

[Solution]

Theoretical Question 2

Motion of an Electric Dipole in a Magnetic Field

(1) Conservation Laws

$$(1a) \quad \vec{r}_{CM} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2), \quad \vec{v}_{CM} = \frac{1}{2}(\vec{v}_1 + \vec{v}_2), \quad \vec{\ell} = \vec{r}_1 - \vec{r}_2, \quad \vec{u} = \dot{\vec{\ell}} = \vec{v}_1 - \vec{v}_2$$

Total force \vec{F} on the dipole is

$$\begin{aligned} \vec{F} &= \vec{F}_1 + \vec{F}_2 = q(\vec{E} + \vec{v}_1 \times \vec{B}) + (-q)(\vec{E} + \vec{v}_2 \times \vec{B}) = q(\vec{v}_1 - \vec{v}_2) \times \vec{B} \\ &= q\dot{\vec{\ell}} \times \vec{B} \end{aligned}$$

so that

$$M\dot{\vec{v}}_{CM} = q\dot{\vec{\ell}} \times \vec{B} \quad (M = 2m) \quad (1)$$

Computing the torque for rotation around the center of mass, we obtain

$$\begin{aligned} I\dot{\vec{\omega}} &= \left(\frac{\vec{\ell}}{2}\right) \times (q\vec{v}_1 \times \vec{B}) + \left(\frac{-\vec{\ell}}{2}\right) \times (-q\vec{v}_2 \times \vec{B}) \\ &= q\vec{\ell} \times (\vec{v}_{CM} \times \vec{B}) \end{aligned} \quad (2)$$

where

$$I = \frac{1}{2} m\ell^2 \quad (3)$$

(1b) From eq.(1), we obtain the conservation law for the momentum:

$$\dot{\vec{P}} = 0, \quad \vec{P} = M\vec{v}_{CM} - q\vec{\ell} \times \vec{B} \quad (4)$$

From eq.(1) and eq.(2), one obtains the conservation law for the energy.

$$\dot{E} = 0, \quad E = \frac{1}{2} Mv_{CM}^2 + \frac{1}{2} I\omega^2 \quad (5)$$

(1c) Using eq.(4) and eq.(2),

$$\begin{aligned} \frac{d}{dt} (\vec{r}_{CM} \times \vec{P}) \cdot \hat{B} &= (\vec{v}_{CM} \times \vec{P}) \cdot \hat{B} = -q\vec{v}_{CM} \times (\vec{\ell} \times \vec{B}) \cdot \hat{B} \\ &= q(\vec{\ell} \times \vec{B}) \times \vec{v}_{CM} \cdot \hat{B} = q(\vec{\ell} \times \vec{B}) \cdot (\vec{v}_{CM} \times \hat{B}) \\ &= q\vec{\ell} \cdot (\vec{B} \times (\vec{v}_{CM} \times \hat{B})) = -q\vec{\ell} \cdot ((\vec{v}_{CM} \times \vec{B}) \times \hat{B}) \\ &= -q\vec{\ell} \times (\vec{v}_{CM} \times \vec{B}) \cdot \hat{B} \\ &= -I\dot{\vec{\omega}} \cdot \hat{B} \end{aligned}$$

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we obtain the conservation law

$$\dot{J}=0 \quad J = (\vec{r}_{CM} \times \vec{P} + I\vec{\omega}) \cdot \hat{B} \quad (6)$$

for the component of the angular momentum along the direction of \vec{B} .

(2) Motion in a Plane Perpendicular to \vec{B}

(2a) Write

$$\vec{\ell} = \ell \{ \cos \varphi(t) \hat{x} + \sin \varphi(t) \hat{y} \}, \quad \varphi(0) = 0, \quad \dot{\varphi}(0) = \omega_0 \quad (7)$$

Note that

$$\vec{\omega} = \dot{\varphi} \hat{z} \quad (8)$$

From eq.(4), we have

$$M\vec{v}_{CM} = \vec{P} + q\ell B (\sin \varphi \hat{x} - \cos \varphi \hat{y}) \quad (9)$$

At $t = 0$, we have $v_{CM} = 0$, $\varphi = 0$ so that

$$\vec{P} = q\ell B \hat{y} \quad (10)$$

Hence from eqs.(9) and (10) we have

$$\dot{x}_{CM} = \left(\frac{q\ell B}{M} \right) \sin \varphi, \quad \dot{y}_{CM} = \left(\frac{q\ell B}{M} \right) (1 - \cos \varphi) \quad (11)$$

From conservation of energy, i.e. Eq.(5), we have

$$\frac{1}{2} I \dot{\varphi}^2 + \frac{(q\ell B)^2}{M} (1 - \cos \varphi) = \frac{1}{2} I \omega_0^2$$

$$\therefore \dot{\varphi}^2 + \frac{1}{2} \omega_c^2 (1 - \cos \varphi) = \omega_0^2 \quad (12)$$

where

$$\omega_c^2 = \frac{4(q\ell B)^2}{MI} = \left(\frac{2qB}{m} \right)^2 \quad (13)$$

In order to make a full turn, $\dot{\varphi}$ can not become zero so that

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$$\omega_0^2 > \omega_c^2 \Rightarrow |\omega_0| > \omega_c = \frac{2qB}{m} \tag{14}$$

(2b) From Eq.(6), we have

$$x_{CM}P + I\omega = J \tag{15}$$

where P is the magnitude of \vec{P} .

At $t = 0$, we have $J = I\omega_0$ so that

$$x_{CM}P + I\omega = I\omega_0 \tag{16}$$

From eq.(12), one can see that $\omega_0^2 \geq \omega^2$ so that $x_{CM} \geq 0$. Thus x_{CM} reaches a maximum d_m when ω takes its minimum value.

When $\omega_0 < \omega_c$, the minimum value of ω is $-\omega_0$ so that

$$d_m = \frac{2I}{P}\omega_0 = \left(\frac{m\omega_0}{qB}\right)\ell, \quad \omega_0 < \omega_c \tag{17}$$

When $\omega_0 > \omega_c$, the minimum value of ω is $\sqrt{\omega_0^2 - \omega_c^2}$ so that

$$d_m = \left(\frac{I}{P}\right)\left(\omega_0 - \sqrt{\omega_0^2 - \omega_c^2}\right) = \frac{m}{2qB}\left(\omega_0 - \sqrt{\omega_0^2 - \omega_c^2}\right)\ell, \quad \omega_0 > \omega_c \tag{18}$$

When $\omega_0 = \omega_c$, $\omega^2 = \frac{1}{2}\omega_c^2(1 + \cos\phi) = \omega_c^2 \cos^2 \frac{\phi}{2}$

$$\therefore \phi = \omega_c \cos \frac{\phi}{2}$$

When ϕ is close to π , let $\phi = \pi - 2\varepsilon$ then

$$\dot{\varepsilon} = -\frac{1}{2}\omega_c \sin \varepsilon \approx -\frac{1}{2}\omega_c \varepsilon$$

$$\therefore \varepsilon \sim e^{-\omega_c t/2}$$

so that it will take $t \rightarrow \infty$ for $\varepsilon \rightarrow 0$, i.e. for ϕ to reach π . Hence

$$d_m = \left(\frac{I}{P}\right)\omega_c = \left(\frac{m\omega_c}{2qB}\right)\ell, \quad \omega_0 = \omega_c \tag{19}$$

(2c) Tension on the rod comes from three sources:

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$$(i) \text{ Coulomb force} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{\ell^2} \quad (20)$$

Positive value means compression on the rod.

$$(ii) \text{ Centrifugal force due to rotation of the rod} = -\frac{1}{2} m\omega^2 \ell \quad (21)$$

(iii) Magnetic force on the particles due to the motion of the center of the mass

$$= q\vec{v}_{CM} \times \vec{B} \cdot (-\hat{\ell}) = q\vec{v}_{CM} \cdot \hat{\ell} \times \vec{B}$$

Taking the square of both sides of eq.(4) and using the initial condition for the value of P^2 , we obtain

$$\frac{1}{2} Mv_{CM}^2 = q\ell\vec{v}_{CM} \cdot \hat{\ell} \times \vec{B} = \frac{1}{2} I(\omega_0^2 - \omega^2) \quad (22)$$

Combining the three forces, we have

$$\text{tension on the rod} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{\ell^2} - \frac{1}{2} m\ell\omega^2 + \frac{1}{4} m\ell(\omega_0^2 - \omega^2) \quad (23)$$

A positive value means compression on the rod.