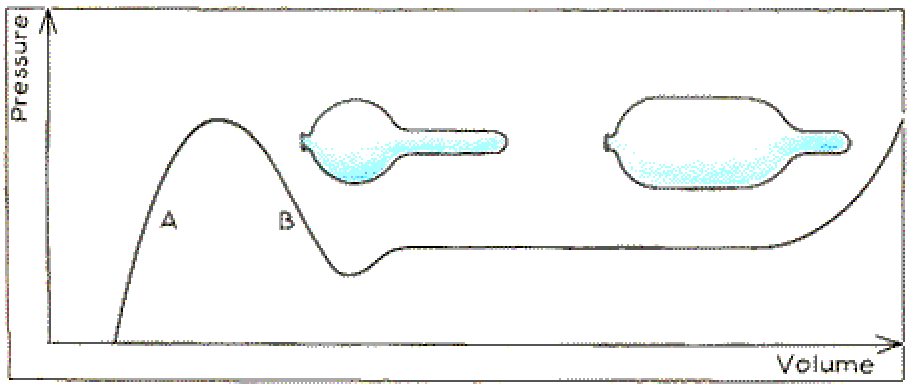
$$\Delta W = P_c \Delta V = \frac{\pi r_1^2 P_c(V_2 - V_1)}{V_1}\Delta L_{thin}$$

Therefore:

$$\frac{\Delta W}{\Delta L_{thin}} = \frac{\pi r_1^2 P_c(V_2 - V_1)}{V_1}$$

Additional discussion (doesn't appear as part of the question):

During a realistic inflation, perturbations are not strong enough to keep the system in global equilibrium at all times. The experimental graph increases up to P_c, continues to increase some way beyond it, reaches a local maximum, then decreases and settles on the plateau at P_c. This over-increase of the pressure is responsible for the fact that inflating a balloon is difficult during the first few puffs. After the plateau, the graph sharply increases as discussed above. The decrease towards the plateau "overshoots" slightly again, reaches a local minimum and rises again to settle on the plateau. This behavior is depicted in the graph below.



The illustration is taken from:
http://www.science-project.com/_members/science-projects/1989/12/1989-12-body.html