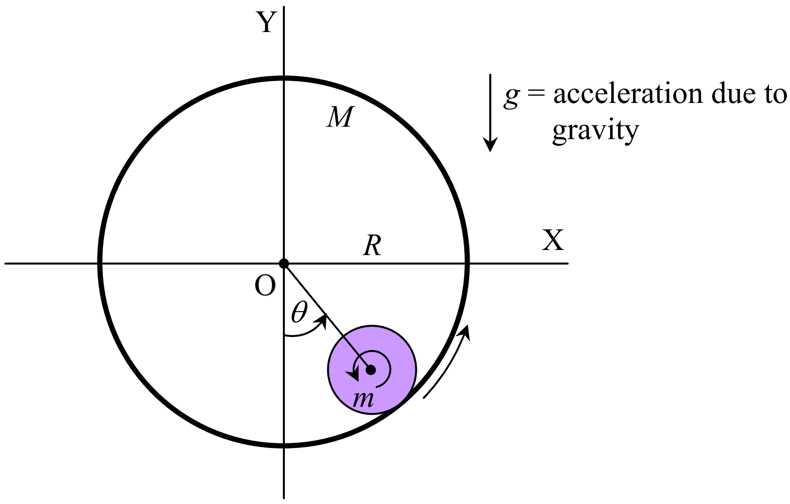
Rolling Cylinders

A thin-walled cylinder of mass M and rough inner surface of radius R can rotate about its fixed central horizontal axis OZ. The Z-axis is perpendicular to and out of the page. Another smaller uniform solid cylinder of mass m and radius r rolls without slipping (except for question 1.8) on the inner surface of M about its own central axis which is parallel to OZ.



1.1) The rotation of M is to be started from rest at the instant $t = 0$ when m is resting at the lowest point. At a later time t the angular position of the centre of mass of m is θ and by then M has turned through an angle ϕ radians. How many radians (designated ψ) would have mass m turned through about its central axis relative to a fixed line (for example, the negative Y-axis)? Give your answer in terms of θ, ϕ, R and r. (0.8 point)

1.2) What is the angular acceleration of m, $\frac{d^2}{dt^2}\psi$, about its own axis through its centre of mass? Give your answer in terms of $R, r,$ and derivatives of θ and ϕ. (0.2 point)

1.3) Derive an equation for the angular acceleration of the centre of mass of m, $\frac{d^2}{dt^2}\theta$, in terms of $m, g, R, r, \theta, \frac{d^2}{dt^2}\phi$, and the moment of inertia I_{CM} of m about its central axis. (1.8 points)

1.4) What is the period of small amplitude oscillation of m when M is constrained to rotate at a constant angular velocity? Give your answer only in terms of $R, r,$ and g. (1.3 point)

1.5) What is the value of θ for the equilibrium position of m in question 1.4? (0.2 point)

1.6) What is the equilibrium position of m when M is rotating with a constant angular acceleration α? Give your answer in terms of $R, g,$ and α. (0.7 point)

1.7) Now M is allowed to rotate (oscillate) freely, without constraint, about its central axis OZ while m is executing a small-amplitude oscillation by pure rolling on the inner surface of M. Find the period of this oscillation. (2.5 points)

1.8) Consider the situation in which M is rotating steadily at an angular velocity Ω and m is rotating (rolling) about its stationary centre of mass, at the equilibrium position found in question 1.5. M is then brought abruptly to a halt. What must be the lowest value of Ω such that m will roll up and reach the highest point of the cylindrical surface of M? The coefficient of friction between m and M is assumed to be sufficiently high that m begins to roll without slipping soon after a short skidding right after M is stopped. (2.5 points)

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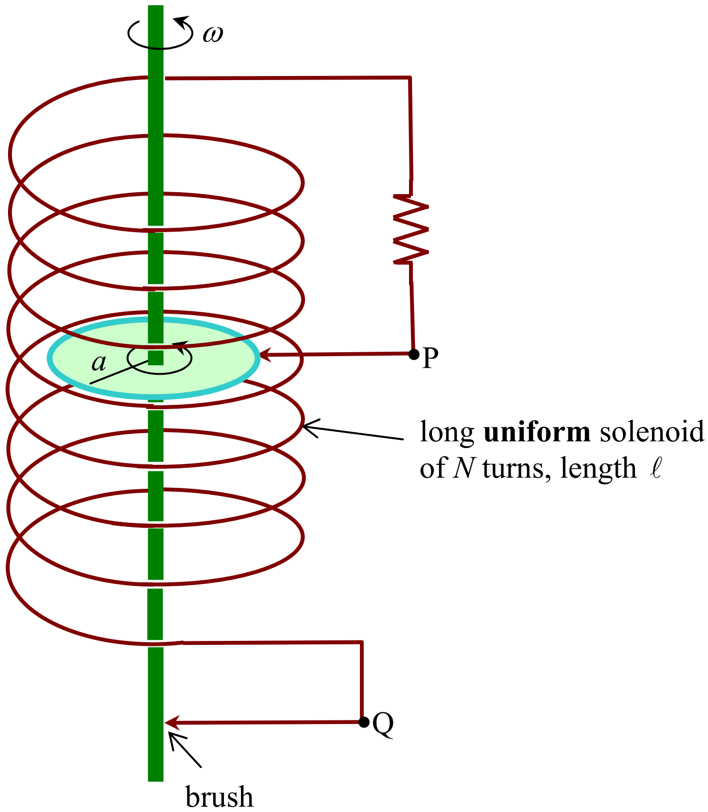
A Self-excited Magnetic Dynamo

A metallic disc of radius a mounted on a slender axle is rotating with a constant angular velocity ω inside a long solenoid of inductance L whose two ends are connected to the rotating disc by two brush contacts as shown. The total resistance of the whole circuit is R. A small magnetic disturbance can initiate the growth of an induced electromotive force across the terminals P, Q.



2.1) Write down the differential equation for $i(t)$, the current through the circuit. Express your answer in terms of $L, R,$ and the induced e.m.f. $(\mathcal{E})$ across the terminals P and Q. (1.0 point)

2.2) What is the value of the magnetic flux density (B) in terms of $i, N, \ell,$ and the permeability of free space μ_0? Ignore the magnetic field generated by the disc and the axle. (1.5 points)

2.3) What is the expression for the induced e.m.f. $(\mathcal{E})$ in terms of $\mu_0, N, a, \ell, i,$ and the angular velocity ω? (2.0 points)

2.4) Solve the equation in question 2.1 for current at any time t in terms of the initial current $i(0)$, and other parameters. (1.5 points)

2.5) What is the minimum value of the angular velocity that will permit the current to grow? Give your answers in terms of $R, \mu_0, N, a,$ and ℓ. (2.0 points)

2.6) In order to maintain a certain steady angular velocity ω, what must be the value of torque applied to the axle at the instant t? (2.0 points)

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The Leidenfrost Phenomenon

The purpose is to estimate the lifetime of a (hemispherical) drop of a liquid sitting on top of a very thin layer of vapour which is thermally insulating the drop from the very hot plate below.

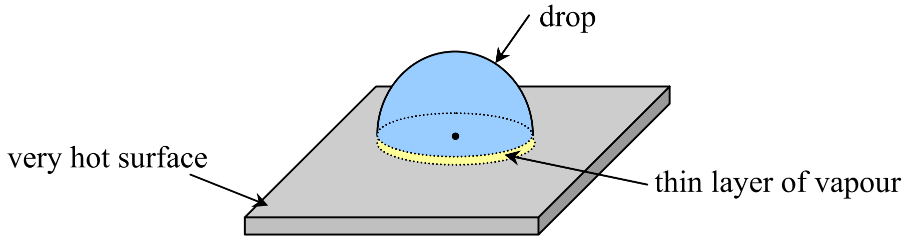

Figure 1

It will be assumed here that the flow of vapour underneath the drop is streamline and behaves as a Newtonian fluid of viscosity coefficient η and of thermal conductivity $\mathcal{K}$. The specific latent heat of vaporization of the liquid is ℓ. And for a Newtonian fluid we have the shear stress $\frac{F}{A} = \eta \times$ the rate of shear $\frac{dv}{dz}$ where v is the flow velocity and z is the perpendicular distance to the direction of flow, and the direction of F is tangential to the surface area A.

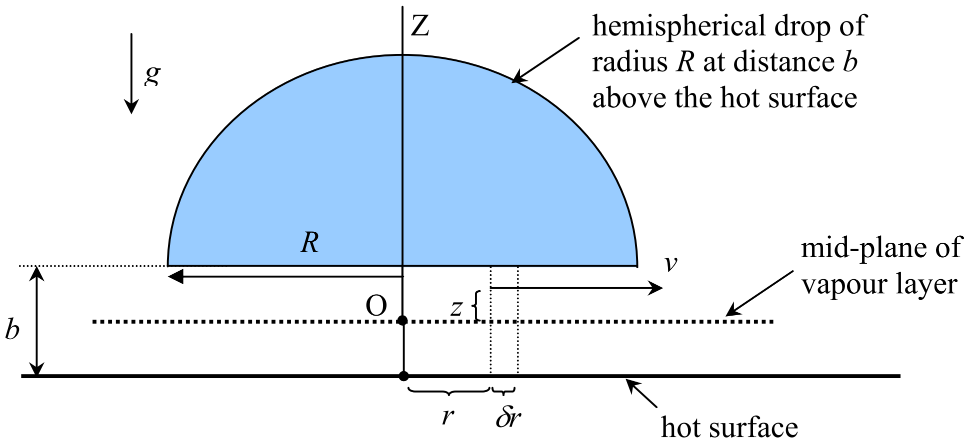

Figure 2

v is the velocity of vapour in the radial direction at the height z above the mid-plane. The pressure P inside the vapour must be higher towards the centre O. This will result in the out-flowing of vapour and in force that holds the drop against the pull of gravity. The thickness of vapour layer under thermal and mechanical equilibria is b.

For a Newtonian flow of vapour we can approximate that

$$\frac{d}{dz}v = \frac{z}{\eta}\frac{d}{dr}P$$

3.1) Show that $$v(z) = \frac{z^2}{2\eta}\frac{d}{dr}P + C$$

where C is an arbitrary constant of integration. (0.5 point)

3.2) Refer to figure 2, find the value of C in terms of $\eta, \frac{d}{dr}P$, and b using the boundary condition $v = 0$ for $z = \pm\frac{b}{2}$. (0.5 point)

3.3) Calculate the volume rate of flow of vapour through the cylindrical surface defined by r. (Hint: the cylinder is of radius r and of height b underneath the drop). (1.0 point)

3.4) By assuming that the rate of production of vapour of density ρ_V is due to heat flow from the hot surface to the drop, find the expression for the pressure $P(r)$. Use P_a to represent the atmospheric pressure, and use ΔT for the temperature difference between those of the hot surface and of the drop. Assume that the system has reached the steady state. (2.0 points)

3.5) Calculate the value of b by equating the weight of the drop to the net force due to pressure difference between the bottom and the top of the drop. The density of the drop is ρ_0. (2.0 points)

3.6) Now, what is the total rate of vaporization? (2.0 points)

3.7) Assume that the drop maintains a hemispherical shape, what is the life-time of the drop? (2.0 points)

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