SOLUTIONS MANUAL

Electricity and Magnetism

Third Edition

Edward M. Purcell and David J. Morin

TO THE INSTRUCTOR: I have tried to pay as much attention to detail in these exercise solutions as I did in the problem solutions in the text. But despite working through each solution numerous times during the various stages of completion, there are bound to be errors. So please let me know if anything looks amiss.

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In addition to any comments you have on these solutions, I welcome any comments on the book in general. I hope you're enjoying using it!

David Morin morin@physics.harvard.edu

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Chapter 1

Electrostatics

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

1.34. Aircraft carriers and specks of gold

The volume of a cube 1 mm on a side is 10^{-3} cm³. So the mass of this 1 mm cube is $1.93 \cdot 10^{-2}$ g. The number of atoms in the cube is therefore

$$6.02 \cdot 10^{23} \cdot \frac{1.93 \cdot 10^{-2} \,\mathrm{g}}{197 \,\mathrm{g}} = 5.9 \cdot 10^{19}. \tag{1}$$

Each atom has a positive charge of $1 e = 1.6 \cdot 10^{-19} \text{ C}$, so the total charge in the cube is $(5.9 \cdot 10^{19})(1.6 \cdot 10^{-19} \text{ C}) = 9.4 \text{ C}$. The repulsive force between two such cubes 1 m apart is therefore

$$F = k \frac{q^2}{r^2} = \left(9 \cdot 10^9 \,\frac{\text{kg m}^3}{\text{s}^2 \,\text{C}^2}\right) \frac{(9.4 \,\text{C})^2}{(1 \,\text{m})^2} = 8 \cdot 10^{11} \,\text{N}.$$
 (2)

The weight of an aircraft carrier is $mg = (10^8 \text{ kg})(9.8 \text{ m/s}^2) \approx 10^9 \text{ N}$. The above F is therefore equal to the weight of 800 aircraft carriers. This is just another example of the fact that the electrostatic force is enormously larger than the gravitational force.

1.35. Balancing the weight

Let the desired distance be d. We want the upward electric force $e^2/4\pi\epsilon_0 d^2$ to equal the downward gravitational force mg. Hence,

$$d^{2} = \frac{1}{4\pi\epsilon_{0}} \frac{e^{2}}{mg} = \left(9 \cdot 10^{9} \frac{\text{kg m}^{3}}{\text{s}^{2} \text{ C}^{2}}\right) \frac{(1.6 \cdot 10^{-19} \text{ C})^{2}}{(9 \cdot 10^{-31} \text{ kg})(9.8 \text{ m/s}^{2})} = 26 \text{ m}^{2},$$
(3)

which gives d = 5.1 m. The non-infinitesimal size of this answer is indicative of the feebleness of the gravitational force compared with the electric force. It takes about $3.6 \cdot 10^{51}$ nucleons (that's roughly how many are in the earth) to produce a gravitational force at an effective distance of $6.4 \cdot 10^6$ m (the radius of the earth) that cancels the electrical force from *one* proton at a distance of 5 m. The difference in these distances accounts for a factor of only $1.6 \cdot 10^{12}$ between the forces (the square of the ratio of the distances). So even if all the earth's mass were somehow located the same distance away from the electron as the single proton is, we would still need about $2 \cdot 10^{39}$ nucleons to produce the necessary gravitational force.

1.36. Repelling volley balls

Consider one of the balls. The vertical component of the tension in the string must equal the gravitational force on the ball. And the horizontal component must equal the electric force. The angle that the string makes with the horizontal is given by $\tan \theta = 10$, so we have

$$\frac{T_y}{T_x} = 10 \implies \frac{F_g}{F_e} = 10 \implies \frac{mg}{q^2/4\pi\epsilon_0 r^2} = 10.$$
(4)

Therefore,

$$q^{2} = \frac{1}{10} (4\pi\epsilon_{0}) mgr^{2} = (0.4)\pi \left(8.85 \cdot 10^{-12} \frac{\mathrm{s}^{2} \mathrm{C}^{2}}{\mathrm{kg \,m}^{3}} \right) (0.3 \,\mathrm{kg}) (9.8 \,\mathrm{m/s}^{2}) (0.5 \,\mathrm{m})^{2}$$

= $8.17 \cdot 10^{-12} \mathrm{C}^{2} \implies q = 2.9 \cdot 10^{-6} \mathrm{C}.$ (5)

1.37. Zero force at the corners

(a) Consider a charge q at a particular corner. If the square has side length ℓ , then one of the other q's is $\sqrt{2} \ell$ away, two of them are ℓ away, and the -Q is $\ell/\sqrt{2}$ away. The net force on the given q, which is directed along the diagonal touching it, is (ignoring the factors of $1/4\pi\epsilon_0$ since they will cancel)

$$F = \frac{q^2}{(\sqrt{2}\ell)^2} + 2\cos 45^\circ \frac{q^2}{\ell^2} - \frac{Qq}{(\ell/\sqrt{2})^2}.$$
 (6)

Setting this equal to zero gives

$$Q = \left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right)q = (0.957)q.$$
(7)

(b) To find the potential energy of the system, we must sum over all pairs of charges. Four pairs involve the charge -Q, four involve the edges of the square, and two involve the diagonals. The total potential energy is therefore

$$U = \frac{1}{4\pi\epsilon_0} \left(4 \cdot \frac{(-Q)q}{\ell/\sqrt{2}} + 4 \cdot \frac{q^2}{\ell} + 2 \cdot \frac{q^2}{\sqrt{2}\ell} \right) = \frac{4\sqrt{2}q}{4\pi\epsilon_0\ell} \left(-Q + \frac{q}{\sqrt{2}} + \frac{q}{4} \right) = 0, \quad (8)$$

in view of Eq. (7). The result in Problem 1.6 was "The total potential energy of any system of charges in equilibrium is zero." With Q given by Eq. (7), the system is in equilibrium (because along with all the q's, the force on the -Q charge is also zero, by symmetry). And consistent with Problem 1.6, the total potential energy is zero.

1.38. Oscillating on a line

If the charge q is at position (x, 0), then the force from the right charge Q equals $-Qq/4\pi\epsilon_0(\ell-x)^2$, where the minus sign indicates leftward. And the force from the left charge Q equals $Qq/4\pi\epsilon_0(\ell+x)^2$. The net force is therefore (dropping terms of

order x^2)

$$F(x) = -\frac{Qq}{4\pi\epsilon_0} \left(\frac{1}{(\ell - x)^2} - \frac{1}{(\ell + x)^2} \right) \\ \approx -\frac{Qq}{4\pi\epsilon_0\ell^2} \left(\frac{1}{1 - 2x/\ell} - \frac{1}{1 + 2x/\ell} \right) \\ \approx -\frac{Qq}{4\pi\epsilon_0\ell^2} \left((1 + 2x/\ell) - (1 - 2x/\ell) \right) \\ = -\frac{Qqx}{\pi\epsilon_0\ell^3}.$$
(9)

This is a Hooke's-law type force, being proportional to (negative) x. The F = ma equation for the charge q is

$$-\frac{Qqx}{\pi\epsilon_0\ell^3} = m\ddot{x} \implies \ddot{x} = -\left(\frac{Qq}{\pi\epsilon_0m\ell^3}\right)x.$$
 (10)

The frequency of small oscillations is the square root of the (negative of the) coefficient of x, as you can see by plugging in $x(t) = A \cos \omega t$. Therefore $\omega = \sqrt{Qq/\pi\epsilon_0 m\ell^3}$. This frequency increases with Q and q, and it decreases with m and ℓ ; these make sense. As far as the units go, $Qq/\epsilon_0\ell^2$ has the dimensions of force F (from looking at Coulomb's law), so ω has units of $\sqrt{F/m\ell}$. This correctly has units of inverse seconds.

ALTERNATIVELY: We can find the potential energy of the charge q at position (x, 0), and then take the (negative) derivative to find the force. The energy is a scalar, so we don't have to worry about directions. We have

$$U(x) = \frac{Qq}{4\pi\epsilon_0} \left(\frac{1}{\ell - x} + \frac{1}{\ell + x}\right).$$
(11)

We'll need to expand things to order x^2 because the order x terms will cancel:

$$U(x) = \frac{Qq}{4\pi\epsilon_0\ell} \left(\frac{1}{1-x/\ell} + \frac{1}{1+x/\ell} \right)$$

$$\approx \frac{Qq}{4\pi\epsilon_0\ell} \left(\left(1 + \frac{x}{\ell} + \frac{x^2}{\ell^2} \right) + \left(1 - \frac{x}{\ell} + \frac{x^2}{\ell^2} \right) \right)$$

$$= \frac{Qq}{4\pi\epsilon_0\ell} \left(2 + \frac{2x^2}{\ell^2} \right).$$
(12)

The constant term isn't important here, because only changes in the potential energy matter. Equivalently, the force is the negative derivative of the potential energy, and the derivative of a constant is zero. The force on the charge q is therefore

$$F(x) = -\frac{dU}{dx} = -\frac{Qqx}{\pi\epsilon_0 \ell^3},$$
(13)

in agreement with the force in Eq. (9).

1.39. Rhombus of charges

We'll do the balancing-the-forces solution first. Let the common length of the strings be ℓ . By symmetry, the tension T is the same in all of the strings. Each of the two charges q is in equilibrium if the sum of the vertical components of the electrostatic forces is equal and opposite to the sum of the vertical components of the tensions. This gives

$$2\left(\frac{qQ}{4\pi\epsilon_0\ell^2}\right)\sin\theta + \frac{q^2}{4\pi\epsilon_0(2\ell\sin\theta)^2} = 2T\sin\theta \implies \frac{q^2}{16\pi\epsilon_0\sin^3\theta} = 2T\ell^2 - \frac{qQ}{2\pi\epsilon_0}.$$
 (14)

Similarly, each charge Q is in equilibrium if

$$2\left(\frac{qQ}{4\pi\epsilon_0\ell^2}\right)\cos\theta + \frac{Q^2}{4\pi\epsilon_0(2\ell\cos\theta)^2} = 2T\cos\theta \implies \frac{Q^2}{16\pi\epsilon_0\cos^3\theta} = 2T\ell^2 - \frac{qQ}{2\pi\epsilon_0}.$$
(15)

The righthand sides of the two preceding equations are equal, so the same must be true of the lefthand sides. This yields $q^2/\sin^3\theta = Q^2/\cos^3\theta$, or $q^2/Q^2 = \tan^3\theta$, as desired.

Some limits: If $Q \gg q$, then $\theta \to 0$. And if $q \gg Q$, then $\theta \to \pi/2$. Also, if q = Q, then $\theta = 45^{\circ}$. These all make intuitive sense.

ALTERNATIVELY: To solve the exercise by minimizing the electrostatic energy, note that the only variable terms in the sum-over-all-pairs expression for the energy are the ones involving the diagonals of the rhombus. The other four pairs involve the sides of the rhombus which are of fixed length. The variable terms are $q^2/4\pi\epsilon_0(2\ell\sin\theta)$ and $Q^2/4\pi\epsilon_0(2\ell\cos\theta)$. Minimizing this as a function of θ yields

$$0 = \frac{d}{d\theta} \left(\frac{q^2}{\sin \theta} + \frac{Q^2}{\cos \theta} \right) = -q^2 \frac{\cos \theta}{\sin^2 \theta} + Q^2 \frac{\sin \theta}{\cos^2 \theta} \implies \frac{q^2}{Q^2} = \tan^3 \theta.$$
(16)

1.40. Zero potential energy

Let's first consider the general case where the three charges don't necessarily lie on the same line. Without loss of generality, we can put the two electrons on the x axis a unit distance apart (that is, at the values $x = \pm 1/2$), as shown in Fig. 1. And we may assume the proton lies in the xy plane. For an arbitrary location of the proton in this plane, let the distances from the electrons be r_1 and r_2 . Then setting the potential energy of the system equal to zero gives

$$U = \frac{1}{4\pi\epsilon_0} \left(\frac{e^2}{1} - \frac{e^2}{r_1} - \frac{e^2}{r_2} \right) \implies \frac{1}{r_1} + \frac{1}{r_2} = 1.$$
(17)

One obvious location satisfying this requirement has the proton on the y axis with $r_1 = r_2 = 2$, that is, with $y = \sqrt{15}/2 \approx 1.94$. In general, Eq. (17) defines a curve in the xy plane, and a surface of revolution around the x axis in space. This surface is the set of all points where the proton can be placed to give U = 0. The surface looks something like a prolate ellipsoid, but it isn't.

Let's now consider the case where all three charges lie on the x axis. Assume that the proton lies to the right of the right electron. We then have $r_1 = x - 1/2$ and $r_2 = x + 1/2$, so Eq. (17) becomes

$$\frac{1}{x-1/2} + \frac{1}{x+1/2} = 1 \implies x^2 - 2x - 1/4 = 0 \implies x = \frac{2 \pm \sqrt{5}}{2}.$$
 (18)

The negative root must be thrown out because it violates our assumption that x > 1/2. (With x < 1/2, the distance r_1 isn't represented by x - 1/2). So we find x = 2.118. The distance from the right electron at x = 1/2 equals $(1 + \sqrt{5})/2$. The ratio of this



Figure 1

5

distance to the distance between the electrons (which is just 1) is therefore the golden ratio. If we assume x < -1/2, then the mirror image at x = -2.118 works equally well. You can quickly check that there is no solution for x between the electrons, that is, in the region -1/2 < x < 1/2. There are therefore two solutions with all three charges on the same line.

1.41. Work for an octahedron

Consider an edge that has two protons at its ends (you can quickly show that at least one such edge must exist). There are two options for where the third proton is. It can be at one of the two vertices such that the triangle formed by the three protons is a face of the octahedron. Or it can be at one of the other two vertices. These two possibilities are shown in Fig. 2.

There are 15 pairs of charges, namely the 12 edges and the 3 internal diagonals. Summing over these pairs gives the potential energy. By examining the two cases shown, you can show that for the first configuration the sum is (the term with the $\sqrt{2}$ comes from the internal diagonals)

$$U = \frac{e^2}{4\pi\epsilon_0} \left(6 \cdot \frac{1}{a} - 6 \cdot \frac{1}{a} - 3 \cdot \frac{1}{\sqrt{2}a} \right) = -(2.121) \frac{e^2}{4\pi\epsilon_0 a} \,. \tag{19}$$

And for the second configuration:

$$U = \frac{e^2}{4\pi\epsilon_0} \left(4 \cdot \frac{1}{a} - 8 \cdot \frac{1}{a} + 2 \cdot \frac{1}{\sqrt{2}a} - 1 \cdot \frac{1}{\sqrt{2}a} \right) = -(3.293) \frac{e^2}{4\pi\epsilon_0 a} \,. \tag{20}$$

Both of these results are negative. This means that energy is released as the octahedron is assembled. Equivalently, it takes work to separate the charges out to infinity. You should think about why the energy is more negative in the second case. (Hint: the two cases differ only in the locations of the leftmost two charges.)

1.42. Potential energy in a 1-D crystal

Suppose the array has been built inward from the left (that is, from negative infinity) as far as a particular negative ion. To add the next positive ion on the right, the amount of external work required is

$$\frac{1}{4\pi\epsilon_0} \left(-\frac{e^2}{a} + \frac{e^2}{2a} - \frac{e^2}{3a} + \cdots \right) = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{a} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \right).$$
(21)

The expansion of $\ln(1 + x)$ is $x - x^2/2 + x^3/3 - \cdots$, converging for $-1 < x \leq 1$. Evidently the sum in parentheses above is just $\ln 2$, or 0.693. The energy of the infinite chain *per ion* is therefore $-(0.693)e^2/4\pi\epsilon_0 a$. Note that this is an exact result; it does not assume that a is small. After all, it wouldn't make any sense to say that "a is small," because there is no other length scale in the setup that we can compare a with.

The addition of further particles on the right doesn't affect the energy involved in assembling the previous ones, so this result is indeed the energy per ion in the entire infinite (in both directions) chain. The result is negative, which means that it requires energy to move the ions away from each other. This makes sense, because the two nearest neighbors are of the opposite sign.

If the signs of all the ions were the same (instead of alternating), then the sum in Eq. (21) would be $(1+1/2+1/3+1/4+\cdots)$, which diverges. It would take an infinite amount of energy to assemble such a chain.





Figure 2

An alternative solution is to compute the potential energy of a given ion due to the full infinite (in both directions) chain. This is essentially the same calculation as above, except with a factor of 2 due to the ions on each side of the given one. If we then sum over all ions (or a very large number N) to find the total energy of the chain, we have counted each pair twice. So in finding the potential energy per ion, we must divide by 2 (along with N). The factors of 2 and N cancel, and we arrive at the above result.

1.43. Potential energy in a 3-D crystal

The solution is the same as the solution to Problem 1.7, except that we have an additional term. We now also need to consider the "half-space" on top of the ion, in addition to the half-plane above it and the half-line to the right of it. In Fig. 12.4 the half-space of ions is on top of the plane of the paper (from where you are viewing the page).

If we index the ions by the coordinates (m, n, p), then the potential energy of the ion at (0, 0, 0) due to the half-line, half-plane, and half-space is

$$U = \frac{e^2}{4\pi\epsilon_0 a} \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{m} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n}}{\sqrt{m^2 + n^2}} + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+n+p}}{\sqrt{m^2 + n^2 + p^2}} \right)$$
(22)

The triple sum takes more computer time than the other two sums. Taking the limits to be 300 instead of ∞ in the triple sum, and 1000 in the other two, we obtain decent enough results via *Mathematica*. We find

$$U = \frac{e^2}{4\pi\epsilon_0 a} (-0.693 - 0.115 - 0.066) = -\frac{(0.874)e^2}{4\pi\epsilon_0 a}, \qquad (23)$$

which agrees with Eq. (1.18) to three digits. This result is negative, which means that it requires energy to move the ions away from each other. This makes sense, because the six nearest neighbors are of the opposite sign.

1.44. Chessboard

W is probably going to be positive, because the four nearest neighbors are all of the opposite sign. Fig. 3 shows a quarter (or actually slightly more than a quarter) of a 7×7 chessboard. Three different groups of charges are circled. The full chessboard consists of four of the horizontal group, four of the diagonal group, and eight of the triangular group. Adding up the work associated with each group, the total work required to move the central charge to a position far away is (in units of $e^2/4\pi\epsilon_0 s$)

$$W = 4\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3}\right) + 4\left(-\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} - \frac{1}{3\sqrt{2}}\right) + 8\left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{13}}\right) \approx 1.4146,$$
(24)

which is positive, as we guessed.

For larger arrays we can use a *Mathematica* program to calculate W. If we have an $N \times N$ chessboard, and if we define H by 2H + 1 = N (for example, H = 50corresponds to N = 101), then the following program gives the work W required to remove the central charge from a 101×101 chessboard.

H=50; 4*Sum[(-1)^(n+1)/n, {n,1,H}] + 4*Sum[Sum[(-1)^(n+m+1)/(n²+m²)^(.5), {n,1,H}], {m,1,H}]



Figure 3



Figure 4





This program involves dividing the chessboard into the regions shown in Fig. 4; the sub-squares have side length H. (If you want, you can reduce the computing time by about a factor of 2 by dividing the chessboard as we did in Fig. 3.) The results for various $N \times N$ chessboards are (in units of $e^2/4\pi\epsilon_0 s$):

N	3	7	101	1001	10,001	100,001
W	1.1716	1.4146	1.6015	1.6141	1.6154	1.6155

The W for an infinite chessboard is apparently roughly equal to $(1.6155)e^2/4\pi\epsilon_0 s$. The prefactor here is double the 0.808 prefactor in the result for Problem 1.7, due to the fact that the latter is the energy *per ion*, so there is the usual issue of double counting.

1.45. Zero field?

The setup is shown in Fig. 5. We know that $E_y = 0$ on the y axis, by symmetry, so we need only worry about E_x . We want the leftward E_x from the two middle charges to cancel the rightward E_x from the two outer charges. This implies that

$$2 \cdot \frac{1}{4\pi\epsilon_0} \frac{q}{y^2 + a^2} \cdot \frac{a}{\sqrt{y^2 + a^2}} = 2 \cdot \frac{1}{4\pi\epsilon_0} \frac{q}{y^2 + (3a)^2} \cdot \frac{3a}{\sqrt{y^2 + (3a)^2}}, \qquad (25)$$

where the second factor on each side of the equation comes from the act of taking the horizontal component. Simplifying this gives

$$\frac{1}{(y^2 + a^2)^{3/2}} = \frac{3}{(y^2 + 9a^2)^{3/2}} \implies y^2 + 9a^2 = 3^{2/3}(y^2 + a^2)$$
$$\implies y = a\sqrt{\frac{9 - 3^{2/3}}{3^{2/3} - 1}} \approx (2.53)a.$$
(26)

In retrospect, we know that there must exist a point on the y axis with $E_x = 0$, by a continuity argument. For small y, the field points leftward, because the two middle charges dominate. But for large y, the field points rightward, because the two outer charges dominate. (This is true because for large y, the distances to the four charges are all essentially the same, but the slope of the lines to the outer charges is smaller than the slope of the lines to the middle charges (it is 1/3 as large). So the x component of the field due to the outer charges is 3 times as large, all other things being equal.) Therefore, by continuity, there must exist a point on the y axis where E_x equals zero.

1.46. Charges on a circular track

Let's work with the general angle θ shown in Fig. 6. In the problem at hand, $4\theta = 90^{\circ}$, so $\theta = 22.5^{\circ}$. The tangential electric field at one of the q's due to Q is

$$\frac{Q}{4\pi\epsilon_0(2R\cos\theta)^2}\,\sin\theta,\tag{27}$$

and the tangential field (in the opposite direction) at one q due to the other q is

$$\frac{q}{4\pi\epsilon_0 (2R\sin 2\theta)^2} \cos 2\theta. \tag{28}$$

Equating these fields gives

$$\frac{Q}{(\cos\theta)^2}\sin\theta = \frac{q}{(\sin 2\theta)^2}\cos 2\theta \implies Q = q\frac{\cos^2\theta\cos 2\theta}{\sin\theta\sin^2 2\theta} = q\frac{\cos 2\theta}{4\sin^3\theta}, \quad (29)$$





where we have used $\sin 2\theta = 2\sin\theta\cos\theta$. Letting $\theta = 22.5^{\circ}$ gives Q = (3.154)q.

Some limits: If all three charges are equally spaced (with $\theta = 30^{\circ}$) then Q = q, as expected. If $\theta \to 0$ then $Q \approx q/(4\theta^3)$. (Two of these powers of θ come from the r^2 in Coulomb's law, and one comes from the act of taking the tangential component of Q's field.) If the q's are diametrically opposite (with $\theta = 45^{\circ}$) then Q = 0, as expected.

1.47. Field from a semicircle

Choose the semicircle to be the top half of a circle with radius R centered at the origin. So the diameter of the semicircle lies along the x axis. Let the angle θ be measured relative to the positive x axis. A small piece of the semicircle subtending an angle $d\theta$ has charge $dQ = Q(d\theta/\pi)$. The magnitude of the field at the center due to this piece is $dQ/4\pi\epsilon_0 R^2$. The x components of the field contributions from the various pieces will cancel in pairs, so only the y component survives, which brings in a factor of $\sin \theta$. The total (vertical) field therefore equals

$$E_y = -\int_0^\pi \frac{Q(d\theta/\pi)}{4\pi\epsilon_0 R^2} \sin\theta = -\frac{Q}{(4\pi\epsilon_0)\pi R^2} \int_0^\pi \sin\theta \,d\theta = -\frac{2Q}{(4\pi\epsilon_0)\pi R^2} \,, \qquad (30)$$

where the minus sign indicates that the field points downward (if Q is positive). This result can be written as $-\lambda/2\pi\epsilon_0 R$, where λ is the linear charge density. Interestingly, it can also be written as $-Q/4\pi\epsilon_0 A$, where $A = \pi R^2/2$ is the area of the semicircle.

1.48. Maximum field from a ring

At (0,0,z) the field due to an element of charge dQ on the ring has magnitude $dQ/4\pi\epsilon_0(b^2+z^2)$. But only the z component survives, by symmetry, and this brings in a factor of $z/\sqrt{b^2+z^2}$. Integrating over the entire ring simply turns the dQ into Q, so we have $E_z = Qz/4\pi\epsilon_0(b^2+z^2)^{3/2}$. Setting the derivative equal to zero to find the maximum gives

$$0 = \frac{(b^2 + z^2)^{3/2}(1) - z(3/2)(b^2 + z^2)^{1/2}(2z)}{(b^2 + z^2)^3} = \frac{b^2 - 2z^2}{(b^2 + z^2)^{5/2}} \implies z = \pm \frac{b}{\sqrt{2}}.$$
 (31)

Since we're looking for a point on the positive z axis, we're concerned with the positive root, $z = b/\sqrt{2}$. Note that we know the field must have a local maximum somewhere between z = 0 and $z = \infty$, because the field is zero at both of these points.

1.49. Maximum field from a blob

(a) Label the points on the curve by their distance r from the origin, and by the angle θ that the line of this distance subtends with the y axis, as shown in Fig. 7. Then a point charge q on the curve provides a y component of the electric field at the origin equal to

$$E_y = \frac{q}{4\pi\epsilon_0 r^2}\cos\theta. \tag{32}$$

If we want this to be independent of the charge's location on the curve, we must have $r^2 \propto \cos \theta$. The curve may therefore be described by the equation,

$$r^2 = a^2 \cos \theta \implies r = a \sqrt{\cos \theta},$$
 (33)

where the constant a is the value of r at $\theta = 0$, that is, the height of the curve along the y axis. We therefore have a family of curves indexed by a.





(b) Assume that the material has been shaped and positioned so that the electric field at the origin takes on the maximum possible value. Assume that the field points in the y direction. Then all the small elements of charge dq on the surface of the material must give equal contributions to the y component of the field at the origin. This is true because if it weren't the case, then we could simply move a tiny piece of the material from one point on the surface to another, thereby increasing the field at the origin, in contradiction to our assumption that the field at the origin is maximum. From part (a), any vertical cross section (formed by the intersection of the surface with a plane containing the y axis) must therefore look like the $r = a\sqrt{\cos\theta}$ curve we found. Equivalently, the desired shape of the material is obtained by forming the surface of revolution of the $r = a\sqrt{\cos\theta}$ curve, around the y axis.

Let's try to get a sense of what the surface looks like. We know that the height is a. To find the width, note that $x = r \sin \theta = a \sin \theta \sqrt{\cos \theta}$. Taking the derivative of this function of θ , you can show that the maximum value of x is achieved when $\tan \theta = \sqrt{2}$; the maximum value is $(4/27)^{1/4}a$. The width is twice this value, or $2(4/27)^{1/4}a \approx 1.24a$. So the ratio of width to height is about 5 to 4.

Let's compare our shape with a sphere of the same volume. The volume of our shape can be obtained by slicing it into horizontal disks. The radius of a disk is $x = r \sin \theta = a \sin \theta \sqrt{\cos \theta}$. And since $y = -r \cos \theta = -a(\cos \theta)^{3/2}$, the height of a disk is $dy = (3/2)a \sin \theta \sqrt{\cos \theta} d\theta$. So the volume is

$$V = \int_{-a}^{0} (\pi x^2) dy = \int_{0}^{\pi/2} \pi \left(a\sin\theta\sqrt{\cos\theta}\right)^2 \cdot \frac{3}{2}a\sin\theta\sqrt{\cos\theta} d\theta$$
$$= \frac{3}{2}\pi a^3 \int_{0}^{\pi/2} \sin^3\theta\cos^{3/2}\theta d\theta.$$
(34)

Writing $\sin^2 \theta$ as $1 - \cos^2 \theta$ yields integrals of $\sin \theta \cos^{3/2} \theta$ and $-\sin \theta \cos^{7/2} \theta$, which give 2/5 and -2/9, respectively. The sum of these is 8/45, so the volume is $V = (4/15)\pi a^3$. Since the diameter of a sphere with volume V is $(6V/\pi)^{1/3}$, we see that a sphere with the same volume would have a diameter of $(8/5)^{1/3}a \approx 1.17a$. Compared with a sphere of the same volume, our shape is therefore stretched by a factor of $(1.24)a/(1.17)a \approx 1.06$ in the x direction, and squashed by a factor of $a/(1.17)a \approx 0.85$ along the y direction. Cross sections of our shape and a sphere with the same volume are shown in Fig. 8.

1.50. Field from a hemisphere

(a) Consider the ring shown in Fig. 9, defined by the angle θ and subtending an angle $d\theta$. Its area is $2\pi(R\cos\theta)(R\,d\theta)$, so its charge is $\sigma(2\pi R^2\cos\theta\,d\theta)$. The horizontal component of the field at the center of the hemisphere is zero, by symmetry. So we need only worry about the vertical component from each piece of the ring, which brings in a factor of $\sin \theta$. Adding up these components from all the pieces of the ring gives the magnitude of the field at the center of the hemisphere, due to the given ring, as

$$dE = \frac{\sigma(2\pi R^2 \cos\theta \, d\theta)}{4\pi\epsilon_0 R^2} \sin\theta = \frac{\sigma \sin\theta \cos\theta \, d\theta}{2\epsilon_0} \,. \tag{35}$$

The field points downward if σ is positive. Integrating over all the rings (θ runs from 0 to $\pi/2$) gives the total field at the center as

$$E = \int_0^{\pi/2} \frac{\sigma \sin \theta \cos \theta \, d\theta}{2\epsilon_0} = \frac{\sigma}{2\epsilon_0} \frac{\sin^2 \theta}{2} \Big|_0^{\pi/2} = \frac{\sigma}{4\epsilon_0} \,. \tag{36}$$





This is half as large as the field at the center of the end face of a half-infinite hollow *cylinder* (see Problem 1.11). You should convince yourself why the cylinder's field must indeed be larger (although the factor of 2 is by no means obvious). *Hint:* consider the field contributions from corresponding bits of the surfaces subtending the same solid angle. The cylinder's surface is tilted with respect to the line to the center of the end face.

In terms of the total charge $Q = 2\pi R^2 \sigma$ on the hemisphere, the result in Eq. (36) can be written as $Q/8\pi\epsilon_0 R^2$. If you solved Exercise 1.47 you will note that the present field is smaller (by a factor of $\pi/4$) than the field at the center of a semicircle with the same charge Q. This is because the semicircle's charge is generally higher up, so the act of taking the vertical component doesn't reduce the field as much as for the hemisphere. Equivalently, building a hemispherical cage out of a large number of semicircular pieces of wire would effectively yield a hemisphere with a larger surface charge density near the top than near the base.

(b) The field due to a *solid* hemisphere with radius R and uniform volume charge density ρ can be found by slicing up the solid hemisphere into concentric hemispherical shells with thickness dR. The effective surface charge density of each shell is ρdR , so the result from part (a) tells us that the field from each shell is $(\rho dR)/4\epsilon_0$. Integrating over R simply turns the dR into an R, so the total field is $\rho R/4\epsilon_0$. Again, this is half as large as the field from a half-infinite solid cylinder (see Problem 1.11).

1.51. N charges on a circle

Let Q_0 be the point charge at the top of the circle, and consider the *n*th point charge away from it (call it Q_n), as shown in Fig. 10 for n = 4 and N = 12. The angle θ equals $n(2\pi/N)$, so the distance from Q_0 to Q_n is $r_n = 2R\sin(\theta/2) = 2R\sin(n\pi/N)$. The horizontal components of all the fields at Q_0 cancel in pairs, so we're concerned only with the vertical component, which brings in a factor of $\sin(\theta/2)$. The vertical component of the field at Q_0 due to Q_n is therefore

$$E_n = \frac{Q/N}{4\pi\epsilon_0 \left(2R\sin(n\pi/N)\right)^2} \sin(n\pi/N). \tag{37}$$

The total field at Q_0 from all the Q_n charges is then

$$E = \frac{Q}{16\pi\epsilon_0 R^2} \sum_{n=1}^{N-1} \frac{1}{N\left(\sin(n\pi/N)\right)}.$$
 (38)

If you compute this sum numerically for various values of N, you will find that it grows with N, although very slowly (like a log). The sum does in fact diverge as $N \to \infty$, due to the behavior of the terms with small n (and likewise for n close to N, because the terms are symmetric around n = N/2). If $n \ll N$, we can write $\sin(n\pi/N) \approx n\pi/N$, so for small n the field from Q_n behaves like $1/(N(n\pi/N)) = 1/n\pi$. And since the sum $\sum_{n=1}^{M} 1/n$ diverges (like $\ln M$), we see that the total field diverges. We are assuming that N is large enough so that n can become very large and still have the approximation $\sin(n\pi/N) \approx n\pi/N$ be valid. This is of course true in the $N \to \infty$ limit.

Note that the only possible cause for the divergence of the total field is the behavior of the fields from nearby Q_n . There is a finite total charge on the ring, so the field from the non-infinitesimally-close charges must be finite, because those charges don't



Figure 10

involve any infinitesimal distances that would make the fields diverge. Equivalently, once the $\sin(n\pi/N)$ term in the denominator of the field becomes non-infinitesimal, the fields go like 1/N, so the sum (which involves fewer than N terms) is bounded.

We saw above that the vertical field contribution from each of the nearby charges equals $1/n\pi$, which is finite. In short, in Eq. (37) the small charge Q/N and the small factor of $\sin(n\pi/N)$ from the vertical component cancel the square of the small distance $2R\sin(n\pi/N)$ in the denominator. But the *total* field ends up diverging because there are so many charges that are very close to Q_0 .

Even though the field at Q_0 diverges in the $N \to \infty$ limit, the actual *force* on Q_0 goes to zero. The force equals the field times the charge Q/N, and since the field only diverges like $\ln N$, the force behaves like $(\ln N)/N$, which goes to zero for large N.

A continuous circle of charge is equivalent to the $N \to \infty$ limit. So if an additional point charge with finite (non-infinitesimal) charge q were placed exactly on the circumference of the (infinitesimally thin) circle, the force on it would be infinite, due to the infinite field. However, in reality there are no true point charges or infinitesimally thin distributions of charge.

1.52. An equilateral triangle

(a) Let F_0 be the force between two charges of $q = 10^{-6}$ C each, at a distance of a = 0.2 m. Then $F_0 = q^2/4\pi\epsilon_0 a^2 = 0.225$ N, as you can verify. The force between B and C has magnitude $(2)(2)F_0 = 4F_0$, and the force between A and either B or C has magnitude $(3)(2)F_0 = 6F_0$. From Fig. 11, the magnitude of the force on A is

$$F_A = 2\cos 30^\circ \cdot 6F_0 = 2.34 \text{ N.}$$
 (39)

The magnitude of the force on C is (squaring and adding the horizontal and vertical components)

$$F_C = \left[(4 + 6\cos 60^\circ)^2 + (6\sin 60^\circ)^2 \right]^{1/2} F_0 = (8.72)F_0 = 1.96 \text{ N.}$$
(40)

And the force on B has the same magnitude.

(b) Three equal charges of $2 \cdot 10^{-6}$ C would yield zero field at the center, by symmetry. So the field at the center is due to the excess charge of $q = 10^{-6}$ C at A. Since A is a distance $a/\sqrt{3}$ from the center, the magnitude of the field at the center of the triangle is

$$E = \frac{q}{4\pi\epsilon_0 (a/\sqrt{3})^2} = \left(9 \cdot 10^9 \,\frac{\text{kg m}^3}{\text{s}^2 \,\text{C}^2}\right) \frac{10^{-6} \,\text{C}}{(0.2 \,\text{m})^2/3} = 6.75 \cdot 10^5 \,\text{N/C.}$$
(41)

1.53. Concurrent field lines

Consider a point at height z above the semicircle. All points on the wire are a distance $\ell = \sqrt{R^2 + z^2}$ from this point, so a small piece of the wire with charge $dq = \lambda R d\theta$ yields a field with magnitude

$$dE = \frac{dq}{4\pi\epsilon_0\ell^2} = \frac{\lambda R \, d\theta}{4\pi\epsilon_0(R^2 + z^2)} \,. \tag{42}$$

Let the x axis split the semicircle in half. Then the net E_y field is zero, by symmetry. The (magnitudes of the) z and x components of the dE field in Eq. (42) are obtained by multiplying it by z/ℓ and x/ℓ . (The latter of these can be obtained in two steps:



multiply by R/ℓ to get the component in the x-y plane, then multiply by x/R to get the x component.) So we have

$$dE_z = \frac{\lambda R z \, d\theta}{4\pi\epsilon_0 (R^2 + z^2)^{3/2}}, \quad \text{and} \quad dE_x = \frac{\lambda R x \, d\theta}{4\pi\epsilon_0 (R^2 + z^2)^{3/2}} = \frac{\lambda R (R\cos\theta) \, d\theta}{4\pi\epsilon_0 (R^2 + z^2)^{3/2}},$$
(43)

where θ runs from $-\pi/2$ to $\pi/2$. The net E_z and E_x components are obtained by integrating over θ . In E_z the integration simply brings in a factor of π . In E_x it brings in a factor of $\int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta = 2$. Therefore,

$$E_z = \frac{\lambda R z}{4\epsilon_0 (R^2 + z^2)^{3/2}}, \quad \text{and} \quad E_x = \frac{\lambda R^2}{2\pi\epsilon_0 (R^2 + z^2)^{3/2}}. \quad (44)$$

Hence $E_z/E_x = \pi z/2R$, which can be written a little more informatively as $E_z/E_x = z/(2R/\pi)$. This is the slope of the **E** vector at a point at height z on the z axis. The slope covers a horizontal distance $2R/\pi$ while covering a vertical distance z. The straight line that points in the direction of the electric field at the point (0, 0, z) therefore passes through the point $(2R/\pi, 0, 0)$ in the plane of the semicircle. This point is independent of z, as desired.

This point also happens to be the "center of charge" of the semicircle, or equivalently the center of mass of a semicircle made out of a piece of wire (we'll leave it to you to verify this). So the result of this exercise is consistent with the following fact (which you may want to try to prove): Far away from a distribution of charges, the electric field points approximately toward the center of charge of the distribution. For nearby points it generally doesn't, although it happens to (exactly) point in that direction for points on the axis of the present setup.

1.54. Semicircle and wires

(a) The charge dq in the piece at A is $\lambda b \, d\theta$. The magnitude of the field due to this charge is

$$E_A = \frac{\lambda b \, d\theta}{4\pi\epsilon_0 b^2} = \frac{\lambda \, d\theta}{4\pi\epsilon_0 b} \,. \tag{45}$$

The charge dq in the piece at B is λdy . But from Fig. 12 we have $dy = r d\theta / \cos \theta = r d\theta / (b/r) = r^2 d\theta / b$. The magnitude of the field due to this charge is therefore

$$E_B = \frac{\lambda (r^2 \, d\theta/b)}{4\pi\epsilon_0 r^2} = \frac{\lambda \, d\theta}{4\pi\epsilon_0 b} \,. \tag{46}$$

Since the magnitudes E_A and E_B are equal, and since the fields are directed oppositely, the sum of the two fields is zero. The entire filament can be built up from these corresponding pairs, so the total field at C is zero. In short, the field contributions from *equal* point charges located at A and B would be in the ratio r^2/b^2 , due to the inverse-square nature of the Coulomb field. But this effect is canceled by the fact that the *actual* charge at B is larger than at A by a factor $(r/b)^2$. One of these factors of r/b comes from the fact that B is farther from C, and the other comes from the fact that the B segment is tilted with respect to the line to C.

This result is consistent with the results from Problem 1.10 (which says that the upward component of the field at C due to each of the straight segments is $\lambda/4\pi\epsilon_0 b$) and Exercise 1.47 (which says that the downward field at C due to the semicircle is $\lambda/2\pi\epsilon_0 b$).





(b) Imagine cutting the two-dimensional setup into thin strips defined by a large number of vertical planes, rotated at small angles with respect to each other, all passing through the axis of the cylinder. Each thin strip is similar to the one-dimensional setup in part (a), except for the fact that the strip gets narrower as it approaches the top of the hemisphere (it is somewhat like a curved pie piece). The linear charge density is therefore effectively smaller at the top. We know from part (a) that a *uniform* linear charge density leads to the downward field from the circular part of a strip canceling the upward field from the straight part. A smaller density on the circular part therefore means that its downward field can't fully cancel the upward field from the straight part. The net field therefore points upward.

This result is consistent with the results from Problem 1.11 (which says that the upward field at C due to the cylinder is $\sigma/2\epsilon_0$) and either Problem 1.12 or Exercise 1.50 (which say that the downward field at C due to the hemisphere is $\sigma/4\epsilon_0$).

1.55. Field from a finite rod

In Fig. 13, define the distances: $\ell = 0.05 \text{ m}$, a = 0.03 m, and b = 0.05 m. The linear charge density of the rod is $\lambda = (8 \cdot 10^{-9} \text{ C})/(0.1 \text{ m}) = 8 \cdot 10^{-8} \text{ C/m}$. At point A the field points leftward and has magnitude

$$E_{A} = \frac{1}{4\pi\epsilon_{0}} \int_{a}^{a+2\ell} \frac{\lambda \, dx}{x^{2}} = \frac{\lambda}{4\pi\epsilon_{0}} \left(\frac{1}{a} - \frac{1}{a+2\ell}\right) = \frac{\lambda}{4\pi\epsilon_{0}} \left(\frac{2\ell}{a(a+2\ell)}\right)$$
$$= \left(9 \cdot 10^{9} \frac{\text{kg m}^{3}}{\text{s}^{2} \text{ C}^{2}}\right) \left(8 \cdot 10^{-8} \frac{\text{C}}{\text{m}}\right) \left(\frac{2(.05 \,\text{m})}{(.03 \,\text{m})(.13 \,\text{m})}\right)$$
$$= 1.85 \cdot 10^{4} \frac{\text{N}}{\text{C}}. \tag{47}$$

As a check, if $a \gg \ell$ this result approaches $(1/4\pi\epsilon_0)(2\ell\lambda/a^2)$, which is correctly the field from a point charge $2\ell\lambda$ at a distance a.

At point B, only the vertical component of the field survives, by symmetry. So the field points downward and has magnitude

$$E_B = \frac{1}{4\pi\epsilon_0} \int_{-\ell}^{\ell} \frac{\lambda \, dx}{b^2 + x^2} \cdot \frac{b}{\sqrt{b^2 + x^2}},\tag{48}$$

where the second factor gives the vertical component. This integral can be evaluated with a trig substitution, $x = b \tan \theta \Longrightarrow dx = b d\theta / \cos^2 \theta$ (or you can just look it up), which yields

$$E_B = \frac{1}{4\pi\epsilon_0} \int_{-\tan^{-1}(\ell/b)}^{\tan^{-1}(\ell/b)} \frac{\lambda b^2 \, d\theta/\cos^2 \theta}{b^3 (1+\tan^2 \theta)^{3/2}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda}{b} \int_{-\tan^{-1}(\ell/b)}^{\tan^{-1}(\ell/b)} \cos \theta \, d\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{b} \sin \theta \Big|_{-\tan^{-1}(\ell/b)}^{\tan^{-1}(\ell/b)} = \frac{1}{4\pi\epsilon_0} \frac{\lambda}{b} \frac{2\ell}{\sqrt{\ell^2 + b^2}}$$

$$= \left(9 \cdot 10^9 \frac{\text{kg m}^3}{\text{s}^2 \text{ C}^2}\right) \frac{(8 \cdot 10^{-8} \text{ C/m})}{(.05 \text{ m})} \frac{2(.05 \text{ m})}{\sqrt{(.05 \text{ m})^2 + (.05 \text{ m})^2}}$$

$$= 2.04 \cdot 10^4 \frac{\text{N}}{\text{C}}.$$
(49)

The $\int \cos\theta \, d\theta$ integral here is just what you would obtain if you parameterized the rod in terms of θ ; see Eq. (1.38).





As a check, if $b \gg \ell$ this result approaches $(1/4\pi\epsilon_0)(2\ell\lambda/b^2)$, which is correctly the field from a point charge $2\ell\lambda$ at a distance b.

1.56. Flux through a cube

- (a) The total flux through the cube is q/ϵ_0 , by Gauss's law. The flux through every face of the cube is the same, by symmetry. Therefore, over any one of the six faces we have $\int \mathbf{E} \cdot d\mathbf{a} = q/6\epsilon_0$.
- (b) Because the field due to q is parallel to the surface of each of the three faces that touch q, the flux through these faces is zero. The total flux through the other three faces must therefore add up to $q/8\epsilon_0$, because our cube is one of eight such cubes surrounding q. Since the three faces are symmetrically located with respect to q, the flux through each must be $(1/3)(q/8\epsilon_0) = q/24\epsilon_0$.

Note: if the charge were a true point charge, and if it were located just inside or just outside the cube, then the field would *not* be parallel to each of the three faces that touch the given corner. The flux would depend critically on the exact location of the point charge. Replacing the point charge with a small sphere, whose center lies at the corner, eliminates this ambiguity.

1.57. Escaping field lines

=

(a) You can quickly show that the desired point with E = 0 must satisfy x > a. Equating the magnitudes of the fields from the two given charges then gives

$$\frac{2q}{4\pi\epsilon_0 x^2} = \frac{q}{4\pi\epsilon_0 (x-a)^2} \implies 2(x-a)^2 = x^2$$
$$\implies \sqrt{2}(x-a) = x \implies x = \frac{\sqrt{2}a}{\sqrt{2}-1} = (2+\sqrt{2})a \approx (3.414)a.$$
(50)

A few field lines, are shown in Fig. 14.¹ Note that the field points in four different directions near the E = 0 point. This is consistent with the fact that the zero vector is the only vector that can simultaneously point in different directions.

(b) Consider a field line that emerges from the 2q charge and ends up at the x =(3.414)a point where the field is zero. (There are actually no field lines that end up right at this point, but we can pick a line infinitesimally close.) Field lines that emerge at a smaller angle (with respect to the x axis) end up at the -qcharge, and field lines that emerge at a larger angle end up at infinity. Consider the Gaussian surface indicated in Fig. 15; the surface is formed by rotating the black curve around the x axis. This surface follows the field lines except very close to the 2q charge, where it takes the form of a small spherical cap. The total charge enclosed within this surface is simply -q, so from Gauss's law there must be an electric-field flux of q/ϵ_0 pointing in to the surface. By construction, the only place where there is flux is the spherical cap, so all of the q/ϵ_0 flux must occur there. But the total flux emanating from the charge 2q is $2q/\epsilon_0$, so the spherical cap must represent half of the total area of a small sphere surrounding the charge 2q. (Very close to the 2q charge, that charge dominates the electric field, so the field is essentially spherically symmetric.) The cap must therefore be a hemisphere, so the desired angle is 180° .

¹This figure technically isn't a plot of field lines, because you can see that some of the lines begin in empty space, whereas we know that field lines can begin and end only at charges or at infinity. So the density of the lines on the page doesn't indicate the field strength. (Well, it fails to do that even if we



Figure 14





If the charges take on the more general values of Nq and -q, then the spherical cap represents 1/N of the complete sphere. So the task is to find the angle subtended by a cap with 1/N of the total area. This requires an integral (whereas in the above case we could simply say we had a hemisphere). But one nice case is N = 4, which leads to an angle of 60° .

1.58. Gauss's law at the center of a ring

(a) Let the ring lie in the horizontal plane. A small piece of the ring with charge dq produces a field $dq/4\pi\epsilon_0 R^2$ at the center. At a small vertical distance z above the center, the magnitude of the field due to the dq piece is essentially the same (it differs only at order z^2/R^2 , by the Pythagorean theorem), so the vertical component is obtained by simply tacking on a $\sin \theta$ type factor, which is z/R here. Integrating over the whole ring turns the dq into Q, so the desired vertical

don't have any lines that abruptly end, because a 2D picture can't mimic the actual density in 3D space.) However, every curve shown is at least part of a field line, so the figure is still helpful in visualizing the flow of the actual field lines.

field is $Qz/4\pi\epsilon_0 R^3$. Alternatively, you can calculate the field exactly (as in the solution to Exercise 1.48) and then take the $z \ll R$ limit.

(b) The solution to Problem 1.8 tells us that in the plane of the ring, the field near the center, at radius r, points radially inward (assuming Q is positive) with magnitude $\lambda r/4\epsilon_0 R^2$. But since $\lambda = Q/2\pi R$, this can be written as $Qr/8\pi\epsilon_0 R^3$. Consider a point near the center, with a nonzero r value, and also a nonzero z value. To leading order, the horizontal component of the field is still $Qr/8\pi\epsilon_0 R^3$, and the vertical component is still $Qz/4\pi\epsilon_0 R^3$, from part (a). That is, these results are actually valid for all points in space near the origin, not just in the plane of the fields. For example, in part (a) the effective values of R change slightly if the point is off the axis, but this doesn't change the field, to leading order. Alternatively, note that due to symmetry, the horizontal component $E_r(r, z)$ is an even function of z. This means that $E_r(r, z)$ has no linear dependence on z. The variation with z therefore starts only at order z^2 , which is negligible for small z. So E_r is essentially independent of z near the axis. Similar reasoning works with E_z as a function of r.

For simplicity, let's define $A \equiv Q/8\pi\epsilon_0 R^3$. Then the horizontal and vertical field components have magnitudes Ar and 2Az, respectively. The top and bottom faces of the small cylinder have a combined area of $2\pi r_0^2$. And the vertical cylindrical side has an area of $(2\pi r_0)(2z_0)$. There is outward flux through the top and bottom, and inward flux through the side, so the net outward flux equals

$$(2\pi r_0^2)(2Az_0) - (4\pi r_0 z_0)(Ar_0) = 0, (51)$$

as desired. If we work backwards, this exercise actually provides a much quicker method, compared with the one in Problem 1.8, for finding the horizontal component of the field near the center of the ring, assuming that we know the vertical component.

1.59. Zero field inside a cylindrical shell

The solution to this exercise is essentially the same as the solution to Problem 1.17. The main points are that areas are proportional to lengths squared, and that the relevant surfaces make the same angle with respect to the line to the point in question.

As in the solution to Problem 1.17, let a be the distance from point P to patch A, and let b be the distance from P to patch B; see Fig. 16. (Since the cones are assumed to be thin, it doesn't matter exactly which points in the patches we use to define these distances.) Draw the "perpendicular" bases of the cones, and call them A' and B'. The ratio of the areas of A' and B' is a^2/b^2 , because areas are proportional to lengths squared. And the angle between the planes of A and A' is the same as the angle between the planes of B and B'. The ratio of the areas of A and B is therefore also equal to a^2/b^2 . So the charge on patch A is a^2/b^2 times the charge on patch B.

The magnitudes of the fields due to the two patches take the general form of $q/4\pi\epsilon_0 r^2$. We just found that the q for A is a^2/b^2 times the q for B. But we also know that the r^2 for A is a^2/b^2 times the r^2 for B. So the values of $q/4\pi\epsilon_0 r^2$ for the two patches are equal. The fields at P due to A and B (which can be treated essentially like point charges, because the cones are assumed to be thin) are therefore equal in magnitude; and opposite in direction, of course. If we draw enough cones to cover the whole cylinder, the contributions to the field from the little patches over the whole cylinder cancel in pairs, so we are left with zero field at P. This holds for any point P inside the cylinder.



Figure 16

1.60. Field from a hollow cylinder

The cylindrical tube of charge (the bold circle) in Fig. 17 has perfect axial symmetry. So inside the tube, E_1 and E_2 must be radial and equal in magnitude. Applying Gauss's law to a cylinder of radius a and length ℓ yields $2\pi a\ell E_1 = 0$, because there is no charge inside the tube. Therefore E = 0 inside the tube. Outside the tube, symmetry also demands that $E_3 = E_4$. Applying Gauss's law to a cylinder of radius rand length ℓ yields $2\pi r\ell E_3 = \lambda \ell / \epsilon_0$, where λ is the charge per unit length. This gives $E = \lambda/2\pi\epsilon_0 r$ outside the tube, just as if the charge were concentrated on the axis.

For a square tube of charge the integral over any cylinder must equal $1/\epsilon_0$ times the charge enclosed, but nothing requires that $E_1 = E_2$ or $E_3 = E_4$ in Fig. 18. The integral of E over the small inner circle vanishes (as it does for any cross-sectional shape), but it can do so with $E_1 \neq E_2$ if, as is the case, these two fields point in opposite directions at the locations shown. By comparing this tube with a square charged *conducting* tube, within which the field is in fact zero (see Chapter 3), you can deduce that E_2 must point inward (if the charge is positive).

1.61. Potential energy of a sphere

The charge inside a sphere of radius r (with r < R) is $q = (4\pi r^3/3)\rho$. The external field of this sphere is the same as if all of the charge were at the center. So the sphere acts like a point charge, as far as the potential energy of an external object is concerned. The next shell to be added, with thickness dr, contains charge $dq = (4\pi r^2 dr)\rho$. The work done in bringing in this dq (which is the same as the potential energy of the shell due to the sphere) is therefore

$$dW = \frac{1}{4\pi\epsilon_0} \frac{q \cdot dq}{r} = \frac{1}{4\pi\epsilon_0} \frac{(4\pi\rho)^2}{3} r^4 \, dr.$$
(52)

Building up the whole sphere this way, from r = 0 to r = R, requires the work:

$$W = \int_0^R \frac{1}{4\pi\epsilon_0} \frac{(4\pi\rho)^2}{3} r^4 dr = \frac{1}{4\pi\epsilon_0} \frac{(4\pi\rho)^2}{3} \frac{R^5}{5}.$$
 (53)

The charge in the complete sphere is $Q = (4\pi R^3/3)\rho$, which gives $4\pi\rho = 3Q/R^3$. Thus the potential energy U, which is the same as the work W, can be written as $U = (3/5)Q^2/4\pi\epsilon_0 R$. Note that $Q^2/4\pi\epsilon_0 R$ has the proper energy dimensions of $(\text{charge})^2/(\epsilon_0 \cdot \text{distance})$. Indeed, we could have predicted that much of the result without any calculation. The only question is what the numerical factor out front is. It happens to be 3/5.

Note that we don't have to worry about the self energy of each infinitesimally thin shell, because by dimensional analysis this energy is proportional to $(dq)^2$. So it is a second-order small quantity and hence can be ignored.

1.62. Electron self energy

Setting the potential energy $(3/5)e^2/4\pi\epsilon_0 r_0$ from Exercise 1.61 equal to mc^2 gives

$$r_{0} = \frac{3}{5} \frac{1}{4\pi\epsilon_{0}} \frac{e^{2}}{mc^{2}}$$

$$= \frac{3}{5} \left(9 \cdot 10^{9} \frac{\text{kg m}^{3}}{\text{s}^{2} \text{ C}^{2}}\right) \frac{(1.6 \cdot 10^{-19} \text{ C})^{2}}{(9.1 \cdot 10^{-31} \text{ kg})(3 \cdot 10^{8} \text{ m/s})^{2}} = 1.69 \cdot 10^{-15} \text{ m.}$$
(54)

It is interesting to ask what happens to the mass if the charge density is kept constant but the radius is doubled. You might think that since there is $2^3 = 8$ times as much



Figure 17



Figure 18

stuff (charge) present, the mass should be 8 times as large. However, the charge e is squared in the above formula for the potential energy, so this yields a factor of $8^2 = 64$. But there is also one power of r_0 in the denominator, so this cuts the result down to 32. Equivalently, the result for the energy in Eq. (53) in the solution to Exercise 1.61 is proportional to R^5 , and $2^5 = 32$.

1.63. Sphere and cones

(a) There is no change in speed inside the shell, because the electric field is zero inside. So we just need to find the speed of the particle when it reaches the surface. The charge on the shell is $4\pi R^2 \sigma$, so the potential energy of the particle at the surface of the shell is

$$V(R) = \frac{(4\pi R^2 \sigma)(-q)}{4\pi\epsilon_0 R} = -\frac{R\sigma q}{\epsilon_0}.$$
(55)

The initial potential energy was zero, so this loss in potential energy shows up as kinetic energy. Hence, $mv^2/2 = R\sigma q/\epsilon_0 \Longrightarrow v = \sqrt{2R\sigma q/\epsilon_0 m}$.

(b) Let's find the potential energy U of the particle, due to one of the cones, when it is located at the tip of the cones. We'll slice the cone into rings and then integrate over the rings. Consider a thin ring around the cone, located at a slant distance x away from the tip. The charge in this ring is $dQ = \sigma 2\pi r dx$, where the radius r is given by $r/x = R/L \implies r = xR/L$. Every point in the ring is the same distance x from the tip, so

$$dU = \frac{dQ(-q)}{4\pi\epsilon_0 x} = -\frac{\left(\sigma 2\pi (xR/L)\,dx\right)q}{4\pi\epsilon_0 x} = -\frac{R\sigma q\,dx}{2\epsilon_0 L}\,.$$
(56)

Integrating from x = 0 to x = L simply turns the dx into an L, so we have $U = -R\sigma q/2\epsilon_0$. We need to double this because there are two cones, so we end up with the same potential energy of $-R\sigma q/\epsilon_0$ as in part (a). We therefore obtain the same speed of $v = \sqrt{2R\sigma q/\epsilon_0 m}$, independent of L.

1.64. Field between two wires

The electric field from a single wire is $\lambda/2\pi\epsilon_0 r$. Between the wires the fields from the two wires point in the same direction, so we have

$$15,000 \text{ N/C} = 2\frac{\lambda}{2\pi\epsilon_0 r} \implies \lambda = (15,000 \text{ N/C})\pi\epsilon_0 r$$
$$= (15,000 \text{ N/C})(3.14) \left(8.85 \cdot 10^{-12} \frac{\text{s}^2 \text{ C}^2}{\text{kg m}^3}\right) (1.5 \text{ m})$$
$$= 6.3 \cdot 10^{-7} \text{ C/m}. \tag{57}$$

The amount of excess charge on 1 km of the positive wire is then $(1000 \text{ m})\lambda = 6.3 \cdot 10^{-4} \text{ C}.$

1.65. Building a sheet from rods

In Fig. 19, the horizontal line represents the sheet, which extends into and out of the page (and also to the left and right). The short segment represents a rod extending into and out of the page, with small width dx. The field at point P due to the rod is $\lambda/2\pi\epsilon_0 r$, where the effective linear charge density of the rod is $\lambda = \sigma dx$. This is true because the amount of charge in a length ℓ of the rod can be written as both $\lambda \ell$ (by



definition) and $\sigma \ell dx$ (because ℓdx is the relevant area). The horizontal component of the field cancels with the horizontal component of the field arising from the rod located symmetrically on the left side of P. So (as expected) we care only about the vertical component. This brings in a factor of $\cos \theta$. And since $x = y \tan \theta$, we have $dx = y d\theta / \cos^2 \theta$. The (vertical) field at P therefore equals

$$E = \int_{-\infty}^{\infty} \frac{\sigma \, dx}{2\pi\epsilon_0 r} \cos\theta = \int_{-\pi/2}^{\pi/2} \frac{\sigma(y \, d\theta/\cos^2\theta)}{2\pi\epsilon_0(y/\cos\theta)} \cos\theta$$
$$= \frac{\sigma}{2\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} d\theta = \frac{\sigma}{2\epsilon_0} \,, \tag{58}$$

as desired. Alternatively, you can write the integral in terms of x. Since $\cos \theta = y/r$ we have

$$E = \int_{-\infty}^{\infty} \frac{\sigma \, dx}{2\pi\epsilon_0 r} \cdot \frac{y}{r} = \frac{\sigma y}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dx}{x^2 + y^2}$$
$$= \frac{\sigma y}{2\pi\epsilon_0} \cdot \frac{1}{y} \tan^{-1} \left(\frac{x}{y}\right) \Big|_{-\infty}^{\infty} = \frac{\sigma y}{2\pi\epsilon_0} \cdot \frac{\pi}{y} = \frac{\sigma}{2\epsilon_0}.$$
 (59)

1.66. Force between two strips

(a) Let's slice one of the strips into narrow rods, and then integrate over the rods. Consider a rod with width dr at a distance r from a given point P, which itself is a distance x away from the edge of the strip; see Fig. 20. The electric field at Pdue to the rod is $\lambda/2\pi\epsilon_0 r$, where $\lambda = \sigma dr$. The distance r runs from x to x + b. So the total field at P due to the strip is

$$E(x) = \int_{x}^{x+b} \frac{\sigma \, dr}{2\pi\epsilon_0 r} = \frac{\sigma}{2\pi\epsilon_0} \ln\left(\frac{x+b}{x}\right). \tag{60}$$

For $x \to 0$ this result diverges, but slowly like $\ln x$. For $x \to \infty$ we can use the Taylor series $\ln(1 + b/x) \approx b/x$, which gives $E \approx (\sigma b)/2\pi\epsilon_0 x$. This makes sense, because from far away the strip looks essentially like a rod with linear charge density σb .

(b) Consider a rod with finite height h and width dx within one of the strips. The force on this rod due to the other strip is $(\sigma h dx)E(x)$. Since the strips are right next to each other, the distance x runs from 0 to b. So the total force on a height h of one of the strips, due to the other strip, is

$$F_{h} = \int_{0}^{b} (\sigma h \, dx) E$$

$$= \frac{\sigma^{2} h}{2\pi\epsilon_{0}} \int_{0}^{b} \ln\left(\frac{x+b}{x}\right) \, dx$$

$$= \frac{\sigma^{2} h}{2\pi\epsilon_{0}} \left(\int_{0}^{b} \ln(x+b) \, dx - \int_{0}^{b} \ln x \, dx\right)$$

$$= \frac{\sigma^{2} h}{2\pi\epsilon_{0}} \left(\int_{b}^{2b} \ln y \, dy - \int_{0}^{b} \ln x \, dx\right)$$

$$= \frac{\sigma^{2} h}{2\pi\epsilon_{0}} \left((y \ln y - y)\Big|_{b}^{2b} - (x \ln x - x)\Big|_{0}^{b}\right)$$

$$= \frac{\sigma^{2} h}{2\pi\epsilon_{0}} (2b \ln 2). \tag{61}$$



Figure 20



Figure 21





You should verify the algebra leading to the last line. The force per unit height is therefore $F_h/h = \sigma^2 b(\ln 2)/\pi\epsilon_0$, which is finite. The field diverges as $x \to 0$, but only like a log. This isn't large enough to outweigh the fact that there is only a small range of x that is very close to the strip. Basically, the area under the $\ln x$ curve near x = 0 is finite.

1.67. Field from a cylindrical shell, right and wrong

(a) Let the rods be parameterized by the angle θ shown in Fig. 21. The width of a rod is $R d\theta$, so its effective charge per unit length is $\lambda = \sigma(R d\theta)$. The rod is a distance $2R \sin(\theta/2)$ from the point P in question, which is infinitesimally close to the top of the cylinder. Only the vertical component of the field from the rod survives, and this brings in a factor of $\sin(\theta/2)$, as you can check. Using the fact that the field from a rod is $\lambda/2\pi\epsilon_0 r$, we find that the field at the top of the cylinder is (apparently)

$$2\int_0^\pi \frac{\sigma R \,d\theta}{2\pi\epsilon_0 \left(2R\sin(\theta/2)\right)}\sin(\theta/2) = \frac{\sigma}{2\pi\epsilon_0}\int_0^\pi d\theta = \frac{\sigma}{2\epsilon_0}\,.$$
 (62)

Interestingly, we see that for a given angular width of a rod, all rods yield the same contribution to the vertical electric field at P.

(b) As noted in the statement of the exercise, it is no surprise that the above result is incorrect, because the same calculation would supposedly yield the field just inside the cylinder too, where it is zero instead of σ/ϵ_0 . The calculation does, however, give the next best thing, namely the average of these two values. We'll see why shortly.

The reason why the calculation is invalid is that it doesn't correctly describe the field due to rods on the cylinder very close to the given point, that is, for rods characterized by $\theta \approx 0$. It is incorrect for two reasons. The closeup view in Fig. 22 shows that the distance from a rod to the given point is *not* equal to $2R\sin(\theta/2)$. Additionally, it shows that the field does *not* point along the line from the rod to the top of the cylinder. It points more vertically, so the extra factor of $\sin(\theta/2)$ in Eq. (62) isn't valid.

What is true is that if we remove a thin strip from the top of the cylinder (so we now have a gap in the circle representing the cross sectional view), then the above integral is valid for the remaining part of the cylinder. The thin strip contributes negligibly to the $\int d\theta$ integral, so we can say that the field due to the remaining part of the cylinder is equal to the above result of $\sigma/2\pi$. By superposition, the total field due to the entire cylinder is this field of $\sigma/2\pi$ plus the field due to the thin strip. But if the point in question is infinitesimally close to the cylinder, then the thin strip looks like an infinite plane, the field of which we know is $\sigma/2\epsilon_0$. The desired total field is then

$$E_{\text{outside}} = E_{\text{cylinder minus strip}} + E_{\text{strip}} = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} \,. \tag{63}$$

By superposition we also obtain the correct field just inside the shell:

$$E_{\text{inside}} = E_{\text{cylinder minus strip}} - E_{\text{strip}} = \frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = 0.$$
(64)

The relative minus sign arises because the field from the cylinder-minus-strip is continuous across the gap, but the field from the strip is not; it points in different directions on either side of the strip.

1.68. Uniform field strength

Let Q_r be the charge inside radius r. Since Gauss's law tell us that $E_r \propto Q_r/r^2$, we therefore want $Q_r \propto r^2$ if E_r is to be constant. That is, we want

$$\int_0^r 4\pi x^2 \rho(x) \, dx \propto r^2,\tag{65}$$

for any value of r up to the radius of the sphere. We quickly see that this proportionality holds if $\rho(x) \propto 1/x$, because $\int (x^2/x) dx = x^2/2$. So this is the desired form of ρ ; it is inversely proportional to the radius. Although ρ diverges at the origin, the charge there is still finite (the amount of charge within a sphere with radius r is proportional to r^2) because of the $4\pi x^2$ in the volume element in the above integral. Note that the field right at the center isn't well defined.

Alternatively (or rather, equivalently), since Gauss's law tells us that we want $Q_r = Br^2$, for some constant B, we have $dQ_r/dr = 2Br$. But a general expression for dQ_r/dr is $4\pi r^2 \rho$ (if we imagine adding on a thin shell). Equating these two expressions for dQ_r/dr gives $\rho = B/2\pi r \propto 1/r$, as desired.

The solution for the case of a cylinder is similar. Again let $Q_{r,\ell}$ be the charge inside radius r, for a length ℓ of the cylinder. Since Gauss's law tell us that $E_r \propto Q_{r,\ell}/r\ell$, we therefore want $Q_{r,\ell} \propto r\ell$ if E_r is to be constant. That is, we want

$$\int_0^r 2\pi x \ell \rho(x) \, dx \propto r\ell,\tag{66}$$

for any value of r up to the radius of the cylinder. This proportionality holds if $\rho(x) \propto 1/x$, because $\int (x/x) dx = x$. So the answer is the same as for the sphere.

Alternatively, Gauss's law tells us that we want $Q_{r,\ell} = Br\ell$, for some constant B, so $dQ_{r,\ell}/dr = B\ell$. But $dQ_{r,\ell}/dr$ also equals $2\pi r\ell\rho$ (if we imagine adding on a thin shell). Equating these two expressions for $dQ_{r,\ell}/dr$ gives $\rho = B/2\pi r \propto 1/r$, as desired.

Interestingly, the answer changes if we kick the dimension down one more step and consider a slab. Since the field due to a charged *sheet* is uniform, we know that the magnitude of the field will be independent of position inside the slab if the slab has $\rho = 0$ everywhere except for a sheet of charge at its center plane. Mathematically, let $Q_{r,A}$ be the charge inside a sub-slab with area A and thickness 2r centered with respect to the given slab. Gauss's law tells us that we want $Q_{r,A} = BA$ for some constant B, so $dQ_{r,A}/dr = 0$. But $dQ_{r,A}/dr$ also equals $2A\rho$ (if we imagine adding on two thin surfaces on the two faces of the sub-slab). Equating these two expressions for $dQ_{r,A}/dr$ gives $\rho = 0$. This holds everywhere except at the center plane at r = 0.

The reasoning in the above three cases (indexed by n = 2, 1, 0, respectively) can be concisely summarized as follows: $Q_r = Br^n \implies dQ_r/dr = nBr^{n-1}$. But also $dQ_r/dr = kr^n\rho$. Therefore $\rho \propto n/r$. The *n* in the numerator makes it clear why the n = 0 case is different from the others.

1.69. Carved-out sphere

The given setup is equivalent to the superposition of a sphere with radius a and density ρ , plus an off-center sphere with radius a/2 and density $-\rho$. The desired fields at A and B are the sums of the fields from these two objects.

The charge in the big sphere is $Q_{\rm b} = (4/3)\pi a^3 \rho$, while the charge in the small sphere is $Q_{\rm s} = (4/3)\pi (a/2)^3 (-\rho) = -Q_{\rm b}/8$. For convenience, let the field at B due to the

big sphere be labeled E_0 . Then

$$E_0 \equiv E_{b,B} = \frac{Q_b}{4\pi\epsilon_0 a^2} = \frac{(4/3)\pi a^3 \rho}{4\pi\epsilon_0 a^2} = \frac{a\rho}{3\epsilon_0},$$
 (67)

and the field is directed downward. The field at A due to the big sphere is $E_{b,A} = 0$. The field at A due to the small (negative) sphere has magnitude

$$E_{s,A} = \frac{Q_s}{4\pi\epsilon_0 (a/2)^2} = \frac{(4/3)\pi(a/2)^3\rho}{4\pi\epsilon_0 (a/2)^2} = \frac{a\rho}{6\epsilon_0} = \frac{E_0}{2},$$
(68)

and is directed upward. The field at B due to the small sphere has magnitude

$$E_{s,B} = \frac{Q_s}{4\pi\epsilon_0 (3a/2)^2} = \frac{(4/3)\pi(a/2)^3\rho}{4\pi\epsilon_0 (3a/2)^2} = \frac{a\rho}{54\epsilon_0} = \frac{E_0}{18},$$
(69)

and is directed upward.

The total field at A is therefore directed upward with magnitude $0 + E_0/2 = a\rho/6\epsilon_0$. And the total field at B is directed downward with magnitude $E_0 - E_0/18 = 17E_0/18 = 17a\rho/54\epsilon_0$.

1.70. Field from two sheets

The field from an infinite sheet with charge density σ has magnitude $\sigma/2\epsilon_0$. It is directed away from the sheet if σ is positive, and toward it if σ is negative. The total field in the given setup equals the superposition of the fields from each sheet; the result is shown in Fig. 23. The field has magnitude $(3\sigma_0 + 2\sigma_0)/2\epsilon_0 = 5\sigma_0/2\epsilon_0$ inside the sheets and $(3\sigma_0 - 2\sigma_0)/2\epsilon_0 = \sigma_0/2\epsilon_0$ outside the sheets. In all regions it is directed away from the $3\sigma_0$ sheet.

If the sheets intersect at right angles, the field is again obtained by superposition, but now the two individual fields are orthogonal. Fig. 24 shows the results in the four regions. The magnitude of the field everywhere is $\sqrt{3^2 + 2^2}(\sigma_0/2\epsilon_0) \approx (1.8)\sigma_0/\epsilon_0$. In all regions it is directed at least partially away from the $3\sigma_0$ sheet and partially toward the $-2\sigma_0$ sheet.

1.71. Intersecting sheets

(a) The electric field from a given sheet points away from the sheet and has uniform magnitude $\sigma/2\epsilon_0$. The three fields at a given point therefore take the form shown in Fig. 25. At the location shown, the net field is directed exactly rightward and has magnitude

$$E = E_2 + (E_1 + E_3)\cos 60^\circ = \frac{\sigma}{2\epsilon_0} + 2\frac{\sigma}{2\epsilon_0}\cos 60^\circ = \frac{\sigma}{\epsilon_0}.$$
 (70)

The field has the same magnitude and direction (to the right) everywhere in the rightmost "pie piece," because the field due to a sheet doesn't depend on the distance from the sheet. Similarly, in the other five pie pieces the magnitude is $\sigma/2\epsilon_0$, and the direction is parallel to the line that bisects the angle of the pie piece.













23

- Figure 26
- (b) Fig. 26 shows the field at points on a symmetrically-located hexagon. Let the "radius" of the hexagon be r, and consider a hexagonal tube with length ℓ perpendicular to the page. The surface area of this tube is $6r\ell$, and the charge enclosed is $6r\ell\sigma$. Since the electric field is everywhere perpendicular to the surface, Gauss's law gives

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0} \implies E \cdot 6r\ell = \frac{6r\ell\sigma}{\epsilon_0} \implies E = \frac{\sigma}{\epsilon_0}, \quad (71)$$

in agreement with the result in part (a). Again, note that E is independent of r. While Gauss's law is always valid, it was actually useful in the present setup because we were able to find a simple surface that is everywhere perpendicular to the electric field (because the electric field is uniform in each pie piece).

(c) For general N, the electric field is everywhere perpendicular to a regular 2N-gon. The surface area of this 2N-gon is $(2N)(2\sin(\pi/2N))r\ell$, and the charge enclosed is $(2N)r\ell\sigma$. So Gauss's law gives

$$E \cdot (2N) \left(2\sin(\pi/2N) \right) r\ell = \frac{(2N)r\ell\sigma}{\epsilon_0} \implies E = \frac{\sigma}{2\epsilon_0 \sin(\pi/2N)} .$$
(72)

As expected, this is independent of r. And it agrees with the above result when N = 3. For large N, we have $\sin(\pi/2N) \approx \pi/2N$, so $E \approx N\sigma/\pi\epsilon_0$. In the case of large N, the sheets are very close to each other, so we effectively have a continuous volume charge distribution that depends on r. The separation between adjacent sheets grows linearly with r, so we have $\rho(r) \propto 1/r$. More precisely, you can show that $\rho(r) = N\sigma/\pi r$. This is consistent with the result from Exercise 1.68, where we found that a cylinder with a density of the form $\rho(r) \propto 1/r$ produces a field whose magnitude is independent of r (inside the cylinder).

1.72. A plane and a slab

The total effective charge per unit area (looking perpendicular to the sheet/slab) is $\sigma + \rho d$, because $\rho(Ad)$ is the charge contained within an area A of the slab. Let x = 0 be defined to be the location of the plane. Then for x < 0 the field is $E = -(\sigma + \rho d)/2\epsilon_0$, and for x > d it is $E = (\sigma + \rho d)/2\epsilon_0$. At a general point inside the slab (that is, for $0 < x \le d$), there is a charge density $\sigma + \rho x$ to the left of the point and $(d - x)\rho$ to the right. So for $0 < x \le d$ the field is

$$\frac{\sigma + \rho x}{2\epsilon_0} - \frac{(d-x)\rho}{2\epsilon_0} = \frac{\sigma - \rho d + 2\rho x}{2\epsilon_0}.$$
(73)

The plot of E as a function of x is shown in Fig. 27. E is continuous at x = d but not at x = 0. If the plane had a nonzero thickness, then the field would be continuous at x = 0. The case shown in the plot has $\rho d > \sigma$. If we instead had $\sigma > \rho d$, then at the discontinuity at x = 0, E would jump to a positive value.

1.73. Sphere in a cylinder

From the reasoning in the solution to Problem 1.27, the electric field inside a uniform cylinder is $\mathbf{E} = \rho \mathbf{r}/2\epsilon_0$, where \mathbf{r} points away from the axis. And the electric field inside a uniform sphere is $\mathbf{E} = \rho \mathbf{r}/3\epsilon_0$, where \mathbf{r} points away from the center.

The given setup may be considered to be the superposition of a uniform cylinder with density ρ and a uniform sphere with density $-3\rho/2$. This produces the desired net density of $-\rho/2$ within the sphere.





Figure 27

Consider a point inside the sphere. Let its position with respect to the axis of the cylinder be \mathbf{r}_{c} , and let its position with respect to the center of the sphere be \mathbf{r}_{s} . Then using the above forms of the fields, along with superposition, we find that the field at this point is

$$\mathbf{E} = \mathbf{E}_{\rm c} + \mathbf{E}_{\rm s} = \frac{\rho \mathbf{r}_{\rm c}}{2\epsilon_0} + \frac{(-3\rho/2)\mathbf{r}_{\rm s}}{3\epsilon_0} = \frac{\rho}{2\epsilon_0}(\mathbf{r}_{\rm c} - \mathbf{r}_{\rm s}).$$
(74)

Let's look at this vector $\mathbf{r}_{c} - \mathbf{r}_{s}$. If the given point lies in the *x-y* plane, then we have the situation shown in Fig. 28; the axis of the cylinder (the *z* axis) points out of the page. The difference $\mathbf{r}_{c} - \mathbf{r}_{s}$ is simply the vector $a\hat{\mathbf{x}}$. If the given point does *not* lie in the *x-y* plane, then \mathbf{r}_{c} will still be parallel to the *x-y* plane, but \mathbf{r}_{s} will now have a *z* component. However, this *z* component doesn't affect the *x-y* component of the difference $\mathbf{r}_{c} - \mathbf{r}_{s}$, so the *x-y* component still equals $a\hat{\mathbf{x}}$. In short, a "top" view of the setup (looking along the *z* axis) makes it clear that the projection of $\mathbf{r}_{c} - \mathbf{r}_{s}$ onto the *x-y* plane always equals $a\hat{\mathbf{x}}$.

From Eq. (74), the x-y component of the field inside the sphere therefore equals $(\rho a/2\epsilon_0)\hat{\mathbf{x}}$, which is uniform, as desired. In the special case where the sphere is centered on the z axis (so that a = 0), there is additionally zero x component, so the field points only in the z direction inside the sphere, with a magnitude dependent only on z. This dependence is linear, so a charge will undergo simple harmonic motion if its initial velocity is in the z direction.

1.74. Zero field in a sphere

Gauss's law says $\int_{S} \mathbf{E} \cdot d\mathbf{a} = Q/\epsilon_0$, where Q is the charge enclosed by the surface S. If we draw a spherical Gaussian surface with radius r inside the given sphere, Gauss's law gives

$$E \cdot 4\pi r^2 = \frac{(4\pi r^3/3)\rho}{\epsilon_0} \implies E = \frac{\rho r}{3\epsilon_0}.$$
 (75)

The full vector form of this field is $\mathbf{E} = \hat{\mathbf{r}}\rho r/3\epsilon_0$. But $r\hat{\mathbf{r}}$ is simply the vector (x, y, z), so $\mathbf{E}_{\text{sphere}} = (\rho/3\epsilon_0)(x, y, z)$.

Similarly, if we draw a cylindrical Gaussian surface with radius r and length ℓ inside the given cylinder, Gauss's law gives

$$E \cdot 2\pi r\ell = \frac{(\pi r^2 \ell)\rho}{\epsilon_0} \implies E = \frac{\rho r}{2\epsilon_0}.$$
(76)

The full vector form of this field is $\mathbf{E} = \hat{\mathbf{r}} \rho r / 2\epsilon_0$, where the $\hat{\mathbf{r}}$ vector here represents the direction away from the z axis. So $r\hat{\mathbf{r}}$ is the vector (x, y, 0). Hence $\mathbf{E}_{cyl} = (\rho/2\epsilon_0)(x, y, 0)$.

Finally, if we draw a slab Gaussian surface with area A and thickness 2z, inside the slab and centered in the slab, Gauss's law gives

$$E \cdot 2A = \frac{(2zA)\rho}{\epsilon_0} \implies E = \frac{\rho z}{\epsilon_0}.$$
 (77)

This field points in the z direction, so it is just $\mathbf{E}_{slab} = (\rho/\epsilon_0)(0, 0, z)$.

If we now give the three objects the densities ρ_1 , ρ_2 , ρ_3 , we find that the total field at a given point (x, y, z) inside all three objects (which means inside the sphere) equals

$$\mathbf{E}_{\text{total}} = \frac{1}{\epsilon_0} \left(\frac{\rho_1}{3}(x, y, z) + \frac{\rho_2}{2}(x, y, 0) + \rho_3(0, 0, z) \right).$$
(78)





y

We want this to be zero. The x and y components will equal zero if

$$\frac{\rho_1}{3} + \frac{\rho_2}{2} = 0 \implies \rho_1 = -\frac{3}{2}\rho_2.$$
(79)

And the z component will equal zero if

$$\frac{\rho_1}{3} + \rho_3 = 0 \implies \rho_1 = -3\rho_3. \tag{80}$$

These equations will be satisfied if the densities are in the ratio of $\rho_1 : \rho_2 : \rho_3 = 3 : -2 : -1$.

As a double check, the sum of these three densities (which is the net density inside the sphere) equals zero. This is correct, because if the field inside the sphere is zero, then any volume we choose inside the sphere must contain zero charge, because there is zero flux through its surface. The only way this can happen is if there is no charge anywhere inside the sphere. Hence $\rho = 0$. This is consistent with the differential form of Gauss's law, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, which we'll get to in Chapter 2; $\mathbf{E} = 0$ implies $\rho = 0$.

1.75. Ball in a sphere

(a) At a radius r inside a sphere with charge density ρ , the electric field is effectively due to the charge inside radius r. So the field is

$$E = \frac{(4\pi r^3/3)\rho}{4\pi\epsilon_0 r^2} = \frac{\rho r}{3\epsilon_0}.$$
 (81)

The field points radially outward (for positive ρ), so we can write the **E** vector compactly as $\mathbf{E} = \rho \mathbf{r}/3\epsilon_0$.

The force on the smaller ball due to the larger sphere is found by integrating the effect of the **E** field on all the pieces of the ball. This integral is easy to do if we group the pieces in pairs that are symmetrically located on either side of the center of the ball (at position **a**). Consider two pieces located at positions $\mathbf{a} + \mathbf{r'}$ and $\mathbf{a} - \mathbf{r'}$. Then because the above **E** field is *linear* in **r**, the **r'** parts of the sum cancel, and we end up with two pieces effectively located at the center of the ball. We can build up the entire ball from such pairs, so we see that the ball can be effectively treated like a point charge at its center, as far as the force from the larger sphere goes. The force is therefore the same as if it were a point charge of the same charge q.

It's no surprise that the ball acts effectively like a point charge at its center, because what we just did is basically find the average value of \mathbf{r} over the ball, which is the same thing we do when we find the center of mass of a uniform object. And the CM of a sphere is at its center. So more generally, if we have an oddly-shaped charged object instead of a nice ball (and even if it isn't uniform), and if it is located entirely within a uniform sphere of charge, then the force on it is the same as if all of its charge were located at its "center of charge."

(b) Now consider the slightly different setup where we remove the charge in the larger sphere where the ball is. This is a more realistic scenario, because it corresponds to hollowing out a cavity and then filling it up with another material. It turns out that the force on the ball doesn't change. This follows from the fact that the larger-sphere-with-cavity is the superposition of the complete larger sphere plus a sphere of negative charge density located where the ball is. So the total force on the ball is the sum of the forces from the original complete sphere plus the new sphere of negative charge. But the latter provides no force on the ball, by symmetry, because it is located in exactly the same place as the ball. The total force is therefore the same as before.

The result of this exercise is consistent with the result in Problem 1.27, which tells us that the electric field is uniform inside a spherical cavity within a uniform sphere.

1.76. Hydrogen atom

The fraction of the negative charge that lies inside a sphere of radius r_1 is

$$\frac{\int_{0}^{r_{1}} \rho \, dv}{\int_{0}^{\infty} \rho \, dv} = \frac{\int_{0}^{r_{1}} C e^{-2r/a_{0}} \, 4\pi r^{2} \, dr}{\int_{0}^{\infty} C e^{-2r/a_{0}} \, 4\pi r^{2} \, dr} = \frac{\int_{0}^{r_{1}} r^{2} e^{-2r/a_{0}} \, dr}{\int_{0}^{\infty} r^{2} e^{-2r/a_{0}} \, dr} = \frac{\int_{0}^{x_{1}} x^{2} e^{-x} \, dx}{\int_{0}^{\infty} x^{2} e^{-x} \, dx}, \tag{82}$$

where we have made the change of variables $x \equiv 2r/a_0$. Note that we don't need to know C, because it cancels. You can verify the integral:

$$\int_{0}^{x_{1}} x^{2} e^{-x} dx = -(x^{2} + 2x + 2)e^{-x}\Big|_{0}^{x_{1}} = 2 - (x_{1}^{2} + 2x_{1} + 2)e^{-x_{1}}.$$
 (83)

For $x_1 = \infty$ (that is, $r_1 = \infty$) the integral is 2. And for $x_1 = 2$ (that is, $r_1 = a_0$) the integral is $2 - 10e^{-2}$. The fraction of the full electron charge -e that lies inside radius a_0 is therefore $(2 - 10e^{-2})/2 = 1 - 5/(2.718)^2 = 0.323$. The net positive charge inside $r = a_0$ is then $q = (1 - 0.323)(1.6 \cdot 10^{-19} \text{ C}) = 1.08 \cdot 10^{-19} \text{ C}$. This field at this radius is

$$\frac{1}{4\pi\epsilon_0} \frac{q}{a_0^2} = \left(9 \cdot 10^9 \,\frac{\mathrm{kg}\,\mathrm{m}^3}{\mathrm{s}^2\,\mathrm{C}^2}\right) \frac{1.08 \cdot 10^{-19}\,\mathrm{C}}{(0.53 \cdot 10^{-10}\,\mathrm{m})^2} \approx 3.5 \cdot 10^{11}\,\mathrm{N/C},\tag{84}$$

which is huge!

1.77. Electron jelly

The force on a proton, at radius r, from the electron jelly is effectively due to the jelly that is inside radius r. The force points toward the center of the sphere. If the net force on a proton is zero, the force from the other proton must also point along the line (away) from the center. The two protons must therefore lie on the same diameter. They also must be the same distance r from the center; this is true because they feel the same force (in magnitude) from each other, so they must also feel the same force from the jelly, which implies that they must have the same value of r.

Since volume is proportional to r^3 , the negative charge inside radius r equals $-2e(r^3/a^3)$. The field at radius r due to the jelly is therefore

$$-\frac{2e(r^3/a^3)}{4\pi\epsilon_0 r^2} = -\frac{er}{2\pi\epsilon_0 a^3}.$$
 (85)

The field at one proton due to the other is $e/(4\pi\epsilon_0(2r)^2)$. So the total field at one of the protons will equal zero if

$$\frac{er}{2\pi\epsilon_0 a^3} = \frac{e}{4\pi\epsilon_0 (2r)^2} \implies r^3 = \frac{a^3}{8} \implies r = \frac{a}{2}.$$
 (86)

This factor of 1/2 is reasonably clear in retrospect. If all of the -2e electron charge were located in a point charge at the center, it would provide a force on one of the protons that is 8 times the force due to the other proton (because the other proton is twice as far away and half as big). So the forces will balance if we reduce the effective electron charge by a factor of 8. This is accomplished by reducing the effective radius of the jelly by a factor of 2.

1.78. Hole in a shell

FIRST SOLUTION: We can solve this exercise by direct integration. Let's slice up the spherical shell (minus the hole) into rings parameterized by the angle θ shown in Fig. 29. The width of a ring is $R d\theta$, and its circumferential length is $2\pi (R \sin \theta)$. So its area is $2\pi R^2 \sin \theta d\theta$. All points on the ring are a distance $2R \sin(\theta/2)$ from the center of the hole. Only the vertical component of the field survives, and this brings in a factor of $\sin(\theta/2)$, as you can check. If the edge of the hole is at the small angle $\theta = \epsilon$, the total field at the middle of the hole is (writing $\sin \theta$ as $2\sin(\theta/2)\cos(\theta/2)$)

$$\frac{1}{4\pi\epsilon_0} \int_{\epsilon}^{\pi} \frac{\sigma 2\pi R^2 \sin\theta \, d\theta}{\left(2R\sin(\theta/2)\right)^2} \sin(\theta/2) = \frac{\sigma}{4\epsilon_0} \int_{\epsilon}^{\pi} \cos(\theta/2) \, d\theta$$
$$= \frac{\sigma}{2\epsilon_0} \sin(\theta/2) \Big|_{\epsilon}^{\pi} \approx \frac{\sigma}{2\epsilon_0} \,, \tag{87}$$

in the limit where $\epsilon \to 0$.

SECOND SOLUTION: The given setup with the hole is the superposition of a complete spherical shell with density σ plus a small disk with density $-\sigma$. And very close to the center of the disk, the disk looks essentially like an infinite plane. The fields due to these two objects, at the point in question, are shown in Fig. 30. The sum of the fields at the center of the hole is therefore a field with magnitude $\sigma/2\epsilon_0$, directed radially outward. Note that we obtain an outward $\sigma/2\epsilon_0$ field independent of whether we look at a point just inside or just outside the shell; these points yield $0 + \sigma/2\epsilon_0$ or $\sigma/\epsilon_0 - \sigma/2\epsilon_0$, respectively. In other words, even though the field isn't continuous across the original complete shell or across the disk, it *is* continuous across the hole. It must be continuous, of course, because there is nothing but vacuum in the hole.

Since the field inside the complete sphere is zero, the field inside the sphere-plus-hole is exactly the same as the field due to the negative disk. The field lines due to a disk are shown in Fig. 2.12. Near the edge of the hole, the tangential component of the field diverges. But at points in the hole exactly on the (removed) surface of the sphere, the *radial* component of the field is exactly $\sigma/2\epsilon_0$, over the entire area of the hole.

THIRD SOLUTION: We can also solve this exercise by considering the force on the little disk, while it is still in place in the shell. If A is the area of the disk, then we know from Eq. (1.49) that the force on it is

$$A\sigma \frac{E_1 + E_2}{2} = A\sigma \frac{0 + \sigma/\epsilon_0}{2} = A\sigma \cdot \frac{\sigma}{2\epsilon_0}.$$
(88)

But the force on the disk equals the charge on the disk times the field at the location of the disk, due to all the *other* charge in the system (that is, the shell with the disk removed). Equation (88) therefore tells us that the (radial) field of the shell-minus-disk must be $\sigma/2\epsilon_0$, as desired.

1.79. Forces on three sheets

From Eq. (1.49) the force per unit area on a sheet is $(E_1 + E_2)\sigma/2$, where E_1 and E_2 are the electric fields on either side. The fields in the various regions can be found by the superpositions of the fields from the individual sheets, using the fact that the field due to a given sheet is $\sigma/2\epsilon_0$. With $\sigma_0 \equiv 10^{-5} \,\mathrm{C/m^2}$, the fields in the two middle regions are $4\sigma_0/\epsilon_0$ upward and $3\sigma_0/\epsilon_0$ downward, as shown in Fig. 31. Above and below all three plates the field is zero. The forces per unit area on the three sheets





Figure 31

hole

R



are therefore

$$F_{A} = \frac{1}{2}(E_{1} + E_{2})\sigma = \frac{1}{2}\left(0 + \frac{4\sigma_{0}}{\epsilon_{0}}\right)(-4\sigma_{0}) = -\frac{8\sigma_{0}^{2}}{\epsilon_{0}} = -90.4 \frac{N}{m^{2}},$$

$$F_{B} = \frac{1}{2}(E_{1} + E_{2})\sigma = \frac{1}{2}\left(\frac{4\sigma_{0}}{\epsilon_{0}} - \frac{3\sigma_{0}}{\epsilon_{0}}\right)(7\sigma_{0}) = \frac{7\sigma_{0}^{2}}{2\epsilon_{0}} = 39.5 \frac{N}{m^{2}},$$

$$F_{C} = \frac{1}{2}(E_{1} + E_{2})\sigma = \frac{1}{2}\left(0 - \frac{3\sigma_{0}}{\epsilon_{0}}\right)(-3\sigma_{0}) = \frac{9\sigma_{0}^{2}}{2\epsilon_{0}} = 50.8 \frac{N}{m^{2}}.$$
(89)

The sum of these forces per area is $(\sigma_0^2/\epsilon_0)(-8 + 7/2 + 9/2)$. This equals zero as it must, because a system can't exert a net force on itself (otherwise momentum wouldn't be conserved).

Alternatively, we can find the forces by calculating the field at the location of a given plate due to the *other* two plates. For example, the bottom two plates produce a field of $(7-3)\sigma_0/2\epsilon_0 = 2\sigma_0/\epsilon_0$ at the location of the top plate, which therefore feels a force per unit area equal to $(2\sigma_0/\epsilon_0)(-4\sigma_0) = -8\sigma_0^2/\epsilon_0$, as above. The $(E_1 + E_2)/2$ averages in Eq. (89) are simply a way of finding the field at the location of one sheet due to the others.

1.80. Force in a soap bubble

The field just outside the balloon is $E = Q/4\pi\epsilon_0 R^2$, and the field inside is zero. So the average field at the surface is E/2. The charge density is $\sigma = Q/4\pi R^2$. From Eq. (1.49), the force per unit area (that is, the pressure) is therefore

$$P \equiv \frac{F}{A} = \sigma \frac{E}{2} = \frac{Q}{4\pi R^2} \cdot \frac{1}{2} \frac{Q}{4\pi\epsilon_0 R^2} = \frac{Q^2}{32\pi^2\epsilon_0 R^4} \,. \tag{90}$$

Consider the two hemispheres defined by the horizontal great circle. The horizontal components of the forces on the different parts of the upper hemisphere cancel by symmetry, so we care only about the vertical components. The vertical component of the force on a little patch with area A is $PA\cos\theta$, where θ is the angle that the plane of the patch makes with the horizontal. If we write this as $P(A\cos\theta)$, we see that the vertical force on a patch equals P times the projection of the area of the patch onto the horizontal plane. Since P is constant, and since the sum of all the projections of the patches in a hemisphere is simply the great-circle area πR^2 , we find that the total upward force on the hemisphere is

$$P \cdot \pi R^2 = \frac{Q^2}{32\pi\epsilon_0 R^2} \,. \tag{91}$$

By comparison with Coulomb's force law, this has the correct units of $(\text{charge})^2/[\epsilon_0 \cdot (\text{length})^2]$. It makes sense that it grows with Q and decreases with R.

If you don't want to use the above reasoning involving the projection, you can do an integral. Slice the sphere into rings, with θ being the angle of a ring down from the top of the sphere. The area of a ring is $da = 2\pi (R \sin \theta) (R d\theta)$, and the vertical component of the force on the ring is $(P da) \cos \theta$. Integrating from $\theta = 0$ to $\theta = \pi/2$ gives the total vertical force on the upper hemisphere as

$$\int_{0}^{\pi/2} (P \, da) \cos \theta = \int_{0}^{\pi/2} \left(\frac{Q^2}{32\pi^2 \epsilon_0 R^4} \right) (2\pi R^2 \sin \theta \, d\theta) \cos \theta$$
$$= \frac{Q^2}{16\pi \epsilon_0 R^2} \int_{0}^{\pi/2} \sin \theta \cos \theta \, d\theta$$
$$= \frac{Q^2}{16\pi \epsilon_0 R^2} \frac{\sin^2 \theta}{2} \Big|_{0}^{\pi/2} = \frac{Q^2}{32\pi \epsilon_0 R^2}, \qquad (92)$$

in agreement with the above result.

1.81. Energy around a sphere

The energy density is $\epsilon_0 E^2/2$, where $E = Q/4\pi\epsilon_0 r^2$. The field is zero inside the sphere of radius R, so the energy contained within a sphere of radius R_1 is

$$\int_{R}^{R_{1}} \frac{\epsilon_{0} E^{2}}{2} 4\pi r^{2} dr = \frac{Q^{2}}{8\pi\epsilon_{0}} \int_{R}^{R_{1}} \frac{dr}{r^{2}} = \frac{Q^{2}}{8\pi\epsilon_{0}} \left(\frac{1}{R} - \frac{1}{R_{1}}\right) = \frac{Q^{2}}{8\pi\epsilon_{0}} \frac{1}{R} \left(1 - \frac{R}{R_{1}}\right).$$
(93)

The total amount of energy out to infinity is obtained by setting $R_1 = \infty$, which gives $Q^2/8\pi\epsilon_0 R$. We therefore want the factor $(1 - R/R_1)$ to equal 9/10, which implies that $R_1 = 10R$.

1.82. Energy of concentric shells

(a) The field is nonzero only in the region a < r < b, where it equals $E = Q/4\pi\epsilon_0 r^2$. The total energy is therefore

$$U = \int_{a}^{b} \frac{\epsilon_{0} E^{2}}{2} dv = \frac{\epsilon_{0}}{2} \left(\frac{Q}{4\pi\epsilon_{0}}\right)^{2} \int_{a}^{b} \frac{1}{r^{4}} 4\pi r^{2} dr = \frac{Q^{2}}{8\pi\epsilon_{0}} \left(\frac{1}{a} - \frac{1}{b}\right).$$
(94)

If b = a then U = 0, of course, because the shells are right on top of each other, so the charges cancel and we effectively have no charge anywhere in the system. If $b \to \infty$ then $U = Q^2/8\pi\epsilon_0 a$, which is correctly the energy of a single shell with charge Q (see Problem 1.32). If $a \to 0$ then U correctly goes to infinity, because the field diverges (sufficiently quickly) near a point charge. Equivalently, it takes an infinite amount of energy to compress a given amount of charge down to a point.

The result in Eq. (94) can be interpreted as follows. As mentioned above, the energy stored in a system consisting of one spherical shell of radius r is $Q^2/8\pi\epsilon_0 r$. Given this result, consider building up the present two-sphere system from scratch (that is, by bringing charges in from infinity) in two steps. It takes an energy $Q^2/8\pi\epsilon_0 a$ to construct the shell of radius a. Then, with that shell in place, it takes an energy $Q^2/8\pi\epsilon_0 b - Q^2/4\pi\epsilon_0 b$ to construct the outer shell of radius b. The first term comes from the self energy of this outer shell. The second term comes from the potential energy of the negative outer shell due to the positive inner shell already in place (which acts like a point charge at its center). The sum of the energies of these two steps yields the result in Eq. (94).

(b) Now let's imagine starting with two neutral shells and then gradually transferring positive charge from the outer shell to the inner shell. At the start, there is no electric field between the shells, so it takes no work to transfer an initial bit of charge dq. But as more charge is piled onto the inner shell, the field grows, and it takes more work to bring in the successive bits dq.

At a moment when there is charge q on the inner shell, the field between the shells is $q/4\pi\epsilon_0 r^2$, so the force on a little charge dq is $q dq/4\pi\epsilon_0 r^2$. The work you must do on this dq is the integral of your force times the displacement, or

$$dW = \int_{b}^{a} \frac{-q \, dq}{4\pi\epsilon_0 r^2} \, dr = \frac{q \, dq}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b}\right),\tag{95}$$

where we have included the minus sign in the force because your force points inward (it is opposite to the electric force). However, you can always put the sign in by hand at the end; you certainly have to do positive work to move the positive charge dq toward the positively charged inner shell.

We must now integrate the above work dW over all the bits dq that we bring in. This gives

$$W = \int_0^Q \frac{q \, dq}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q^2}{8\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b}\right),\tag{96}$$

in agreement with the result in part (a). It may seem mysterious that the potential energy of a system can be found by integrating $\epsilon_0 E^2/2$ over the volume. But the agreement of the two above methods, applied to our setup involving two shells, should help convince you that the $\epsilon_0 E^2/2$ method does indeed give the energy.

1.83. Potential energy of a cylinder

The electric field at radius r outside the given cylinder equals $\lambda/2\pi\epsilon_0 r$, where $\lambda = \pi a^2 \rho$. Consider a length ℓ of the cylinder. The energy stored in the external field within this length, out to a radius R, equals

$$U_{\text{ext}} = \frac{\epsilon_0}{2} \int_a^R \left(\frac{\pi a^2 \rho}{2\pi\epsilon_0 r}\right)^2 (2\pi r \, dr)\ell = \frac{\pi \rho^2 a^4 \ell}{4\epsilon_0} \int_a^R \frac{dr}{r} = \frac{\pi \rho^2 a^4 \ell}{4\epsilon_0} \ln\left(\frac{R}{a}\right). \tag{97}$$

The field at radius r inside the cylinder is due only to the charge inside radius r. This charge has linear density $\pi r^2 \rho$, so the field equals $(\pi r^2 \rho)/2\pi\epsilon_0 r = \rho r/2\epsilon_0$. The energy stored in the internal field, within a length ℓ , is then

$$U_{\rm int} = \frac{\epsilon_0}{2} \int_0^a \left(\frac{\rho r}{2\epsilon_0}\right)^2 (2\pi r \, dr)\ell = \frac{\pi \rho^2 \ell}{4\epsilon_0} \int_0^a r^3 \, dr = \frac{\pi \rho^2 a^4 \ell}{16\epsilon_0} \,. \tag{98}$$

Finding the energy per unit length simply involves erasing the ℓ . Using $\rho = \lambda/\pi a^2$, we can write the sum of U_{ext} and U_{int} , per unit length, as $(\lambda^2/4\pi\epsilon_0)(\ln(R/a) + 1/4)$, as desired. As mentioned in the statement of the exercise, this diverges as $R \to \infty$. It also diverges as $a \to 0$.

Chapter 2

The electric potential

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

2.31. Finding the potential

The line integral along the first path is (we'll suppress the z component of the argument)

$$\int_{(0,0)}^{(x_1,y_1)} \mathbf{E} \cdot d\mathbf{s} = \int_0^{x_1} E_x(x,0) \, dx + \int_0^{y_1} E_y(x_1,y) \, dy$$
$$= 0 + \int_0^{y_1} (3x_1^2 - 3y^2) \, dy = 3x_1^2 y_1 - y_1^3. \tag{99}$$

The line integral along the second path is

$$\int_{(0,0)}^{(x_1,y_1)} \mathbf{E} \cdot d\mathbf{s} = \int_0^{y_1} E_y(0,y) \, dy + \int_0^{x_1} E_x(x,y_1) \, dx$$
$$= \int_0^{y_1} (0-3y^2) \, dy + \int_0^{x_1} 6xy_1 \, dx = -y_1^3 + 3x_1^2y_1. \quad (100)$$

These two results are equal, as desired. The electric potential ϕ , if taken to be zero at (0,0), is just the negative of our result, because we define ϕ by $\phi = -\int \mathbf{E} \cdot d\mathbf{s}$, or equivalently $\mathbf{E} = -\nabla \phi$. Hence $\phi(x, y) = y^3 - 3x^2y$. The negative gradient of this is

$$-\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (6xy, 3x^2 - 3y^2, 0), \tag{101}$$

which does indeed equal the given \mathbf{E} .

An alternative method of finding ϕ is to integrate the components of **E** to find the general form that ϕ must take. Since $-\partial \phi/\partial x$ equals $E_x = 6xy$, we see that $-\phi$ must take the form of $3x^2y + f(y, z)$, where f(y, z) is an arbitrary function of y and z. Likewise, since $-\partial \phi/\partial y$ equals $E_y = 3x^2 - 3y^2$, we see that $-\phi$ must take the form of $3x^2y - y^3 + g(x, z)$. Finally, since $-\partial \phi/\partial z$ equals $E_z = 0$, we see that $-\phi$ must take the form of 0 + h(x, y), that is, ϕ is a function of only x and y. The only function consistent with all three of these forms is $-\phi = 3x^2y - y^3$ (plus a constant), in agreement with the above result.

D

32

Figure 32

2.32. Line integral the easy way

The charges are shown in Fig. 32. The line integral of \mathbf{E} equals the negative of the change in potential. The potentials at C and D are

$$\phi_C = \frac{1}{4\pi\epsilon_0} \left(-\frac{3q}{\ell} + \frac{2q}{\sqrt{2\ell}} \right) = \frac{q}{4\pi\epsilon_0\ell} (-1.586),$$

$$\phi_D = \frac{1}{4\pi\epsilon_0} \left(\frac{2q}{\ell} - \frac{3q}{\sqrt{2\ell}} \right) = \frac{q}{4\pi\epsilon_0\ell} (-0.121).$$
(102)

So the line integral from C to D equals

$$\int_{C}^{D} \mathbf{E} \cdot d\mathbf{s} = -(\phi_{D} - \phi_{C}) = \phi_{C} - \phi_{D} = \frac{q}{4\pi\epsilon_{0}\ell}(-1.464).$$
(103)

With the given values of q and ℓ , this becomes

$$\frac{1}{4\pi\epsilon_0} \left(\frac{10^{-9} \text{C}}{0.05 \,\text{m}}\right) (-1.464) = -264 \,\text{V}.$$
(104)

The negative result makes sense, because the field between C and D points at least partly upward, while the ds in the line integral from C to D points downward.

If you actually want to calculate the line integral, the y component of the field, as a function of y, is (taking the origin to be at the lower left corner, and ignoring the $4\pi\epsilon_0$'s):

$$E_y = \frac{2q}{y^2 + \ell^2} \cdot \frac{y}{\sqrt{y^2 + \ell^2}} + \frac{3q}{(\ell - y)^2 + \ell^2} \cdot \frac{\ell - y}{\sqrt{(\ell - y)^2 + \ell^2}}.$$
 (105)

The integral $\int_{C}^{D} \mathbf{E} \cdot d\mathbf{s} = \int_{\ell}^{0} E_{y} dy$ is readily calculated, and you can show that the result is $\phi_{C} - \phi_{D}$, where these ϕ 's are given in Eq. (102).

2.33. Plot the potential

The potential as a function of z is

$$\phi(z) = \frac{1}{4\pi\epsilon_0} \left(\frac{12\,\mathrm{C}}{|z|} - \frac{6\,\mathrm{C}}{|z-(3\,\mathrm{m})|} \right) \implies 4\pi\epsilon_0 \phi = \frac{12}{|z|} - \frac{6}{|z-3|}, \quad (106)$$

where we have ignored the units (which are C/m) in the second expression. The plot of $4\pi\epsilon_0\phi$ is shown in Fig. 33. ϕ goes to $+\infty$ at z=0, and $-\infty$ at z=3. You can quickly verify that $\phi = 0$ at z = 2 and z = 6. You can also show that ϕ achieves a local maximum of $\phi \approx (0.34)/4\pi\epsilon_0$ at $z = 6 + 3\sqrt{2} \approx 10.24$. Since $E_z = -\partial \phi/\partial z$, the field is zero at z = 10.24, so a charge placed there will be in equilibrium (unstable with regard to motion in the z direction if the charge is positive, stable if it is negative).

2.34. Extremum of ϕ

By symmetry, the **E** field at points on the y axis has no x or z component. And we know that E_y equals $-\partial \phi / \partial y$. So if ϕ has a local maximum or minimum at some point on the y axis, then $\partial \phi / \partial y$, and hence E_y , equals zero. The full vector **E** therefore also equals zero.

At the point (0, y, 0) with y > 1, the E_y component equals (ignoring the factor of $1/4\pi\epsilon_0$, along with the units of the various quantities)

$$E_y = \frac{2}{y^2} - 2 \cdot \frac{1}{(y-1)^2 + 1^2} \cdot \frac{y-1}{\sqrt{(y-1)^2 + 1^2}},$$
(107)



Figure 33

where the last factor gives the y component of the field from the two negative charges. Setting $E_y = 0$, moving one of the terms to the other side of the equation, and squaring, we find

$$y^{4} = \frac{(y^{2} - 2y + 2)^{3}}{(y - 1)^{2}}.$$
(108)

Another way of obtaining this relation is to (as you can check) write down the potential (again ignoring the factor of $1/4\pi\epsilon_0$ and units),

$$\phi(0, y, 0) = \frac{2}{y} - 2 \cdot \frac{1}{\sqrt{(y-1)^2 + 1^2}},$$
(109)

and then set $\partial \phi / \partial y = 0$. The result is Eq. (108), of course, because $E_y = -\partial \phi / \partial y$. We can solve Eq. (108) numerically; *Mathematica* gives the numerical result of y = 1.621. Plots of $\phi(y)$ and $E_y(y)$ (times $4\pi\epsilon_0$) for points on the y axis are shown in Fig. 34. For large y, we know that both ϕ and E_y must be negative, because for large y we have a charge 2 C at a distance y, and two -1 C charges at a distance essentially equal to

y-1. So the negative charges win out.

You can show that E_y reaches a maximum negative value at y = 2.153; you will again need to solve an equation numerically. The existence of such a point between y = 1.621and $y = \infty$ follows from a continuity argument similar to the one involving ϕ : Since $E_y = 0$ at these two points, E_y must have a local maximum or minimum somewhere between.

2.35. Center vs. corner of a square

Dimensional analysis tells us that for a given charge density σ , the potential at the center of a square of side s must be proportional to Q/s, where Q is the total charge, σs^2 . (This is true because the potential has the units of $q/4\pi\epsilon_0 r$, and Q is the only charge in the setup, and s is the only length scale.) Hence ϕ is proportional to $\sigma s^2/s = \sigma s$. So for fixed σ it is proportional to s.

Equivalently, if we imagine increasing the size of the square by a factor f in each direction, the integral $\phi \propto \int (\sigma da)/r$ picks up a factor of f^2 in the da and a factor of f in the r, yielding a net factor of f in the numerator.

Said in yet another (equivalent) way, if we imagine increasing the size of the square by a factor f in each direction, and if we look at corresponding pieces of the small and large versions, then the piece in the large version has f^2 times as much charge, but it is f times as far away from a given point, so its contribution to the potential is $f^2/f = f$ times the contribution in the small version.

If we assemble 4 squares of side b, we make a square of side 2b. The potential at the center is $4\phi_1$ (the sum of 4 corner potentials of the side b square). But from the above reasoning that $\phi \propto s$, this potential of $4\phi_1$ must also be 2 times the center potential of the side-b square. So we have $4\phi_1 = 2\phi_0$, or $\phi_0 = 2\phi_1$. Hence the desired ratio is 2.

It therefore takes twice as much work to bring a charge in from infinity to the center, as it does to a corner. From Eqs. (2.25) and (2.30), the analogous statement for a disk of charge is that it takes $\pi/2$ as much work to bring a charge in from infinity to the center, as it does to the edge. But that problem can't be solved via the above scaling/superposition argument.

The above result for the square actually holds more generally for any rectangle with uniform charge density. All of the steps in the above logic are still valid, so the potential at the center is twice the potential at a corner.



Figure 34

2.36. Escaping a cube, toward an edge

Let the cube have side length 2ℓ . Then the center is a distance $\sqrt{3}\ell$ from each corner. So the potential at the center (ignoring the factor of $e/4\pi\epsilon_0\ell$) is $8/\sqrt{3} = 4.6188$. Let's calculate the potential as a function of x, where x is the distance from the center to the midpoint of an edge (at which point $x = \sqrt{2}\ell$). There are three different distances involved. You can obtain these by considering the plane that contains four corners and also the displacement x, which yields two distances each of $\sqrt{1 + (\sqrt{2} \pm x)^2}$; and also the plane that contains four corners and is perpendicular to the displacement x, which yields four distances of $\sqrt{3 + x^2}$. The potential is therefore (ignoring the $e/4\pi\epsilon_0\ell$)

$$\phi(x) = \frac{2}{\sqrt{1 + (\sqrt{2} - x)^2}} + \frac{2}{\sqrt{1 + (\sqrt{2} + x)^2}} + \frac{4}{\sqrt{3 + x^2}}.$$
 (110)

As a double check, this does equal $8/\sqrt{3}$ when x = 0. Plugging in $x = \sqrt{2}$ gives the potential at the midpoint of an edge as 4.4555. Although this is smaller than the 4.6188 potential at the center, the plot in Fig. 35 shows that ϕ achieves a maximum value about halfway out to the edge. The maximum happens to be located at x = 0.761, and the value is $\phi_{\text{max}} = 4.6242$. Since this is larger than the value of ϕ at the center, the proton will *not* escape if it heads directly toward the midpoint of an edge.

2.37. Field on the earth

The radius of the earth is $r = 6.4 \cdot 10^6$ m, so we have

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} = \left(9 \cdot 10^9 \,\frac{\text{kg m}^3}{\text{s}^2 \,\text{C}^2}\right) \frac{1 \,\text{C}}{(6.4 \cdot 10^6 \,\text{m})^2} = 2.2 \cdot 10^{-4} \,\frac{\text{V}}{\text{m}} \,,$$

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} = \left(9 \cdot 10^9 \,\frac{\text{kg m}^3}{\text{s}^2 \,\text{C}^2}\right) \frac{1 \,\text{C}}{(6.4 \cdot 10^6 \,\text{m})} = 1400 \,\text{V}. \tag{111}$$

This value of 1400 V is larger than you might think it should be, given the large radius of the earth. The point is that a coulomb is a large quantity of charge, on an everyday scale. Or equivalently ϵ_0 has a small value in the system of units we use.

2.38. Interstellar dust

The potential is

$$\frac{Q}{4\pi\epsilon_0 R} = -0.15 \text{ V} \implies Q = (-0.15 \text{ V})4\pi \left(8.85 \cdot 10^{-12} \frac{\text{s}^2 \text{ C}^2}{\text{kg m}^3}\right) (3 \cdot 10^{-7} \text{ m})$$
$$= -5 \cdot 10^{-18} \text{ C}. \tag{112}$$

The charge of an electron is $-1.6 \cdot 10^{-19}$ C, so Q corresponds to $n = (5 \cdot 10^{-18} \text{ C})/(1.6 \cdot 10^{-19} \text{ C}) = 31$ electrons. The field at the surface is $Q/4\pi\epsilon_0 R^2$. We could plug in the value of Q we just found, or we could just realize that

$$E = \frac{(Q/4\pi\epsilon_0 R)}{R} = \frac{\phi}{R} = \frac{-0.15 \text{ V}}{3 \cdot 10^{-7} \text{ m}} = -5 \cdot 10^5 \frac{\text{V}}{\text{m}}, \qquad (113)$$

which is a rather large field. Basically, the small R matters more in E than in ϕ , because it is squared in E.

Interestingly, note that the relation $\phi = ER$ says that it takes the same amount of work to drag a test charge out to infinity from the surface of a sphere, as it takes to drag the charge a distance R at full field strength (the value at the surface).




2.39. Closest approach

The size of the electron cloud around the nucleus is much larger than the nucleus. This implies that the potential associated with the cloud is much smaller than the potential associated with the nucleus (due to the 1/r factor). We may therefore ignore the potential due to the electron cloud in the following calculation.

Closest approach is reached in a head-on approach, because in that case the proton ends up instantaneously at rest, which means that all of the initial kinetic energy has been converted into potential energy. The initial kinetic energy is of the proton is eV_0 , where $V_0 = 5 \cdot 10^6$ V. The potential energy at closest approach is $e(47e)/4\pi\epsilon_0 r$. These are equal when

$$V_0 = \frac{47e}{4\pi\epsilon_0 r} \implies r = \frac{47e}{4\pi\epsilon_0 V_0} = \frac{47(1.6\cdot10^{-19}\,\mathrm{C})}{(1.11\cdot10^{-10}\,\frac{\mathrm{s}^2\,\mathrm{C}^2}{\mathrm{kg\,m^3}})(5\cdot10^6\,\mathrm{V})} = 1.35\cdot10^{-14}\,\mathrm{m}.$$
(114)

This is larger than the radius of the silver nucleus, which is about $5 \cdot 10^{-15}$ m, so it was reasonable to consider coulomb repulsion only. The strength of the electric field at this position is

$$E = \frac{47e}{4\pi\epsilon_0 r^2} = \frac{(47e/4\pi\epsilon_0 r)}{r} = \frac{V_0}{r} = \frac{5\cdot 10^6 \,\mathrm{V}}{1.35\cdot 10^{-14} \,\mathrm{m}} = 3.7\cdot 10^{20} \,\mathrm{V/m},\tag{115}$$

which is huge. The acceleration of the proton at this position is

$$a = \frac{F}{m} = \frac{eE}{m} = \frac{(1.6 \cdot 10^{-19} \,\mathrm{C})(3.7 \cdot 10^{20} \,\mathrm{V/m})}{1.67 \cdot 10^{-27} \,\mathrm{kg}} = 3.5 \cdot 10^{28} \,\mathrm{m/s^2}, \tag{116}$$

which is gigantic.

2.40. Gold potential

Since volume is proportional to r^3 , the amount of charge inside radius r is $Q(r^3/a^3)$. The field at radius r is effectively due to the charge inside r, so for $r \leq a$ the field is

$$E = \frac{1}{4\pi\epsilon_0} \frac{Qr^3/a^3}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{Qr}{a^3}.$$
 (117)

Outside the sphere the field is simply $Q/4\pi\epsilon_0 r^2$. The potential at the surface of the sphere relative to infinity is

$$\phi(a) - \phi(\infty) = -\int_{\infty}^{a} E \, dr = -\int_{\infty}^{a} \frac{Q \, dr}{4\pi\epsilon_0 r^2} = \frac{1}{4\pi\epsilon_0} \frac{Q}{a} \,, \tag{118}$$

and the potential at the center of the sphere relative to the surface is

$$\phi(0) - \phi(a) = -\int_{a}^{0} E \, dr = -\int_{a}^{0} \frac{1}{4\pi\epsilon_{0}} \frac{Qr \, dr}{a^{3}} = \frac{1}{4\pi\epsilon_{0}} \frac{Q}{2a} \,. \tag{119}$$

Adding the two preceding equations gives $\phi(0) - \phi(\infty) = (1/4\pi\epsilon_0)(3Q/2a)$. For the problem at hand, this yields (assuming $\phi(\infty) = 0$ as usual)

$$\phi(0) = \frac{1}{4\pi\epsilon_0} \frac{3Q}{2a} = \left(9 \cdot 10^9 \,\frac{\mathrm{kg}\,\mathrm{m}^3}{\mathrm{s}^2\,\mathrm{C}^2}\right) \frac{3(79 \cdot 1.6 \cdot 10^{-19}\,\mathrm{C})}{2(6 \cdot 10^{-15}\,\mathrm{m})} = 2.84 \cdot 10^7\,\mathrm{V},\tag{120}$$

or 28.4 megavolts.

2.41. A sphere between planes

The electric field due to the sheets is nonzero only between them, so the field to the right of the system is due only to the sphere. The potential at the point where the sphere touches the right sheet is therefore $Q_{\rm sphere}/4\pi\epsilon_0 R = 4\pi R^2\sigma/4\pi\epsilon_0 R = \sigma R/\epsilon_0$.

The sphere produces no internal field, so the field in its interior is due only to the sheets. It therefore takes on the constant value of σ/ϵ_0 , pointing to the left. The potential difference between the surface of the sphere and its center is then $-(\sigma/\epsilon_0)R$, with the center at a lower potential. The total potential at the center, relative to $x = +\infty$, is therefore $\sigma R/\epsilon_0 - \sigma R/\epsilon_0 = 0$.

The potential at the point where the sphere touches the left sheet, relative to the center of the sphere, is $-(\sigma/\epsilon_0)R$. And the potential at $x = -\infty$, relative to the contact point on the left sheet, is $-4\pi R^2 \sigma/4\pi\epsilon_0 R = -\sigma R/\epsilon_0$, due to the sphere's field. The potential at $x = -\infty$, relative to the center (which we found has the same potential as $x = +\infty$) is therefore $-2\sigma R/\epsilon_0$.

As far as the potential at $x = -\infty$ goes, there was actually no need to make any mention of the sphere. We could have chosen a path from $x = +\infty$ to $x = -\infty$ that goes nowhere near the sphere, in which case the potential simply decreases by $-(\sigma/\epsilon_0)(2R)$ due to the field between the sheets. In any event, for any path, the potential due to the sphere has equal values at $x = \pm \infty$, so it provides no net difference in the potential at these points. But the sheets do provide a net difference, because there is no way to get from $x = +\infty$ to $x = -\infty$ without passing through the (infinite) sheets. For compact charge distributions (that is, ones that don't extend to infinity), it is possible to set $\phi = 0$ at all points at infinity. But if a charge distribution extends to infinity (as in the present setup), then all points at infinity are not equivalent; we cannot consistently assign them all the value of $\phi = 0$.

2.42. E and ϕ for a cylinder

(a) Consider a coaxial cylinder with length ℓ and radius r < a. The charge contained inside is $\pi r^2 \ell \rho$. The area of the cylindrical part of the surface is $2\pi r \ell$, and since **E** is perpendicular to the surface by symmetry, the flux is $2\pi r \ell E$. So Gauss's law gives the internal electric field as

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0} \implies 2\pi r \ell E = \frac{\pi r^2 \ell \rho}{\epsilon_0} \implies E = \frac{\rho r}{2\epsilon_0} \quad \text{(for } r < a\text{)}. \quad (121)$$

We'll also need the external field for part (b). For this field, consider a cylinder of radius r > a. This contains a fixed amount of charge $\pi a^2 \ell \rho$, so Gauss's law gives

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0} \implies 2\pi r \ell E = \frac{\pi a^2 \ell \rho}{\epsilon_0} \implies E = \frac{\rho a^2}{2\epsilon_0 r} \qquad \text{(for } r > a\text{).}$$
(122)

This is the same as the field from a line of charge (namely $\lambda/2\pi\epsilon_0 r$) with linear density $\lambda = \pi a^2 \rho$. Note that the internal and external fields agree at r = a.

(b) If $\phi = 0$ at r = 0, then we have

For
$$r < a$$
: $\phi(r) = -\int_{0}^{r} E \, dr = -\int_{0}^{r} \frac{\rho r \, dr}{2\epsilon_{0}} = -\frac{\rho r^{2}}{4\epsilon_{0}}$,
For $r > a$: $\phi(r) = -\int_{0}^{a} E \, dr - \int_{a}^{r} E \, dr$
 $= -\frac{\rho a^{2}}{4\epsilon_{0}} - \int_{a}^{r} \frac{\rho a^{2} \, dr}{2\epsilon_{0}r} = -\frac{\rho a^{2}}{4\epsilon_{0}} - \frac{\rho a^{2}}{2\epsilon_{0}} \ln(r/a)$. (123)

This goes to $-\infty$ as $r \to \infty$. It also goes to $-\infty$ for any given value of r if $a \to 0$ while the charge per unit length $(\pi a^2 \rho)$ is held constant.

2.43. Potential from a rod

At point P_1 in Fig. 36 we have

$$\phi_1 = \frac{1}{4\pi\epsilon_0} \int_{-d}^d \frac{\lambda \, dz}{2d-z} = -\frac{\lambda}{4\pi\epsilon_0} \ln(2d-z) \Big|_{-d}^d = -\frac{\lambda}{4\pi\epsilon_0} \ln \frac{d}{3d} = \frac{\lambda}{4\pi\epsilon_0} \ln 3.$$
(124)

At point P_2 with a general x value, we have (using the integral table in Appendix K)

$$\phi_2 = \frac{1}{4\pi\epsilon_0} \int_{-d}^{d} \frac{\lambda \, dz}{\sqrt{x^2 + z^2}} = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\sqrt{x^2 + z^2} + z\right) \Big|_{-d}^{d} = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{\sqrt{x^2 + d^2} + d}{\sqrt{x^2 + d^2} - d}\right). \tag{125}$$

These two potentials are equal when

$$\frac{\sqrt{x^2 + d^2} + d}{\sqrt{x^2 + d^2} - d} = 3 \implies 4d = 2\sqrt{x^2 + d^2} \implies x = \sqrt{3} d. \tag{126}$$

2.44. Ellipse potentials

Let z' specify the location of a point on the rod. Then the potential at the general point (x, 0, z) is (you should verify this integral by differentiating it)

$$\phi = \frac{1}{4\pi\epsilon_0} \int_{-d}^{d} \frac{\lambda \, dz'}{\sqrt{x^2 + (z'-z)^2}} = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\sqrt{x^2 + (z'-z)^2} + (z'-z)\right)\Big|_{-d}^{d}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{\sqrt{x^2 + (d-z)^2} + d-z}{\sqrt{x^2 + (d-z)^2} - d-z}\right). \quad (127)$$

You can verify that if x = 3d/2 and z = d, this equals $(\lambda/4\pi\epsilon_0) \ln 3$, which is the same as the value of ϕ in Exercise 2.43. And the sum of the distances from (3d/2, 0, d) to the ends of the rod at (0, 0, d) and (0, 0, -d) equals 3d/2 + 5d/2 = 4d, which is the same as the sum of the distances for each of the points in Exercise 2.43. So this is all consistent with the equipotential curve in the x-z plane being an ellipse.

If x and z lie on the ellipse described by $x^2/(a^2 - d^2) + z^2/a^2 = 1$, then solving for x^2 gives $x^2 = (a^2 - z^2)(a^2 - d^2)/a^2$. Using this, you can show that the quantity under the square root in Eq. (127) can be written as

$$x^{2} + (d \pm z)^{2} = \left(\frac{a^{2} \pm zd}{a}\right)^{2}.$$
 (128)

We therefore have

$$\phi = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{a^2 - zd + a(d-z)}{a^2 + zd - a(d+z)}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{(a-z)(a+d)}{(a-z)(a-d)}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{a+d}{a-d}\right).$$
(129)

As desired, this result is independent of z and x, so ϕ is constant on the ellipse described by $x^2/(a^2 - d^2) + z^2/a^2 = 1$. The ellipse containing the point (3d/2, 0, d), along with the two points from Exercise 2.43, has its a value (which is the length of the major axis, which lies along the z axis) equal to 2d.



2.45. A stick and a point charge

(a) A little piece dx of the stick at position x (where x is negative) is a distance a - x from the point x = a. Adding up the contributions to the electric field from all the pieces of the stick gives the stick's electric field at x = a as (with $\lambda = Q/\ell$ being the linear charge density)

$$E_{\text{stick}} = \int_{-\ell}^{0} \frac{\lambda \, dx}{4\pi\epsilon_0 (a-x)^2} = \frac{\lambda}{4\pi\epsilon_0} \frac{1}{a-x} \Big|_{-\ell}^{0} = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{a+\ell}\right)$$
$$= \frac{\lambda\ell}{4\pi\epsilon_0 a(a+\ell)} = \frac{Q}{4\pi\epsilon_0 a(a+\ell)} \,. \tag{130}$$

Between the two objects, the field from the stick points rightward, and the field from the point charge points leftward. We want these two fields to cancel. The distance from the point x = a to the point charge is $\ell - a$, so we want

$$\frac{Q}{4\pi\epsilon_0 a(a+\ell)} = \frac{Q}{4\pi\epsilon_0 (\ell-a)^2} \implies (\ell-a)^2 = a(a+\ell)$$
$$\implies \ell^2 - 2a\ell + a^2 = a^2 + a\ell$$
$$\implies a = \ell/3. \tag{131}$$

(b) There are no other points. Ignoring the inside of the stick, the field can't be zero anywhere else on the x axis because to the right of the point charge, both objects produce rightward-pointing fields, so they can't cancel. And to the left of the stick, both objects produce leftward-pointing fields, so again they can't cancel. For points not on the x axis, both objects produce a field that has a nonzero component pointing away from the x axis, so again the sum can't be zero.

The existence of an E = 0 point inside the stick, mentioned in the statement of the exercise, follows from a continuity argument: The field points rightward (and in fact diverges) just to the right of the right end of the stick. And it points leftward just to the left of the left end of the stick. So somewhere in between it must be zero.



Figure 37

(c) Some equipotential curves and field lines are shown in Fig. 37 (but see Footnote 1). The field lines are everywhere perpendicular to the equipotential curves (because $\mathbf{E} = -\nabla \phi$, and the gradient of a function is perpendicular to the level

surfaces). The equipotential curves make the transition from two surfaces (one around each object) to one surface (essentially a sphere at infinity) by intersecting at the E = 0 point at $x = \ell/3$ that we found in part (a). The field is indeed zero wherever equipotential curves intersect, because the field must be perpendicular to both lines at the crossing, and the only vector that is perpendicular to two different directions is the zero vector. Zoomed-in views of the equipotentials and field lines near the E = 0 point are shown in Fig. 38 and Fig. 39, respectively. The field lines that start off heading nearly along the x axis toward the point $x = \ell/3$ end up taking nearly a right turn and then head off to infinity. Their direction is vertical near the x axis due to the symmetric nature of the crossing. And it is vertical at infinity because the two objects have equal charge; you can use Gauss's law to show this. But it isn't vertical in between.

2.46. Right triangle ϕ

The potential at P equals the area integral,

$$\phi_P = \int \frac{\sigma \, da}{4\pi\epsilon_0 r} = \frac{\sigma}{4\pi\epsilon_0} \int_0^b dx \int_0^{ax/b} \frac{dy}{\sqrt{x^2 + y^2}} \,. \tag{132}$$

The dy integral equals

$$\ln\left(\sqrt{x^2 + y^2} + y\right)\Big|_0^{ax/b} = \ln\left(\frac{\sqrt{x^2 + (ax/b)^2} + (ax/b)}{\sqrt{x^2 + 0^2} + 0}\right) = \ln\left(\sqrt{1 + \frac{a^2}{b^2}} + \frac{a}{b}\right).$$
(133)

Note that this is independent of x. That is, all vertical strips with thickness dx give the same contribution to the potential at P. To see intuitively why this is the case, consider two strips, and look at two infinitesimal bits of these strips that subtend the same angle. If the right bit is, say, twice as far from P as the left bit, then it has twice the charge (because the thickness dx is fixed, but the height of the right bit is twice as large). These two factors of 2 cancel in the expression $\phi = (1/4\pi\epsilon_0)(q/r)$, which means that the two bits of the strips give the same contribution to the potential. It then follows that the two complete strips give the same contribution.

Due to the independence of x, the dx integral in Eq. (132) simply brings in a factor of b, so we arrive at

$$\phi_P = \frac{\sigma b}{4\pi\epsilon_0} \ln\left(\frac{\sqrt{a^2 + b^2} + a}{b}\right). \tag{134}$$

Since $\sqrt{a^2 + b^2}/b = 1/\cos\theta$, and $a/b = \tan\theta = \sin\theta/\cos\theta$, we can write ϕ_P as

$$\phi_P = \frac{\sigma b}{4\pi\epsilon_0} \ln\left(\frac{1+\sin\theta}{\cos\theta}\right),\tag{135}$$

as desired. Alternatively, you can derive this result directly, by performing an integral over x and θ , instead of x and y (by substituting $y = x \tan \theta$). You will need to use the integral $\int d\theta / \cos \theta = \ln[(1 + \sin \theta) / \cos \theta]$.

REMARK: It is interesting to compare the potentials at the two vertices of a given very thin right triangle, or equivalently at the left vertices of the two triangles shown in Fig. 40. In the first case we can take the $a \ll b$ limit of Eq. (134), which you can show (with the help of $\ln(1 + \epsilon) \approx \epsilon$) leads to $\phi \approx \sigma a/4\pi\epsilon_0$. (You can verify that this is consistent with the result you would obtain for a thin pie piece. You should also think about why it is



Figure 38



Figure 39



Figure 40

independent of b.) In the second case we have the reverse situation with $b \ll a$, which leads to $\phi \approx (\sigma b/4\pi\epsilon_0) \ln(2a/b)$. So if we take $a = \ell$ and $b = 100\ell$ in the first case, we obtain $\phi \approx \sigma \ell/4\pi\epsilon_0$. And if we take $b = \ell$ and $a = 100\ell$ in the second case, we obtain $\phi \approx (\ln 200)\sigma \ell/4\pi\epsilon_0$. It makes sense that the second case has the larger ϕ , because the charge is generally closer to that vertex P in that case. But the log behavior isn't obvious.

2.47. A square and a disk

The result from Exercise 2.46 is $\phi = (\sigma b/4\pi\epsilon_0) \ln[(1+\sin\theta)/\cos\theta]$, where θ is measured with respect to the side with length b. We can divide the given square into 8 triangles, all of which have $\theta = 45^{\circ}$ and b = s/2. The potential at the center is therefore

$$\phi_{\text{square}} = 8 \frac{\sigma(s/2)}{4\pi\epsilon_0} \ln\left(\frac{1+\sin 45^\circ}{\cos 45^\circ}\right) = \frac{(3.525)\sigma s}{4\pi\epsilon_0} \,. \tag{136}$$

For the disk, we can slice it into concentric rings, which gives the potential at the center as

$$\phi_{\rm disk} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \int_0^{d/2} \frac{2\pi r\sigma \, dr}{r} = \frac{\pi\sigma d}{4\pi\epsilon_0} \,. \tag{137}$$

Setting $\phi_{\text{disk}} = \phi_{\text{square}}$ yields $d/s = (3.525)/\pi = 1.122$. As was to be expected, the disk is larger than the inscribed circle, for which d = s; but smaller than the circumscribed circle, for which $d = \sqrt{2} s = (1.414)s$.

2.48. Field from a hemisphere

As in Problem 2.7, our strategy will be to find the potential at radius r, and then take the derivative to find the field. The calculation is the same as in Problem 2.7, except that the limits of integration are modified. If we define θ in the same way is in Fig. 12.28, it now runs from $\pi/2$ to π . Following the steps in the solution to Problem 2.7, the potential at point P in Fig. 41 is (we'll keep things in terms of the density σ)

$$\phi(r) = \int_{\pi/2}^{\pi} \frac{2\pi R^2 \sigma \sin \theta \, d\theta}{4\pi \epsilon_0 \sqrt{R^2 + r^2 - 2rR \cos \theta}}$$
$$= \frac{\sigma R}{2\epsilon_0 r} \sqrt{R^2 + r^2 - 2rR \cos \theta} \Big|_{\pi/2}^{\pi}$$
$$= \frac{\sigma R}{2\epsilon_0 r} \Big((R+r) - \sqrt{R^2 + r^2} \Big).$$
(138)

We are concerned with small r, because we want to know the field at the center. For small r we can write $\sqrt{R^2 + r^2} = R\sqrt{1 + r^2/R^2} \approx R(1 + r^2/2R^2)$. So the potential near the center is

$$\phi(r) = \frac{\sigma R}{2\epsilon_0 r} \left(r - r^2 / 2R \right) = \frac{\sigma R}{2\epsilon_0} \left(1 - r / 2R \right)$$
(139)

The field at the center is then

$$E(r) = -\frac{d\phi}{dr} = \frac{\sigma}{4\epsilon_0} \,. \tag{140}$$

You can check that you arrive at the same result if you take P to be on the left side of the center. You will need to be careful about the limits of integration and various signs.



Figure 41

2.49. E for a sheet, from a cutoff potential

Let's slice the finite disk into concentric rings. The charge in a given ring is $\sigma(2\pi r dr)$, and all points in the ring are a distance $\sqrt{r^2 + z^2}$ from a point that is a distance z from the center of the disk, on the axis. So the potential (relative to infinity) at this point is

$$\phi(z) = \frac{1}{4\pi\epsilon_0} \int_0^R \frac{2\pi\sigma r \, dr}{\sqrt{r^2 + z^2}} = \frac{\sigma}{2\epsilon_0} \sqrt{r^2 + z^2} \Big|_0^R = \frac{\sigma}{2\epsilon_0} \Big(\sqrt{R^2 + z^2} - z\Big).$$
(141)

In the $R \gg z$ limit we can write

$$\sqrt{R^2 + z^2} = R\sqrt{1 + \frac{z^2}{R^2}} \approx R\left(1 + \frac{z^2}{2R^2}\right) = R + \frac{z^2}{2R}.$$
 (142)

We can ignore the second term because it goes to zero as $R \to \infty$ (so the end result of the Taylor series was a trivial one). We therefore obtain

$$\phi(z) = \frac{\sigma}{2\epsilon_0} (R - z). \tag{143}$$

The constant R term here is irrelevant because it simply introduces a constant additive term to $\phi(z)$, which yields zero when we take the derivative to find E(z). (The $\sigma R/2\epsilon_0$ term is the potential at the center of the disk, where z = 0.) So $\phi(z)$ is effectively equal to $-\sigma z/2\epsilon_0$. The field (which is perpendicular to the disk) is therefore

$$E(z) = -\frac{d\phi}{dz} = \frac{\sigma}{2\epsilon_0}, \qquad (144)$$

in agreement with the result obtained via the standard (and quicker) method involving Gauss's law.

This procedure of truncating the sheet into a disk is valid for the following reason. Since the field from a point charge falls off like $1/r^2$, we know that the field from a very large disk is essentially equal to the field from an infinite sheet. The extra annulus that extends out to infinity gives a negligible contribution because the process of taking the perpendicular component adds another factor of $\sim r$ in the denominator. More precisely, the perpendicular field component from a ring equals $(2\pi\sigma r dr/(r^2 + z^2))(z/\sqrt{r^2 + z^2})$, and for large r this behaves like dr/r^2 , the integral of which converges. Therefore, the field we found for the above finite disk is essentially the same as the field for an infinite sheet, provided that R is large enough to make the approximation in Eq. (142) valid. Mathematically, the field in Eq. (144) is independent of R, so when we finally take the $R \to \infty$ limit, nothing changes.

Equivalently, we showed that although the potential in Eq. (143) depends linearly on R, the field in Eq. (144) is independent of R. As far as the field is concerned, it doesn't matter that increasing the size of the disk changes the potential at every point, because the potential everywhere changes by the same amount (assuming $R \gg z$). The variation with z remains the same, so $E = -d\phi/dz$ doesn't change. As an analogy, we can measure the gravitational potential energy mgy with respect to the floor. If we shift the origin to instead be the ceiling, then the potential energy at every point changes by the same amount, but the gravitational force everywhere is still mg.

2.50. Dividing the charge

The potential is constant over the surface of a given sphere, so we can pull the ϕ outside the integral in Eq. (2.32) and write the potential energy of a sphere as U =

 $(\phi/2) \int \rho \, dv = \phi q/2$. So if the spheres of radii R_1 and R_2 have charge q and Q - q, respectively, the sum of the two potential energies is

$$U = \frac{q}{4\pi\epsilon_0 R_1} \cdot \frac{q}{2} + \frac{Q-q}{4\pi\epsilon_0 R_2} \cdot \frac{Q-q}{2} = \frac{1}{4\pi\epsilon_0} \left(\frac{q^2}{2R_1} + \frac{(Q-q)^2}{2R_2}\right).$$
 (145)

Minimizing this by setting the derivative with respect to q equal to zero yields

$$0 = \frac{dU}{dq} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{R_1} - \frac{(Q-q)}{R_2} \right).$$
(146)

Solving for q gives $q = QR_1/(R_1 + R_2)$. So there is charge $QR_1/(R_1 + R_2)$ on the first sphere and charge $QR_2/(R_1 + R_2)$ on the second sphere.

The two terms in Eq. (146) (without the minus sign in front of the second term) are simply the potentials of the two spheres. So the condition of minimum energy is equivalent to the condition of equal potentials. Note that the second derivative, $d^2U/d^2q = 1/R_1 + 1/R_2$, is positive, so the extremum is indeed a minimum of U, not a maximum. This is consistent with the special case where $R_1 = R_2$; equal division of the charge involves half as much total energy as piling all of Q on one sphere, from Eq. (145).

2.51. Potentials on the axis

We'll slice the cylinder into rings and then integrate over these rings. The charge in a ring with width dx is dQ = Q(dx/b). The ring is a convenient charge element to use in computing the potential at axial points, because all of the charge in the ring is equidistant from a given point on the axis, We'll find the potential at a general axial point, a distance x_0 from the midpoint. With x = 0 chosen to be the midpoint of the axis, we have (the integral in given in Appendix K)

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{dQ}{r} = \frac{1}{4\pi\epsilon_0} \int_{-b/2}^{b/2} \frac{Q\,dx/b}{\sqrt{a^2 + (x - x_0)^2}}$$
$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{b} \ln\left(\sqrt{a^2 + (x - x_0)^2} + (x - x_0)\right)\Big|_{-b/2}^{b/2}$$
$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{b} \ln\left(\frac{\sqrt{a^2 + (b/2 - x_0)^2} + (b/2 - x_0)}{\sqrt{a^2 + (-b/2 - x_0)^2} + (-b/2 - x_0)}\right). (147)$$

At the midpoint we have $x_0 = 0$, so

$$\phi_{\rm mid} = \frac{1}{4\pi\epsilon_0} \frac{Q}{b} \ln \frac{\sqrt{a^2 + b^2/4} + b/2}{\sqrt{a^2 + b^2/4} - b/2} \,. \tag{148}$$

At an endpoint we have $x_0 = b/2$, so

$$\phi_{\text{end}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{b} \ln \frac{a}{\sqrt{a^2 + b^2} - b} = \frac{1}{4\pi\epsilon_0} \frac{Q}{b} \ln \frac{\sqrt{a^2 + b^2} + b}{a}.$$
 (149)

The difference is therefore

$$\phi_{\rm mid} - \phi_{\rm end} = \frac{1}{4\pi\epsilon_0} \frac{Q}{b} \ln \frac{a(\sqrt{a^2 + b^2/4} + b/2)}{(\sqrt{a^2 + b^2} + b)(\sqrt{a^2 + b^2/4} - b/2)}.$$
 (150)

This is a rather messy answer, so let's look at how ϕ_{mid} and ϕ_{end} behave in some limits, to feel better about the result. Consider the limit $b \to 0$, in which case the cylinder reduces to a thin ring. To first order in b, we can ignore the b^2 terms under the square roots in the above expressions. We obtain

$$4\pi\epsilon_0\phi_{\rm mid} \approx \frac{Q}{b}\ln\frac{a+b/2}{a-b/2} \approx \frac{Q}{b}\ln\left(1+\frac{b}{a}\right) \approx \frac{Q}{b} \cdot \frac{b}{a} = \frac{Q}{a}.$$
 (151)

Likewise,

$$4\pi\epsilon_0\phi_{\rm end} \approx \frac{Q}{b}\ln\left(1+\frac{b}{a}\right) \approx \frac{Q}{a}$$
. (152)

Both of these results make sense, because all of the charge on the ring is essentially a distance a from the midpoint of the axis, which is essentially the same as an endpoint. The difference in the potentials is therefore zero.

In the limit $b \to \infty$, both ϕ_{mid} and ϕ_{end} go to zero because of the Q/b coefficient, and because we are holding the total charge Q constant. If we instead let the charge per unit length be constant, so that $Q = \lambda b$, then you can show that $4\pi\epsilon_0\phi_{\text{mid}} \approx 2\lambda \ln(b/a)$ and $4\pi\epsilon_0\phi_{\text{end}} \approx \lambda \ln(2b/a)$. For large b, the "2" inside the log doesn't matter much, so we see that ϕ_{mid} is approximately twice as large as ϕ_{mid} . The reason for this basically comes down to the fact that if you're in the middle of a long cylinder, you see two long cylinders on either side of you (and also the fact that the potential due to a stick grows only like the log of the length, so that a long stick yields roughly the same potential as one that has twice the length).

A rough sketch of the field lines is shown in Fig. 42. The field is zero at the center, by symmetry. We haven't worried about drawing the density of field lines correctly; for a very long cylinder, you can show that the field at the surface is twice the field at the midpoint of an end face.

2.52. Spherical cavity in a slab

(a) The given setup can be considered to be the superposition of the given slab and a sphere with charge density $-\rho$. Our goal is to show that the potential at a point on the surface of the slab at infinity equals the potential at the center of the cavity. From the standard Gauss's-law argument with a pillbox extending from -x to x, the field due to the slab in the interior is $\hat{\mathbf{x}}\rho x/\epsilon_0$. (Alternatively, the relevant part of the slab is effectively a sheet with surface charge density $\rho(2x)$). And outside the slab the field is $\hat{\mathbf{x}}\rho R/\epsilon_0$ (the whole slab is effectively a sheet with surface charge density $\rho(2R)$). Similar Gauss's-law arguments give the field due to the (negative) sphere in the interior as $-\hat{\mathbf{r}}\rho r/3\epsilon_0$, and outside as $-\hat{\mathbf{r}}\rho R^3/3\epsilon_0 r^2$. Integrating the slab and sphere fields to find the total potential relative to the center of the cavity, we find that the potential inside the cavity equals $-\rho x^2/2\epsilon_0 + \rho r^2/6\epsilon_0$.

At the rightmost point of the cavity, we have x = r = R, so the potential of this point (relative to the center) is $(\rho R^2/\epsilon_0)(-1/2 + 1/6) = -\rho R^2/3\epsilon_0$. If we now march along the surface of the slab to infinity, only the field from the sphere matters. The change in potential as we march out is

$$\Delta\phi = -\int_R^\infty \frac{-\rho R^3}{3\epsilon_0 r^2} \, dr = \frac{\rho R^3}{3\epsilon_0 R} = \frac{\rho R^2}{3\epsilon_0} \,. \tag{153}$$

(Alternatively, the change in potential from R to ∞ is just $-Q/4\pi\epsilon_0 R$, with $Q = -4\pi R^3 \rho/3$.) This is correctly positive, because the potential increases as



Figure 42

we move away from the negative sphere. This $\Delta \phi$ exactly cancels the negative potential at the rightmost point of the cavity, so we end up with zero potential on the surface of the slab at infinity, as desired. The algebra here basically boils down to 1/2 = 1/6 + 1/3. The 1/2 is the (magnitude of the) decrease in potential due to the slab. The 1/6 + 1/3 is the increase in potential due to the sphere, inside and outside.

- (b) The potential inside the cavity is $\phi = -\rho x^2/2\epsilon_0 + \rho r^2/6\epsilon_0$. In the plane of the paper, we have $r^2 = x^2 + y^2$, so ϕ becomes $\rho(y^2 2x^2)/6\epsilon_0$. Points on the $\phi = 0$ curve therefore satisfy $y^2 = 2x^2 \Longrightarrow y = \pm \sqrt{2}x$. This is consistent with Fig. 2.49, where the slope of the straight line looks to be a little larger than 1.
- (c) We found in part (a) that as we move from the rightmost point of the cavity out to infinity along the surface of the slab, the potential increases by $\rho R^2/3\epsilon_0$. If we want to end up at the same potential as at the rightmost point of the cavity, we must then move away from the slab by a distance that causes the potential to decrease by $\rho R^2/3\epsilon_0$. Since the field outside the slab equals $\hat{\mathbf{x}}\rho R/\epsilon_0$ (far away, the sphere can be ignored), the change in potential as we move away from the slab is $-(\rho R/\epsilon_0)\Delta x$. This equals $-\rho R^2/3\epsilon_0$ when $\Delta x = R/3$, as desired.

2.53. Field from two shells

By superposition, the electric field outside both shells is that of two point charges located at the centers. And also by superposition, the field inside each shell is that of a point charge at the center of the other shell (because a given spherical shell with uniform charge distribution produces no field in its interior). We therefore obtain the fields shown in Fig. 43. Note that the field lines inside each shell are straight.

The two shells are equivalent (as far as their external fields go) to two point charges Q and -Q at their centers. Therefore we may replace the spheres with these point charges. Since the centers are initially 2a apart, the amount of work required to move them to infinity is $(1/4\pi\epsilon_0)(Q^2/2a)$.

In more detail: Let the positive shell be labeled A, and the negative shell B. The external field of A alone is that of a point charge Q at its center. So the work required to move B to infinity in the given setup is the same as the work required in an alternative setup where A is replaced by a point charge. But the work in this case is the same as if we instead held B fixed and moved the point charge to infinity. But since the external field of B alone is that of a point charge -Q at its center, we may replace B with a point charge. The work is therefore the same as in the case where we have two charges Q and -Q that are initially 2a apart.

2.54. An equipotential for a disk

From Eq. (2.30) the potential on the rim (with $\phi = 0$ at $r = \infty$) is $\sigma a/\pi\epsilon_0$. From Eq. (2.25) the potential on the symmetry axis is $(\sigma/2\epsilon_0)(\sqrt{y^2 + a^2} - y)$. Setting these equal yields

$$\sqrt{y^2 + a^2} - y = \frac{2a}{\pi} \implies y^2 + a^2 = \left(y + \frac{2a}{\pi}\right)^2 \implies y = a\frac{\pi^2 - 4}{4\pi} \approx (0.467)a.$$
(154)

The equipotential surface is (roughly) represented by the curve in Fig. 44 The direction of the curve near the edge of the disk happens to be perpendicular to the plane of the disk. This follows from the result of Exercise 2.57(b) below; the tangential component of the field diverges near the edge of the disk, so the field is essentially parallel to the disk. The equipotential curve is therefore is perpendicular to the disk.



Figure 43





2.55. Hole in a disk

(a) Slicing the disk into concentric rings, we find the potential at the center to be (with $\ell = 1 \text{ cm}$)

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0} \int_{\ell}^{3\ell} \frac{2\pi r\sigma \, dr}{r} = \frac{\sigma\ell}{\epsilon_0} \,. \tag{155}$$

Plugging in the various quantities gives

$$\phi = \frac{\left(-10^{-5} \frac{C}{m^2}\right)(0.01 \text{ m})}{8.85 \cdot 10^{-12} \frac{s^2 C^2}{\text{kg m}^3}} = -11,300 \text{ V}.$$
 (156)

(b) The electron's final kinetic energy at infinity equals the loss in potential energy. This loss has magnitude

$$(-e)\phi = (-1.6 \cdot 10^{-19} \,\mathrm{C})(-11,300 \,\mathrm{V}) = 1.81 \cdot 10^{-15} \,\mathrm{J}.$$
 (157)

Since this is only about 2% of the electron's rest energy, namely $mc^2 = 8.2 \cdot 10^{-14}$ J, a nonrelativistic calculation will suffice:

$$\frac{1}{2}mv^2 = 1.81 \cdot 10^{-15} \,\mathrm{J} \implies v = \left(\frac{2(1.81 \cdot 10^{-15} \,\mathrm{J})}{9.1 \cdot 10^{-31} \,\mathrm{kg}}\right)^{1/2} = 6.3 \cdot 10^7 \,\mathrm{m/s}, \ (158)$$

which is about 20% of the speed of light. This answer is very close to the answer obtained via the correct relativistic calculation: Conservation of energy gives

$$\gamma mc^2 = mc^2 + |\Delta U| \implies \gamma = 1 + \frac{1.81 \cdot 10^{-15} \,\mathrm{J}}{8.2 \cdot 10^{-14} \,\mathrm{J}} = 1.022.$$
 (159)

Hence (with $\beta \equiv v/c$),

$$\beta = \sqrt{1 - 1/\gamma^2} = 0.206 \implies v = \beta c = 6.2 \cdot 10^7 \,\mathrm{m/s.}$$
 (160)

2.56. Energy of a disk

From Eq. (2.30) the potential at a point on the rim of a disk with radius r is $\phi_r = \sigma r/\pi\epsilon_0$. Adding on a ring with charge $dq = \sigma 2\pi r dr$ requires an energy of $dU = \phi_r dq = 2\sigma^2 r^2 dr/\epsilon_0$. The total amount of energy required to assemble the disk of charge is therefore

$$U = \int_0^a dU = \frac{2\sigma^2}{\epsilon_0} \int_0^a r^2 dr = \frac{2\sigma^2 a^3}{3\epsilon_0}.$$
 (161)

But $Q = \pi a^2 \sigma \Longrightarrow \sigma = Q/\pi a^2$, so we can write

$$U = \frac{2}{3\pi^2 \epsilon_0} \frac{Q^2}{a} \approx \frac{0.0675}{\epsilon_0} \frac{Q^2}{a} \,. \tag{162}$$

From Problem 1.32 or Exercise 2.58, the energy required to build up a hollow spherical shell with radius a and charge Q is

$$U = \frac{1}{8\pi\epsilon_0} \frac{Q^2}{a} \approx \frac{0.0398}{\epsilon_0} \frac{Q^2}{a} \,. \tag{163}$$

As expected, this is smaller than the result for the disk, because the charges are generally closer to each other in the case of the disk.

2.57. Field near a disk

(a) A little area element within a wedge in Fig. 2.50 has area $r dr d\theta$. So at the given point P, the magnitude of the field due to this little area is $\sigma(r \, dr \, d\theta)/4\pi\epsilon_0 r^2 =$ $\sigma dr d\theta / 4\pi\epsilon_0 r$. But only the vertical component (in the plane of the page) survives, which brings in a factor of $\cos \theta$, yielding $\sigma \cos \theta \, dr \, d\theta / 4\pi \epsilon_0 r$. If we integrate this from r = 0 up to either the r_1 or r_2 in Fig. 2.50, we obtain an infinite result. However, the divergence from one wedge cancels the divergence from the other, because the field due to the short wedge cancels the field due to the part of the long wedge out to a distance r_1 . The uncanceled part of the long wedge comes from r values ranging from r_1 out to the end at r_2 .¹ The net vertical component of the field from the two opposite wedges in Fig. 2.50 is therefore

$$E_{\parallel} = \int_{r_1}^{r_2} \frac{\sigma \cos\theta \, dr \, d\theta}{4\pi\epsilon_0 r} = \frac{\sigma}{4\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right) \cos\theta \, d\theta. \tag{164}$$

We must now integrate this result over θ . We can integrate from 0 to $\pi/2$ and then double the result; this will end up covering the whole disk.

The task now is to find r_1 and r_2 in terms of θ . We can find r_2 by using the law of cosines in the triangle involving r_2 in Fig. 45. This gives $R^2 = (\eta R)^2 + r_2^2 - r_2^2$ $2(\eta R)r_2\cos\theta$. Solving this quadratic equation for r_2 and choosing the positive root gives

$$r_2 = R\left(\sqrt{1 - \eta^2 \sin^2 \theta + \eta \cos \theta}\right). \tag{165}$$

The process involving r_1 (with the triangle containing the angle $\pi - \theta$) is the same except for the replacement of $\cos \theta$ with $-\cos \theta$. So twice the integration of the result in Eq. (164) from 0 to $\pi/2$ gives

$$E_{\parallel} = \frac{\sigma}{2\pi\epsilon_0} \int_0^{\pi/2} \ln\left(\frac{\sqrt{1-\eta^2 \sin^2\theta} + \eta \cos\theta}{\sqrt{1-\eta^2 \sin^2\theta} - \eta \cos\theta}\right) \cos\theta \, d\theta. \tag{166}$$

(b) Let's look at small values of η first. We can ignore terms of order η^2 , so the log term in Eq. (166) becomes

$$\ln\left(\frac{1+\eta\cos\theta}{1-\eta\cos\theta}\right) = \ln\left(\frac{(1+\eta\cos\theta)^2}{1-\eta^2\cos^2\theta}\right) \approx \ln\left(1+2\eta\cos\theta\right) \approx 2\eta\cos\theta, \quad (167)$$

where we have used $\ln(1+x) \approx x$. Substituting this into Eq. (166) gives

$$E_{\parallel} \approx \frac{\sigma\eta}{\pi\epsilon_0} \int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{\sigma\eta}{\pi\epsilon_0} \left(\frac{\pi}{4}\right) = \frac{\sigma\eta}{4\epsilon_0} \,, \tag{168}$$

where we have used the fact that the average value of $\cos^2 \theta$ equals 1/2. It is reasonable that this result is linear in η , because as the given point P is moved away from the center, the unbalanced part of the disk (the shaded part in Fig. 46) grows linearly with η (for small η). Intuitively, the $\sigma \eta/4\epsilon_0$ field from the unbalanced part should take the same form, up to a numerical factor, as the field from an effective line of charge with small width $2\eta R$ (indicated by the dotted



centered at P

 $2\eta R$

disk

=== circle







¹If the given point doesn't lie exactly in the plane of the disk, then there is actually no divergence. But it is still worth mentioning the canceling divergences, because E_{\parallel} is well defined even if the point does lie in the plane (whereas E_{\perp} isn't, for an infinitesimally thin disk).

lines in Fig. 46). And indeed, you can show that the former is $\pi/4$ times the latter.

Now let's look at values of η very close to 1. With $\eta \equiv 1 - \epsilon$, the square root term in Eq. (166) becomes (dropping terms of order ϵ^2)

$$\sqrt{1 - (1 - \epsilon)^2 \sin^2 \theta} \approx \sqrt{\cos^2 \theta + 2\epsilon \sin^2 \theta} = \cos \theta \sqrt{1 + \frac{2\epsilon \sin^2 \theta}{\cos^2 \theta}}$$
$$\approx \cos \theta \left(1 + \frac{\epsilon \sin^2 \theta}{\cos^2 \theta}\right) = \cos \theta + \frac{\epsilon \sin^2 \theta}{\cos \theta}, \quad (169)$$

where we have used $\sqrt{1+x} \approx 1 + x/2$. In the numerator in the log term in Eq. (166), the leading-order term (which happens to be zeroth order in ϵ , in this case) is simply $2\cos\theta$. But in the denominator, the $\cos\theta$ terms cancel, so the leading-order term is $\epsilon \sin^2\theta/\cos\theta + \epsilon \cos\theta = \epsilon/\cos\theta$. The argument of the log term is therefore $(2\cos\theta)/(\epsilon/\cos\theta) = 2\cos^2\theta/\epsilon$. So Eq. (166) becomes (we'll drop the $\ln(2\cos^2\theta)$ term, because it is small compared with the leading-order $\ln \epsilon$ term; so all that matters is the $1/\epsilon$ behavior of the argument)

$$E_{\parallel} \approx \frac{\sigma}{2\pi\epsilon_0} \int_0^{\pi/2} \ln\left(\frac{2\cos^2\theta}{\epsilon}\right) \cos\theta \,d\theta$$
$$\approx -\frac{\sigma\ln\epsilon}{2\pi\epsilon_0} \int_0^{\pi/2} \cos\theta \,d\theta = -\frac{\sigma\ln\epsilon}{2\pi\epsilon_0} \,. \tag{170}$$

Note that this result is positive since $\epsilon < 1$. We see that the field diverges as we get very close to the edge of the disk, but it diverges slowly, like a log. The log behavior makes sense, because as we get close to the edge, the short wedge in Fig. 2.50 cancels only a small part of the long wedge. So we need to integrate the 1/r field (see Eq. (164)) due to the long wedge almost down to r = 0. And the integral of 1/r diverges like $\ln r$ near r = 0.

You can check, for various values of η near 0 or 1, that Eqs. (168) and (170) correctly agree with a numerical evaluation of Eq. (166).

2.58. Energy of a shell

The relevant volume in the integral in Eq. (2.32) is all located right on the surface of the shell where the potential ϕ takes on the uniform value of $Q/4\pi\epsilon_0 R$. We can therefore take ϕ outside the integral, yielding

$$U = \frac{1}{2}\phi \int \rho \, dv = \frac{1}{2}\phi Q = \frac{1}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)Q = \frac{Q^2}{8\pi\epsilon_0 R}\,.$$
 (171)

2.59. Energy of a cylinder

The electric field outside the cylinder is $\lambda/2\pi\epsilon_0 r$, where $\lambda = \rho\pi a^2$ is the charge per unit length in the cylinder. So the field outside is $E_{\text{out}} = \rho a^2/2\epsilon_0 r$. The field at radius r inside the cylinder is due to the charge within radius r, so the effective charge per unit length is $\lambda_r = \rho \pi r^2$. The field inside is therefore $E_{\text{in}} = \lambda_r/2\pi\epsilon_0 r = \rho r/2\epsilon_0$.

The potential at radius r inside the cylinder, relative to a given radius R outside the cylinder, is the negative integral of the field from R down to r. We must break this integral into two pieces:

$$\phi(r) = -\int_{R}^{a} \frac{\rho a^{2}}{2\epsilon_{0}r'} dr' - \int_{a}^{r} \frac{\rho r'}{2\epsilon_{0}} dr' = \frac{\rho a^{2}}{2\epsilon_{0}} \ln\left(\frac{R}{a}\right) + \frac{\rho}{4\epsilon_{0}}(a^{2} - r^{2}).$$
(172)

Equation (2.32) then gives the energy (relative to the configuration where the charge is distributed over a cylinder with radius R) in a length ℓ of the cylinder as

$$U = \frac{1}{2} \int \rho \phi \, dv = \frac{1}{2} \int_0^a \rho \left[\frac{\rho a^2}{2\epsilon_0} \ln \left(\frac{R}{a} \right) + \frac{\rho}{4\epsilon_0} (a^2 - r^2) \right] 2\pi r \ell \, dr$$

$$\implies \frac{U}{\ell} = \frac{\pi \rho^2}{2\epsilon_0} \int_0^a \left[a^2 \ln \left(\frac{R}{a} \right) + \frac{1}{2} (a^2 - r^2) \right] r \, dr$$

$$= \frac{\pi \rho^2}{2\epsilon_0} \left[\frac{a^4}{2} \ln \left(\frac{R}{a} \right) + \frac{1}{2} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) \right]$$

$$= \frac{\rho^2 \pi^2 a^4}{4\pi\epsilon_0} \left[\ln \left(\frac{R}{a} \right) + \frac{1}{4} \right]$$

$$= \frac{\lambda^2}{4\pi\epsilon_0} \left[\ln \left(\frac{R}{a} \right) + \frac{1}{4} \right], \qquad (173)$$

as desired. If R = a, so that all of the charge is initially distributed over the surface of the cylinder, then it takes an amount of work per unit length equal to $\lambda^2/16\pi\epsilon_0$ to move the charge inward and distribute it uniformly throughout the volume of the cylinder.

2.60. Horizontal field lines

In terms of r and θ , the y coordinate of a point in the plane is $y = r \cos \theta$. (Remember that θ is measured down from the vertical.) Using the given expression for r, we can write y as a function of only θ as $y = (r_0 \sin^2 \theta) \cos \theta$. The curves are horizontal at points where y achieves a maximum value, so we want to maximize y as a function of θ :

$$0 = \frac{dy}{d\theta} = r_0 \left(2\sin\theta\cos\theta \cdot \cos\theta + \sin^2\theta(-\sin\theta) \right) \implies \tan\theta = \pm\sqrt{2}.$$
(174)

The field is therefore horizontal everywhere on the two lines with slopes of $\pm 1/\sqrt{2}$ passing through the origin. (Since θ is measured from the vertical, the slope of a line is $1/\tan \theta$.) More generally, the field is horizontal everywhere in space on the cones generated by rotating these two lines around the y axis.

2.61. Dipole field on the axes

With the charges q and -q located at $z = \ell/2$ and $-\ell/2$, consider a distant point on the positive z axis with z = r. The charge q is slightly closer than the charge -qis to this point, so the upward field due to the charge q is slightly stronger than the downward field due to the charge -q. The net field will therefore point upward, and it has magnitude (with $k \equiv 1/4\pi\epsilon_0$)

$$E = \frac{kq}{(r-\ell/2)^2} - \frac{kq}{(r+\ell/2)^2} = \frac{kq}{r^2} \left(\frac{1}{(1-\ell/2r)^2} - \frac{1}{(1+\ell/2r)^2} \right)$$
$$\approx \frac{kq}{r^2} \left(\frac{1}{1-\ell/r} - \frac{1}{1+\ell/r} \right), \tag{175}$$

where we have dropped terms of order ℓ^2/r^2 . Using $1/(1 \pm \epsilon) \approx 1 \mp \epsilon$, we obtain

$$E \approx \frac{kq}{r^2} \left(\left(1 + \frac{\ell}{r} \right) - \left(1 - \frac{\ell}{r} \right) \right) = \frac{2kq\ell}{r^3} \,. \tag{176}$$

This field points in the positive $\hat{\mathbf{r}}$ direction, so it agrees with the result in Eq. (2.36),

$$\frac{kq\ell}{r^3} \left(2\cos\theta \,\hat{\mathbf{r}} + \sin\theta \,\hat{\boldsymbol{\theta}} \right),\tag{177}$$

when $\theta = 0$.

In the transverse direction, we have the situation shown in Fig. 47. The magnitudes of the two fields are equal. The horizontal components cancel, but the downward components add. The distances from the given point to the two charges are essentially equal to r, so the magnitudes of the fields are kq/r^2 . The (negative) vertical components are obtained by multiplying by $\sin \beta$, which is approximately equal to $(\ell/2)/r$ in the small-angle approximation. The vertical field is therefore directed downward with magnitude

$$E \approx 2\left(\frac{kq}{r^2}\right)\frac{\ell/2}{r} = \frac{kq\ell}{r^3}.$$
(178)

This agrees with the result in Eq. (2.36) when $\theta = \pi/2$, because the $\hat{\theta}$ vector points downward at the given point (in the direction of increasing θ , which is measured down from the vertical). This field is half as large as the field on the vertical axis, for a given value of r.

2.62. Square quadrupole

By symmetry, the desired field is radial. Let the lines to the negative charges make an angle θ with respect to the line to the center of the square, and let the distance to the negative charges be r_1 , as shown in Fig. 48. Then the total field at a distance rfrom the center is (with $k \equiv 1/4\pi\epsilon_0$, $d \equiv \ell/\sqrt{2}$, and $\epsilon \equiv d/r$)

$$E_{r} = \frac{kq}{(r-d)^{2}} + \frac{kq}{(r+d)^{2}} - \frac{2kq}{r_{1}^{2}}\cos\theta$$

$$= \frac{kq}{(r-d)^{2}} + \frac{kq}{(r+d)^{2}} - \frac{2kq}{r^{2}+d^{2}}\frac{r}{\sqrt{r^{2}+d^{2}}}$$

$$= \frac{kq}{r^{2}} \left[\frac{1}{(1-\epsilon)^{2}} + \frac{1}{(1+\epsilon)^{2}} - \frac{2}{(1+\epsilon^{2})^{3/2}} \right]$$

$$\approx \frac{kq}{r^{2}} \left[(1+2\epsilon+3\epsilon^{2}) + (1-2\epsilon+3\epsilon^{2}) - 2(1-3\epsilon^{2}/2) \right]$$

$$= \frac{kq}{r^{2}} \left[9\epsilon^{2} \right] = \frac{9kq\ell^{2}}{2r^{4}}.$$
(179)

There are various ways of obtaining the above Taylor series, but perhaps the easiest is to note that, for example, $1/(1-\epsilon)^2$ equals the derivative of $1/(1-\epsilon)$, which itself is just the sum of the geometric series $1 + \epsilon + \epsilon^2 + \cdots$.

Our result for E_r is positive, so the field points away from the square. Along the other diagonal, it points toward the square. This implies that if we traverse a large circle around the quadrupole, there are four locations where the radial component of the field is zero. This should be contrasted with the field of a dipole, which has only two such locations where the radial component is zero.

2.63. Two-dimensional dipole

Consider the point P show in Fig. 49. The field due to the positive wire takes the form $\lambda/2\pi\epsilon_0 r$. Since the field is radial and has a magnitude that depends only on r,







Figure 48



Figure 49

the potential at P (due to the positive wire) relative to the point midway between the wires is

$$\phi = -\int_{\ell/2}^{r_1} E_r \, dr = -\int_{\ell/2}^{r_1} \frac{\lambda \, dr}{2\pi\epsilon_0 r} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\ell/2}{r_1}\right). \tag{180}$$

As a double check on the sign, if r_1 is very small (although we're not concerned with this case), then ϕ is a large positive quantity, as it should be. Likewise, the potential due to the negative wire is $-(\lambda/2\pi\epsilon_0)\ln((\ell/2)/r_2)$. When we add these two potentials, the ℓ dependence drops out, and we end up with a total potential of $\phi = (\lambda/2\pi\epsilon_0)\ln(r_2/r_1)$. Using the same approximate forms of r_1 and r_2 that we used in Eq. (2.35) in the $r \gg \ell$ limit, we find

$$\frac{r_2}{r_1} = \frac{r + (\ell/2)\cos\theta}{r - (\ell/2)\cos\theta} = \frac{1 + (\ell/2r)\cos\theta}{1 - (\ell/2r)\cos\theta}$$
$$\approx \left(1 + (\ell/2r)\cos\theta\right)^2 \approx 1 + (\ell/r)\cos\theta, \tag{181}$$

where we have used $1/(1-\epsilon) \approx 1+\epsilon$. We can now use the Taylor approximation $\ln(1+\epsilon) \approx \epsilon$ to write

$$\phi(r,\theta) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right) \approx \frac{\lambda}{2\pi\epsilon_0} \left(\frac{\ell\cos\theta}{r}\right) = \frac{\lambda\ell\cos\theta}{2\pi\epsilon_0 r} \,. \tag{182}$$

The 1/r dependence in ϕ is the same as the 1/r dependence in the individual fields from the wires. This is analogous to the fact that in the 3D dipole case in Section 2.7, the $1/r^2$ dependence in ϕ was the same as the $1/r^2$ dependence in the individual fields from the point charges.

We can now find the electric field via $\mathbf{E} = -\nabla \phi$. In polar coordinates the gradient operator is given by $\nabla = \hat{\mathbf{r}}(\partial/\partial r) + \hat{\boldsymbol{\theta}}(1/r)(\partial/\partial \theta)$. So the electric field equals

$$\mathbf{E} = -\hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{\lambda\ell\cos\theta}{2\pi\epsilon_0 r}\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\lambda\ell\cos\theta}{2\pi\epsilon_0 r}\right) = \frac{\lambda\ell}{2\pi\epsilon_0 r^2}\left(\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right). \tag{183}$$

The calculation of the shapes of the field-line curves and the constant-potential curves is nearly the same as in Section 2.7. The equation for the constant- ϕ curves is immediately obtained from Eq. (182). The set of points for which ϕ takes on the constant value ϕ_0 is given by

$$\frac{\lambda\ell\cos\theta}{2\pi\epsilon_0 r} = \phi_0 \implies r = \left(\frac{\lambda\ell}{2\pi\epsilon_0\phi_0}\right)\cos\theta \implies r = r_0\cos\theta, \qquad (184)$$

where $r_0 \equiv \lambda \ell / 2\pi \epsilon_0 \phi_0$ is the radius associated with the angle $\theta = 0$. In the lower half plane, both ϕ_0 and $\cos \theta$ are negative, so r is still positive, as it should be. As an exercise, you can show that $r = r_0 \cos \theta$ describes a circle with diameter r_0 . So the equipotential curves are circles; see the solid lines in Fig. 50.

As explained in Section 2.7, the slope of a given curve at a given point, relative to the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ basis vectors at that point, is $dr/rd\theta$. So the slope of the $r = r_0 \cos \theta$ curve is

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{r_0\cos\theta}(-r_0\sin\theta) = -\frac{\sin\theta}{\cos\theta}\,.$$
(185)

Remember that this is the slope with respect to the local $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}}$ basis (which varies with position), and not the fixed $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}$ basis.



Field lines and constant-potential curves for a dipole. The two sets of curves are orthogonal at all intersections. The solid lines show constant- ϕ curves $(r = r_0 \cos \theta)$, and the dashed lines show **E** field lines $(r = r_0 \sin \theta)$.

Figure 50

Now consider the **E** field. As in Section 2.7, we'll do things in reverse order, first finding the slope of the tangent, and then using that to find the equation of the field-line curves. The slope of the tangent is immediately obtained from the E_r and E_{θ} components given in Eq. (183). We have

$$\frac{E_r}{E_{\theta}} = \frac{\cos\theta}{\sin\theta} \,. \tag{186}$$

This slope is the negative reciprocal of the slope of the tangent to the constant- ϕ curves, given in Eq. (185), as it should be. To find the equation for the field-line curves, we can use the fact that the slope in Eq. (186) must be equal to $dr/rd\theta$. We can then separate variables and integrate to obtain

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{\sin\theta} \implies \int \frac{dr}{r} = \int \frac{\cos\theta \,d\theta}{\sin\theta} \implies \ln r = \ln\sin\theta + C.$$
(187)

Exponentiating both sides gives

$$r = r_0 \sin \theta. \tag{188}$$

where $r_0 \equiv e^C$ is the radius associated with the angle $\theta = \pi/2$. As an exercise, you can show that $r = r_0 \sin \theta$ describes a circle with diameter r_0 . So the field-line curves are also circles; see the dashed lines in Fig. 50.

2.64. Field lines near the equilibrium point

(a) Setting a = 1 and ignoring the factor of $q/4\pi\epsilon_0$, the potential due to the two charges, at locations in the xy plane, is

$$\phi(x,y) = \frac{4}{\sqrt{(x+2)^2 + y^2}} - \frac{1}{\sqrt{(x+1)^2 + y^2}}.$$
(189)

Using the Taylor expansion $1/\sqrt{1+\epsilon} \approx 1-\epsilon/2+3\epsilon^2/8$, and keeping terms up to second order in x and y, we have

$$\phi(x,y) = \frac{4}{\sqrt{4 + (4x + x^2 + y^2)}} - \frac{1}{\sqrt{1 + (2x + x^2 + y^2)}}$$
$$= \frac{2}{\sqrt{1 + (x + x^2/4 + y^2/4)}} - \frac{1}{\sqrt{1 + (2x + x^2 + y^2)}}$$
$$\approx 2\left(1 - \frac{1}{2}(x + x^2/4 + y^2/4) + \frac{3}{8}(x + \cdots)^2\right)$$
$$- \left(1 - \frac{1}{2}(2x + x^2 + y^2) + \frac{3}{8}(2x + \cdots)^2\right)$$
$$= 1 + (1/4)(y^2 - 2x^2).$$
(190)

(Alternatively, you can obtain this from the **Series** operation in *Mathematica*.) If we had included the z dependence, then a z^2 term would appear in the same manner as the y^2 term. That is, the term in parentheses would be $y^2 + z^2 - 2x^2$. In terms of all the given parameters, you can show that

$$\phi(x,y) \approx \frac{q}{4\pi\epsilon_0 a} \left(1 + \frac{y^2 - 2x^2}{4a^2} \right). \tag{191}$$

Some level surfaces of the function $y^2 - 2x^2$ are shown in Fig. 51. The origin is a saddle point; it is a maximum with respect to variations in the x direction, and a minimum with respect to variations in the y direction. The constant- ϕ lines passing through the equilibrium point are given by $y = \pm \sqrt{2}x$ (near the origin). If we zoom in closer to the origin, the curves keep the same general shape; the picture looks the same, with the only change being the ϕ value associated with each curve.

(b) The electric field is the negative gradient of the potential, so we have

$$\mathbf{E} = -\nabla\phi = \frac{q}{8\pi\epsilon_0 a^3} (2x, -y). \tag{192}$$

The field lines are the curves whose tangents are the \mathbf{E} field vectors, by definition. Equating the slope of a curve with the slope of the tangent \mathbf{E} vector, and separating variables and integrating, gives

$$\frac{dy}{dx} = \frac{E_y}{E_x} \implies \frac{dy}{dx} = -\frac{y}{2x} \implies \int \frac{dy}{y} = -\frac{1}{2} \int \frac{dx}{x}$$
$$\implies \ln y = -\frac{1}{2} \ln x + A \implies y = \frac{B}{\sqrt{x}}, \quad (193)$$

where A is a constant of integration, and $B \equiv e^A$. Different values of B give different field lines. Technically, this $y = B/\sqrt{x}$ result is valid only in the first quadrant. But since the setup is symmetric with respect to the yz plane (at least near the origin), and also with respect to rotations around the x axis, the general form of the field lines in the xy plane is shown in Fig. 52. If we zoom in closer to the origin, the lines keep the same general shape.

If we include the z dependence, then the correct expression for the field near the origin has the (2x, -y) vector in Eq. (192) replaced with (2x, -y, -z). As a check on this, the divergence of this vector is zero, which is correct because $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$







and because there are no charges at the origin. Although the abbreviated vector in Eq. (192) is sufficient for making a picture of what the field lines look like, it has nonzero divergence, so its utility goes only so far. See Exercise 2.65 for an extension of this exercise.

2.65. A theorem on field lines

(a) First note that the equilibrium point must lie on the line containing the two charges, because otherwise the fields from the two charges won't point along the same line, which means that there is no way for the fields to exactly cancel.

There are various cases to consider. (1) If both charges are nonzero and have the same sign, then the equilibrium point must lie between them, and it is easy to see that there is only one such point. (2) If both charges are nonzero and have opposite sign, then the equilibrium point must lie outside them, closer to the smaller charge. The one exception to this case is when the charges are equal and opposite, in which case there is no equilibrium point (technically, it is located at infinity). (3) If one of the charges is zero, then there is no equilibrium point. (4) If both charges are zero (a trivial case which probably isn't worth considering), then every point is an equilibrium point.

(b) If we choose the origin of our coordinate system to be the equilibrium point, with the two charges lying on the x axis, and if we Taylor-expand the potential around the origin, then it can't have any terms that are linear in the coordinates. This is true because if it did, the electric field (which is the negative gradient of the potential) would have constant nonzero terms, violating the fact that E = 0 at the equilibrium point. So the potential must take the form of $\phi = A + ax^2 + by^2 + cz^2$ (plus higher order terms, which we can ignore close to the origin). But b = c because the system is symmetric under rotations around the x axis. So $\phi = A + ax^2 + b(y^2 + z^2)$. The relation $\nabla^2 \phi = -\rho/\epsilon_0$ tells us that $\nabla^2 \phi = 0$, because $\rho = 0$ at the equilibrium point (and everywhere else, except at the locations of the charges). Therefore, a = -2b. So the potential must take the form of

$$\phi = A + b(-2x^2 + y^2 + z^2). \tag{194}$$

In the xy plane (that is, for z = 0), the equipotential lines are therefore given by $y = \pm \sqrt{2}x$, as desired.

Note that all of the above reasoning is still valid even if we replace one (or both) of the point charges with a stick (as in Exercise 2.45, for example), as long as all of the charge in the setup lies on a single line.

REMARK: The above reasoning can also be applied to the equipotential lines that cross at the origin in Fig. 12.42 in the solution to Problem 2.19. We claim that the slopes of these lines are equal to $\pm 1/\sqrt{2}$. This follows from Gauss's law and symmetry around the z axis (instead of the x axis in the above setup with two points); the potential must look like

$$\phi = A + b(-2z^2 + x^2 + y^2) \tag{195}$$

near the origin. In the xz plane (that is, for y = 0), the equipotential lines are therefore given by $z = \pm x/\sqrt{2}$. All of the above results can be summarized by saying that if a system is symmetric under rotations around a given axis, then at an equilibrium point the equipotential lines make an angle of $\tan^{-1}(\sqrt{2}) \approx 54.7^{\circ}$ with respect to the symmetry axis.

2.66. Equipotentials for two point charges

(a) From the same reasoning we used in Problem 2.19, the field on the z axis equals

$$E(x) = \frac{2Qz}{4\pi\epsilon_0 (z^2 + R^2)^{3/2}}.$$
(196)

The dependence on z is the same as in the case of the ring in Problem 2.19, so the maximum still occurs at $z = R/\sqrt{2}$. (The form of the answer is actually exactly the same; the total charge appears in the numerator, with the total charge being 2Q in the present setup, whereas it was just Q in Problem 2.19.)

- (b) A sketch of some equipotential curves is shown in Fig. 53; we have chosen R = 1. The full surfaces are obtained by rotating the curves around the x axis. Close to the two point charges, the curves are circles, which means that the equipotentials are spheres in 3D space. Far away, the curves become large circles (or spheres in 3D) around the whole setup. The transition between the double spheres and the single sphere occurs where the equipotentials cross at the origin, as shown. From Exercise 2.65, the slopes of the lines at the crossing are $\pm \sqrt{2}$.
- (c) From Fig. 53, it appears that the transition from concave up to concave down occurs at about z = 3R/2 (you can show that it's actually $\sqrt{2}R$). This isn't equal to the $z = R/\sqrt{2}$ location of the maximum field we found in part (a), so apparently the result analogous to the one in Problem 2.19 does *not* hold. Let's see why.

Consider the point on the z axis where the transition from concave up to concave down occurs. From the reasoning in Problem 2.19, we know that $\partial E_x/\partial x = 0$ at this point. In Problem 2.19 we noted that $\partial E_y/\partial y = 0$ was also zero at this point, due to the symmetry under rotations around the z axis. However, the present scenario with two point charges is *not* symmetric around the z axis. It is symmetric around the x axis instead. So we can't say that $\partial E_y/\partial y = 0$ is zero. And indeed, the cross section of the equipotential surface in the y-z plane is a circle, which is concave down. So $\partial E_y/\partial y = 0$ is positive.

Therefore, since $\partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z = 0$ from Gauss's law, and since $\partial E_x/\partial x$ is zero at the transition point (which happens to be $z = \sqrt{2}R$), and since $\partial E_y/\partial y$ is positive at any point on the z axis, we see that $\partial E_z/\partial z$ must be negative at the transition point. This means that E_z is decreasing; that is, the maximum has already occurred. This is consistent with the fact that the maximum-field z value of $R/\sqrt{2}$ from part (a) is smaller than transition-point z value of $\sqrt{2}R$.

2.67. Product of ρ and ϕ

(a) If we start with the two collections of charges very far apart and then bring collection 1 toward collection 2, a little piece of charge $dq_1 = \rho_1 dv$ in collection 1 picks up an energy of $dq_1\phi_2 = (\rho_1 dv)\phi_2$ in the presence of collection 2, where ϕ_2 is evaluated at the final location of the charge dq_1 . So the total energy of the system is $\int \rho_1\phi_2 dv$. This energy ignores the self energies of the two collections. But these self energies don't change throughout the process, so $\int \rho_1\phi_2 dv$ represents the increase in energy, that is, the work done.

If we instead bring collection 2 toward collection 1, the same reasoning shows that the work done is $\int \rho_2 \phi_1 dv$. But the work can't depend on how the collections are brought together, so we have $\int \rho_1 \phi_2 dv = \int \rho_2 \phi_1 dv$, as desired.



Figure 53

(b) Since $\mathbf{E} = -\nabla \phi$, the given vector identity can be rewritten as

$$\mathbf{E}_1 \cdot \mathbf{E}_2 = (\nabla \cdot \mathbf{E}_1)\phi_2 - \nabla \cdot (\mathbf{E}_1\phi_2). \tag{197}$$

And since $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, the identity becomes $\mathbf{E}_1 \cdot \mathbf{E}_2 = \rho_1 \phi_2/\epsilon_0 - \nabla \cdot (\mathbf{E}_1 \phi_2)$. If we integrate this over all space, we can use the divergence theorem to write the third term as an integral over a surface at infinity. But since our distributions have finite extent, \mathbf{E} falls off like $1/r^2$ (or faster, if the net charge is zero) and ϕ falls off like 1/r (or faster). Since the area of a surface grows only line r^2 , the product $E\phi$ (area) goes to zero. We therefore arrive at $\int \mathbf{E}_1 \cdot \mathbf{E}_2 \, dv = (1/\epsilon_0) \int \rho_1 \phi_2 \, dv$. The same procedure with the 1's and 2's reversed yields $\int \mathbf{E}_1 \cdot \mathbf{E}_2 \, dv = (1/\epsilon_0) \int \rho_2 \phi_1 \, dv$. Hence $\int \rho_1 \phi_2 \, dv = \int \rho_2 \phi_1 \, dv$, as desired.

2.68. E and ρ for a sphere

As in the example in Section 2.10, the Cartesian components of the electric field are given by $E_x = (x/r)E_r$, and likewise for y and z.

Inside the sphere, the field is radial with $E_r = \rho r/3\epsilon_0$, so we quickly find the Cartesian components to be $(E_x, E_y, E_z) = (\rho/3\epsilon_0)(x, y, z)$. Equation (2.59) therefore gives div $\mathbf{E} = (\rho/3\epsilon_0)(1+1+1) = \rho/\epsilon_0$, as desired.

Outside the sphere, the field is radial with $E_r = \rho R^3/3\epsilon_0 r^2$. The Cartesian x component is

$$E_x = \frac{x}{r} \frac{\rho R^3}{3\epsilon_0 r^2} \propto \frac{x}{r^3} = \frac{x}{(x^2 + y^2 + z^2)^{3/2}}.$$
 (198)

The constant of proportionality doesn't matter because the end result will be zero. The $\partial E_x/\partial x$ term in Eq. (2.59) is then

$$\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{x(-3/2)(2x)}{(x^2+y^2+z^2)^{5/2}} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}},$$
(199)

with similar expressions for the $\partial E_y/\partial y$ and $\partial E_z/\partial z$ terms. The sum of all three terms is zero, because the coefficient of x^2 is (-2+1+1), etc. This is consistent with div $\mathbf{E} = \rho/\epsilon_0$ because $\rho = 0$ outside the sphere.

2.69. E and ϕ for a slab

(a) At position x inside the slab, there is a slab with thickness $\ell - x$ to the right of x, which acts effectively like a sheet with surface charge density $\sigma_{\rm R} = (\ell - x)\rho$. Likewise, to the left of x we effectively have a sheet with surface charge density $\sigma_{\rm L} = (\ell + x)\rho$. Since the electric field from a sheet is $\sigma/2\epsilon_0$, the net field at position x inside the slab is

$$E = \frac{(\ell+x)\rho}{2\epsilon_0} - \frac{(\ell-x)\rho}{2\epsilon_0} = \frac{\rho x}{\epsilon_0}, \qquad (200)$$

and it is directed away from the center plane (if ρ is positive). You can also quickly obtain this by using a Gaussian surface that extends a distance x on either side of the center plane.

Outside the slab, the slab acts effectively like a sheet with surface charge density $\rho(2\ell)$, so the field has magnitude $(2\rho\ell)/2\epsilon_0 = \rho\ell/\epsilon_0$ and is directed away from the slab. E(x) is continuous at $x = \pm \ell$, as it should be since there are no surface charge densities in the setup.

(b) The potential relative to x = 0 is $\phi = -\int_0^x E \, dx$. Inside the slab this gives

$$\phi_{\rm in}(x) = -\int_0^x \frac{\rho x}{\epsilon_0} = -\frac{\rho x^2}{2\epsilon_0} \,. \tag{201}$$

Outside the slab, we must continue the integral past $x = \pm \ell$. On the right side of the slab, where $x > \ell$, the potential is

$$\phi(x) = -\int_{0}^{\ell} E_{x} dx - \int_{\ell}^{x} E_{x} dx = -\int_{0}^{\ell} \frac{\rho x}{\epsilon_{0}} dx - \int_{\ell}^{x} \frac{\rho \ell}{\epsilon_{0}} dx$$
$$= -\frac{\rho \ell^{2}}{2\epsilon_{0}} - \frac{\rho \ell}{\epsilon_{0}} (x - \ell) = \frac{\rho \ell^{2}}{2\epsilon_{0}} - \frac{\rho \ell x}{\epsilon_{0}}.$$
(202)

On the left side of the slab, where $x < -\ell$, you can show that the only change in ϕ is that there is a relative "+" sign between the terms (basically, just change ℓ to $-\ell$). So the potential outside the slab equals

$$\phi_{\rm out}(x) = \frac{\rho\ell^2}{2\epsilon_0} - \frac{\rho\ell|x|}{\epsilon_0} \,. \tag{203}$$

From Eqs. (201) and (203) we see that $\phi(x)$ is continuous at the boundaries at $x = \pm \ell$, as it should be. Plots of E(x) and $\phi(x)$ are shown in Fig. 54.

(c) For a single Cartesian direction, we have $\nabla \cdot \mathbf{E} = \partial E_x / \partial x$ and $\nabla^2 \phi = \partial^2 \phi / \partial x^2$. The following four relations are indeed all true:

Inside :
$$\rho(x) = \epsilon_0 \nabla \cdot \mathbf{E} \iff \rho = \epsilon_0 \partial (\rho x/\epsilon_0)/\partial x,$$
 (204)
Outside : $\rho(x) = \epsilon_0 \nabla \cdot \mathbf{E} \iff 0 = \epsilon_0 \partial (\rho \ell/\epsilon_0)/\partial x,$
Inside : $\rho(x) = -\epsilon_0 \nabla^2 \phi \iff \rho = -\epsilon_0 \partial^2 (-\rho x^2/2\epsilon_0)/\partial x^2,$
Outside : $\rho(x) = -\epsilon_0 \nabla^2 \phi \iff 0 = -\epsilon_0 \partial^2 (\rho \ell^2/2\epsilon_0 \pm \rho \ell x/\epsilon_0)/\partial x^2.$

We also have $\mathbf{E} = -\nabla \phi$ both inside and outside, which is true by construction due to the line integrals we calculated in part (b).

2.70. Triangular E

The slopes in the triangular part of the plot of E are $\pm E_0/a$, so we quickly find that in the left and right regions near the origin, E(x) takes the form of

$$E_{\rm L}(x) = (E_0/a)x + E_0$$
 and $E_{\rm R}(x) = -(E_0/a)x + E_0.$ (205)

Gauss's law in differential form is $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$, which in one dimension becomes simply $\rho = \epsilon_0 \partial E_x / \partial x$. So the charge densities in the left and right regions are

$$\rho_{\rm L} = \epsilon_0 E_0 / a \quad \text{and} \quad \rho_{\rm R} = -\epsilon_0 E_0 / a.$$
(206)

And $\rho = 0$ outside the $-a \le x \le a$ region. So we have two slabs with opposite charge densities, with the positive slab on the left.

Since $\mathbf{E} = -\nabla \phi$ (which in one dimension becomes $\mathbf{E} = -\hat{\mathbf{x}} \partial \phi / \partial x$), we simply need to integrate E(x) to obtain $\phi(x)$. We find

$$\phi_{\rm L}(x) = -E_0 x^2 / 2a - E_0 x$$
 and $E_{\rm R}(x) = E_0 x^2 / 2a - E_0 x.$ (207)

There is technically a constant of integration in each of these expressions, but the constants are zero if we take $\phi = 0$ at x = 0. Since $\mathbf{E} = 0$ outside the $-a \leq x \leq a$





Figure 55

region, ϕ is constant, taking on the values it has at the boundaries, namely $\pm E_0 a/2$. The plots of ρ , E, and ϕ are shown in Fig. 55.

A double check: At x = 0, the two slabs act effectively like sheets with charge densities $\pm \sigma = \pm \rho a$. They each create a field pointing to the right with magnitude $\sigma/2\epsilon_0$, so the total field at x = 0 is $2(\rho a)/2\epsilon_0 = \rho a/\epsilon_0$. And since we found above that $\rho = \epsilon_0 E_0/a$, this field equals E_0 , in agreement with the given value.

2.71. A one-dimensional charge distribution

The given potential is shown in Fig. 56(a). The electric field is found via $\mathbf{E} = -\nabla \phi$. For $|x| > \ell$ the potential is constant, so E = 0 there. For $|x| < \ell$ the potential depends only on x, so we have

$$E_x = -\frac{d\phi}{dx} = 2Bx. \tag{208}$$

The plot of E_x is shown in Fig. 56(b). Note that it is discontinuous at $x = \pm \ell$. This implies that there must be a surface charge density on the planes at $x = \pm \ell$.



Figure 56

The charge density is given by $\rho = -\epsilon_0 \nabla^2 \phi$, or equivalently by $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$. For $|x| > \ell$ we have $\rho = 0$, and for $|x| < \ell$ we have

$$\rho = -\epsilon_0 \frac{d^2 \phi}{dx^2} = 2\epsilon_0 B. \tag{209}$$

So we have a uniform slab of charge between $x = -\ell$ and $x = \ell$. However, we aren't quite done, because as mentioned above, there is also a surface charge density on the planes at $x = \pm \ell$. This is consistent with $\rho = -\epsilon_0 \nabla^2 \phi$, because if you tried to calculate

 $-\epsilon_0 \nabla^2 \phi$ or $\epsilon_0 \nabla \cdot \mathbf{E}$ there, you would obtain an infinite result (due to the discontinuity in E_x), consistent with the fact that a surface charge occupies zero volume.

To determine the surface charge density σ on the two planes, we can look at the discontinuity in the field across them. From Fig. 56(b), E has a downward jump of $-2B\ell$ at both planes. Gauss's law tells us that the change in the field at a surface is equal to σ/ϵ_0 . Hence $\sigma = -2\epsilon_0 B\ell$. You can quickly check that the sign here makes the discontinuity in E work out properly. The plot of ρ is shown in Fig. 56(c), where the arrows indicate the negative infinite volume densities associated with the surface charges.

Our system therefore consists of a thick slab with positive volume charge density $\rho = 2\epsilon_0 B$ sandwiched between two sheets with negative surface charge density $\sigma = -2\epsilon_0 B\ell$. Note that the total charge per unit area in the system is $2\sigma + \rho(2\ell)$, which equals zero. This is consistent with the fact that the electric field is zero for $|x| > \ell$.

2.72. A spherical charge distribution

The given potential, shown in Fig. 57(a), arises from a spherically symmetric charge distribution. The potential is more briefly described in spherical coordinates by

$$\phi(r) = \begin{cases} \frac{\rho_0 r^2}{4\pi\epsilon_0} & \text{(for } r < a), \\ \frac{\rho_0}{4\pi\epsilon_0} \left(-a^2 + \frac{2a^3}{r} \right) & \text{(for } r > a). \end{cases}$$
(210)

Note that $\phi(r)$ is continuous at r = a, where it takes on the value $\rho a^2/4\pi\epsilon_0$. Also note that $\phi = -\rho a^2/4\pi\epsilon_0$ at $r = \infty$; it is not necessary to have $\phi = 0$ at infinity.





The electric field is given by $\mathbf{E} = -\nabla \phi$, which reduces to $E_r = -d\phi/dr$ for a function that depends only on r. This gives

$$E_{r}(r) = \begin{cases} \frac{-\rho_{0}r}{2\pi\epsilon_{0}} & \text{(for } r < a), \\ \frac{\rho_{0}a^{3}}{2\pi\epsilon_{0}r^{2}} & \text{(for } r > a). \end{cases}$$
(211)

A plot of E_r is shown in Fig. 57(b). You should verify that you obtain the same field if you work with Cartesian coordinates, where $\nabla \phi = (\partial \phi / \partial x, \partial \phi / \partial y, \partial \phi / \partial z)$. Note that $E_r(r)$ is not continuous at r = a; it jumps from $-\rho a/2\pi\epsilon_0$ to $\rho a/2\pi\epsilon_0$. This implies that there must be a surface charge density on the sphere with radius a.

To obtain the charge distribution, we can use $\rho = -\epsilon_0 \nabla^2 \phi$, or equivalently $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$. In spherical coordinates, the Laplacian of a function that depends only on r is given by $\nabla^2 \phi = (1/r^2)(d/dr)(r^2 d\phi/dr)$, and the divergence of a vector function that depends only on r is given by given by $\nabla \cdot \mathbf{E} = (1/r^2)\partial(r^2 E_r)/\partial r$. Either of these gives

$$\rho(r) = \begin{cases}
-\frac{3\rho_0}{2\pi} & \text{(for } r < a), \\
0 & \text{(for } r > a).
\end{cases}$$
(212)

Again, you should verify that you obtain these same results in Cartesian coordinates, where $\nabla^2 \phi = \partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 + \partial^2 \phi / \partial z^2$ and $\nabla \cdot \mathbf{E} = \partial E_x / \partial x + \partial E_y / \partial y + \partial E_z / \partial z$.

We therefore have a uniform charge density inside r = a (the total charge there is $4\pi a^3 \rho/3 = -2\rho_0 a^3$), and zero charge outside. But as mentioned above, there is also a surface charge density σ on the sphere with radius a. Equation (212) doesn't contradict this fact, because that equation has nothing to say about $\rho(r)$ right at r = a. If you tried to calculate $-\epsilon_0 \nabla^2 \phi$ or $\epsilon_0 \nabla \cdot \mathbf{E}$ there, you would obtain an infinite result due to the discontinuity in \mathbf{E} , consistent with the fact that a surface charge occupies zero volume.

To determine σ , we can look at the discontinuity in the field across the sphere at r = a. From Eq. (211) the field just inside this sphere is $-\rho_0 a/2\pi\epsilon_0$, and the field just outside is $\rho_0 a/2\pi\epsilon_0$. Gauss's law (with a little pillbox) tells us that the change in the field at the surface, which is $\rho_0 a/\pi\epsilon_0$, must equal σ/ϵ_0 . Hence $\sigma = \rho_0 a/\pi$. The total charge on the surface is then $4\pi a^2 \sigma = 4\rho_0 a^3$, which is twice as large as the $-2\rho_0 a^3$ charge distributed throughout the inside the sphere. The external field of the entire sphere is therefore the field of a net charge $4\rho_0 a^3 - 2\rho_0 a^3 = 2\rho_0 a^3$, which is $E_r(r) = (2\rho_0 a^3)/4\pi\epsilon_0 r^2 = \rho_0 a^3/2\pi\epsilon_0 r^2$ in agreement with Eq. (211). Indeed, working backward from this external field would be another way of finding the surface density σ .

A plot of $\rho(r)$ is shown in Fig. 57(c). The spike indicates the infinite volume charge density associated with the surface charge density. Looking back at the plot of $\phi(r)$ Fig. 57(a), note that the first derivative of ϕ (which is related to the field) is not well defined at r = a, consistent with the discontinuity in E. Also, the second derivative of ϕ (which is related to the density) is infinite at r = a, consistent with the infinite ρ .

2.73. Satisfying Laplace

In $f(x, y) = x^2 + y^2$, then

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 + 2 \neq 0.$$
(213)

If $g(x, y) = x^2 - y^2$, then $\nabla^2 f = 2 - 2 = 0$. So g satisfies Laplace's equation, but f does not. The plot of g(x, y) looks like the saddle shown in Fig. 58. It is a positive parabola along the x axis, and a negative parabola along the y axis.



Figure 58









The gradient of g is $\nabla g = (\partial g/\partial x, \partial g/\partial y) = (2x, -2y)$. So the gradients at the points (0, 1), (1, 0), (0, -1), (-1, 0), are, respectively, the vectors (0, -2), (2, 0), (0, 2), (-2, 0). These are indicated in Fig. 59. Not that the gradient points in the direction of steepest ascent, as it should.

Since the divergence of (2x, -2y) equals zero (or equivalently since the Laplacian of ϕ equals zero), there is zero net flux of the vector field (2x, -2y) out of any closed volume. You should convince yourself that this is believable by looking at the full vector field in Fig. 60.

2.74. Oscillating exponential ϕ

(a) Given $\phi = \phi_0 e^{-kz} \cos kx$, we have

$$\frac{\partial \phi}{\partial x} = -k\phi_0 e^{-kz} \sin kx, \qquad \qquad \frac{\partial^2 \phi}{\partial x^2} = -k^2 \phi_0 e^{-kz} \cos kx, \qquad (214)$$

and

$$\frac{\partial \phi}{\partial z} = -k\phi_0 e^{-kz} \cos kx, \qquad \qquad \frac{\partial^2 \phi}{\partial z^2} = k^2 \phi_0 e^{-kz} \cos kx. \tag{215}$$

Therefore,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$
(216)

(b) From the derivatives in part (a), we have

$$\mathbf{E} = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \left(k\phi_0 e^{-kz} \sin kx, 0, k\phi_0 e^{-kz} \cos kx\right)$$
$$= k\phi_0 e^{-kz} (\sin kx, 0, \cos kx). \tag{217}$$

To get a sense of what this field looks like, note that $E_z/E_x = 1/\tan kx$. This is independent of z, so for a given value of x, the slopes of all the field-line curves are the same. This slope is infinite for x = 0, $\pm \pi/k$, $\pm 2\pi/k$, etc, and it is zero for $x = \pm \pi/2k$, $\pm 3\pi/2k$, etc. Fig. 61 shows a few of the curves. The density of the field lines in this figure does *not* indicate the strength of the field, so we have also drawn Fig. 62, in which the relative field strengths are accurately presented; this figure shows only half of a cycle in x, for the sake of clarity. The constant nature of the slope for a given value of x is clear from this figure. (Equivalently, Fig. 61 looks the same at any height.) Since the surface charge density on the sheet is proportional to the normal component of the field, it is evident from Fig. 61 that the charge density is high at x = 0, $\pm \pi/k$, etc., while it is zero at $x = \pm \pi/2k$, etc. This is consistent with the result we will find in part (c).

If you want to calculate the actual shape of the field-line curves, you can do this as follows. If a curve is described by the function z(x), then the slope dz/dx is given by the ratio E_z/E_x that we found above (because the **E** field is by definition tangent to the field-line curve). Therefore, if a particular field line crosses the xaxis at $x = x_0$ (assume $0 < x < \pi/2k$), we have

$$\frac{dz}{dx} = \frac{1}{\tan kx} \implies \int_0^z dz = \int_{x_0}^x \frac{\cos kx \, dx}{\sin kx}$$
$$\implies z = \frac{1}{k} \ln(\sin kx) \Big|_{x_0}^x = \frac{1}{k} \ln\left(\frac{\sin kx}{\sin kx_0}\right). \tag{218}$$

It is actually slightly more informative to write this as $z = (1/k) [\ln(\sin kx) - \ln(\sin kx_0)]$. This form makes it clear that the different curves have the same shape but are simply shifted vertically relative to each other by an amount $-(1/k) \ln(\sin kx_0)$. The closer x_0 is to zero, the larger this term is, so the higher the curve goes in the xz plane. This is consistent with Fig. 61. This form also makes it clear that all the different curves associated with different values of x_0 have the same slope for a given value of x.

(c) The component of the field perpendicular to the sheet, very close to the sheet, satisfies $E_z = \sigma/2\epsilon_0$. Therefore,

$$\sigma = 2\epsilon_0 E_z \Big|_{z=0} = 2\epsilon_0 k\phi_0 e^{-0} \cos kx = 2\epsilon_0 k\phi_0 \cos kx.$$
(219)

Note that since we are told that the only charges in the system are on the sheet, the system is symmetric with respect to the sheet. So the potential and field below the sheet are the mirror images of the potential and field above the sheet.

2.75. Curls and divergences

In Cartesian coordinates,

$$(\nabla \times \mathbf{F}) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right),$$

$$\nabla \cdot \mathbf{E} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

$$(220)$$

- (a) If $\mathbf{F} = (x + y, -x + y, -2z)$ we quickly find $\nabla \times \mathbf{F} = (0, 0, -2)$ and $\nabla \cdot \mathbf{F} = 1 + 1 2 = 0$. Since the curl isn't zero, there is no associated potential ϕ .
- (b) If $\mathbf{G} = (2y, 2x + 3z, 3y)$ then we find $\nabla \times \mathbf{G} = (0, 0, 0)$ and $\nabla \cdot \mathbf{G} = 0 + 0 + 0 = 0$. Since $\nabla \times \mathbf{G} = 0$ there exists a g such that $\mathbf{G} = \nabla g$. To determine g, we can compute the line integral of **G** from a fixed point, say (0, 0, 0), to a general point (x_0, y_0, z_0) over any path. Using the path composed of the three segments in the x, then y, then z directions, we have

$$g(x_0, y_0, z_0) = \int_{(0,0,0)}^{(x_0, y_0, z_0)} \mathbf{G} \cdot d\mathbf{s}$$

= $\int_0^{x_0} G_x(x, 0, 0) \, dx + \int_0^{y_0} G_y(x_0, y, 0) \, dy + \int_0^{z_0} G_z(x_0, y_0, z) \, dz$
= $\int_0^{x_0} 0 \, dx + \int_0^{y_0} 2x_0 \, dy + \int_0^{z_0} 3y_0 \, dz$
= $2x_0y_0 + 3y_0z_0.$ (221)

Since (x_0, y_0, z_0) is a general point, we can drop the subscripts and write g(x, y, z) = 2xy + 3yz. You can quickly check that the gradient of g is indeed **G**. A quicker method of obtaining g is the following. The x component of $\nabla g = \mathbf{G}$ tells us that $\partial g/\partial x = 2y$. So g must be a function of the form $2xy + f_1(y, z)$. Similarly, the y component tells us that g must take the form $3yz + f_2(x, z)$, and the z component tells us that g must take the form $3yz + f_3(x, y)$. You can quickly verify that the only function satisfying all three of these forms is 2xy+3yz (plus a constant).

(c) If $\mathbf{H} = (x^2 - z^2, 2, 2xz)$ then we find $\nabla \times \mathbf{H} = (0, -4z, 0)$ and $\nabla \cdot \mathbf{H} = 2x + 0 + 2x = 4x$. Since the curl isn't zero, there is no associated potential ϕ .

2.76. Zero curl

We are given $E_x = 6xy$, $E_y = 3x^2 - 3y^2$, $E_z = 0$. So we have

$$(\nabla \times \mathbf{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0,$$

$$(\nabla \times \mathbf{E})_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0,$$

$$(\nabla \times \mathbf{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 6x - 6x = 0.$$
(222)

The divergence is

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 6y - 6y = 0.$$
(223)

The zero here implies that the associated charge density is zero.

2.77. Zero dipole curl

The dipole **E** field in Eq. (2.36) has no angular ϕ dependence, and also no $\hat{\phi}$ component. So we quickly see that only the $\hat{\phi}$ component of the spherical-coordinate expression for $\nabla \times \mathbf{A}$ in Eq. (F.3) in Appendix F survives. Using the values of E_r and E_{θ} from Eq. (2.36) we have

$$\nabla \times \mathbf{E} = \frac{1}{r} \left(\frac{\partial (rE_{\theta})}{\partial r} - \frac{\partial E_r}{\partial \theta} \right) \hat{\phi} = \frac{q\ell}{4\pi\epsilon_0} \frac{1}{r} \left(\frac{\partial (\sin\theta/r^2)}{\partial r} - \frac{\partial (2\cos\theta/r^3)}{\partial \theta} \right) \hat{\phi}$$
$$= \frac{q\ell}{4\pi\epsilon_0} \frac{1}{r} \left(\frac{-2\sin\theta}{r^3} - \frac{-2\sin\theta}{r^3} \right) \hat{\phi} = 0.$$
(224)

REMARK: Let's look at what's going on physically in the special case of $\theta = \pi/2$. Consider the circulation of the field around the loop shown in Fig. 63, which consists of radial and tangential segments. The tangential piece on the right is longer than the piece on the left, being proportional to r. If the field fell off like 1/r, these effects would cancel in the line integral, and there would be no net circulation from the tangential parts. But for our dipole, the field falls off like $1/r^3$, so the contribution from the left piece dominates, yielding a net counterclockwise circulation from the tangential pieces. This has the correct sign to cancel with the clockwise circulation from the radial parts (which simply add; from Eq. (2.36) there is a very small positive E_r just above the $\theta = \pi/2$ line, and a very small negative E_r just below). So it's believable that things work out, although the above calculation is needed to show quantitatively that the curl is exactly zero.

2.78. Divergence of the curl

(a) In Cartesian coordinates the divergence of the curl is

$$\nabla \cdot (\nabla \times \mathbf{A}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$
$$= \frac{\partial}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$
$$= \left(\frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z}\right) + \left(\frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x}\right) + \left(\frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}\right)$$
$$= 0. \tag{225}$$

We have used the fact that partial differentiation commutes, for any function with continuous derivatives.





(b) The derivation can be summed up by the relations,

$$\int_{C} \mathbf{A} \cdot d\mathbf{s} = \int_{S} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_{V} \nabla \cdot (\nabla \times \mathbf{A}) \, dv.$$
(226)

The first equality is the statement of Stokes' theorem, and the second is the statement of Gauss's theorem (the divergence theorem) applied to the vector " $\nabla \times \mathbf{A}$." The logic of the derivation is as follows. The line integral of \mathbf{A} around the curve C in Fig. 2.52 is zero because the curve backtracks along itself. (We can make the two "circles" of C be arbitrarily close to each other, and they run in opposite directions.) Stokes' theorem then tells us that the surface integral of $\nabla \times \mathbf{A}$ over S is also zero. The surface S is essentially the same as the closed surface S' consisting of S plus the tiny area enclosed by C. So the surface integral of $\nabla \times \mathbf{A}$ over S' is zero. But S' encloses the volume V, so Gauss's theorem tells us that the volume integral of $\nabla \cdot (\nabla \times \mathbf{A})$ over V is also zero. Since this result holds for any arbitrary volume V, the integrand $\nabla \cdot (\nabla \times \mathbf{A})$ must be identically zero, as we wanted to show.²

This logic here basically boils down to the mathematical fact that the boundary of a boundary is zero. More precisely, the volume integral of $\nabla \cdot (\nabla \times \mathbf{A})$ equals (by Gauss) the surface integral of $\nabla \times \mathbf{A}$ over the boundary S' of the volume V, which in turn equals (by Stokes) the line integral of \mathbf{A} over the boundary C of the boundary S' of the volume V. But S' has no boundary, so C doesn't exist. That is, C has zero length. The line integral over C is therefore zero, which means that the original volume integral of $\nabla \cdot (\nabla \times \mathbf{A})$ is also zero.

In view of this, there actually wasn't any need to pick the curve C to be of the specific stated form. We could have just picked a very tiny circle. The first step in the above derivation, namely that the line integral of **A** around the curve C is zero, still holds (but now simply because C has essentially no length), so the derivation proceeds in exactly the same way.

2.79. Vectors and squrl

If squrl **F** is uniquely defined at a given point in space, then if we reverse the direction of $\hat{\mathbf{n}}$ (that is, if we reverse the orientation of the integration around the little loop), the lefthand side of the given equation changes sign. But the righthand side can only be positive. Therefore, squrl **F** cannot be uniquely defined at any point. (In the original definition of the curl, the righthand side does indeed change sign, because the direction of the integration around the loop reverses.)

 $^{{}^{2}}$ If $\nabla \cdot (\nabla \times \mathbf{A})$ were different from zero at some point, then the integral over a small volume containing this point would be nonzero. This is true because we can pick the volume to be small enough so that $\nabla \cdot (\nabla \times \mathbf{A})$ is essentially constant, so there is no possibility of cancelation.

CHAPTER 2. THE ELECTRIC POTENTIAL

Chapter 3

Electric fields around conductors

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

3.31. In or out

The charge distributions are shown in Fig. 64. In both cases, there is negative charge on the inside surface of the inner "ring" (due to the attraction to the charge q) and positive charge on the outside surface of the outer ring (due to the self repulsion of the leftover positive charge).

In the first case, the charge q is *outside* the conducting shell, so there must be zero field inside. Consistent with this, there is no charge on the inner side of the surface that touches the hollow interior of the conductor. (If there were such a charge, we could draw a Gaussian surface that lies partially inside the metal of the conductor, and partially in the interior of the conducting shell, to show that there would be a nonzero field in the interior.)

In the second case, the charge q is *inside* the conducting shell, so there is a nonzero field in the interior. Consistent with this, there is nonzero charge (the negative charge shown) on the inner side of the surface that touches the hollow interior of the conductor. The positive charge is still outside, as it was in the first case.

The first case is consistent with the fact that there is no electric field in the hollow interior of a conducting shell containing no charge, while the second case doesn't apply because there is charge inside.

3.32. Gravity screen

The electric field can be "blocked" because of the existence of charges of opposite signs. As seen in Fig. 3.8, these oppositely-signed charges set up a compensating electric field. If only positive charges existed, conductors couldn't block electric fields. For this reason, since only one sign of gravitational mass exists, it is impossible to block the gravitational field.

If negative-mass particles existed, then in theory it would be possible to block a gravitational field. For example, if we have a point mass located outside a spherical shell, we could put some negative mass on the near size of the shell. This mass would



Figure 64

repel a positive-mass particle in the interior, canceling the attraction from the original positive point mass.

However, there is one aspect in which the gravitational case differs fundamentally from the electrical case. In the latter, like charges repel; whereas in the former, like masses attract. This means that in the gravitational case, like masses would attract each other and collapse down to a point, if they were allowed to move freely. So we would have to bolt them down in the desired distribution.

3.33. Two concentric shells

(a) The charge distributions and field lines are shown (roughly) in Fig. 65. Let the four surfaces be labeled 1, 2, 3, 4, starting from the innermost one. There is charge -q on surface 1. This is true because the field is zero inside the metal of the conductor, so a spherical Gaussian surface drawn inside the metal of the inner conductor has no flux, so the net charge enclosed in the sphere must be zero. The negative surface charge density on surface 1 is higher near the off-center point charge. Since the inner conductor is neutral, a charge +q must reside on surface 2. This surface charge density is spherically symmetric, because it feels no field from the charges inside (or outside) of it, due to the zero field inside the metal of the conductors.



By the same Gauss's-law reasoning, there must be a charge -q on surface 3, because there is zero field inside the metal of the outer conductor. This surface charge density is spherically symmetric. A charge +q is left for surface 4. This surface charge density is actually negative near the outer charge q if that charge is located close enough to the shells; see Exercise 3.49. But in any case the total charge on surface 4 is +q. Between the shells, the field is spherically symmetric, consistent with the spherically symmetric charge densities on surfaces 2 and 3.

(b) The shells are now at the same potential, so the field between them must be zero. Therefore, the only difference from the scenario in part (a) is that we just need to erase the field between the shells and erase the charges on surfaces 2 and 3. The charges on surfaces 1 and 4 aren't affected by this change, because the surfaces 2 and 3 together produced zero field everywhere except between them. Basically, when we connect the shells, the charges on surfaces 2 and 3 simply neutralize each other.

3.34. Equipotentials

Assume that the point charge is positive (the general result is the same if it is negative). Then the near part of the sphere ends up negatively charged, and the far part ends up positively charged. (The sizes of these two regions depend on the distance from the point charge to the sphere.) By continuity, there must therefore be a circle on the sphere where the charge density is zero. But the electric field near the sphere (which is perpendicular to the conducting surface) is given by σ/ϵ_0 . So if $\sigma = 0$ on the circle, then E = 0 also.

The general shapes of the equipotentials are shown in Fig. 66. (The various curves have been chosen to indicate the general features; their potential values aren't equally spaced.) In this specific case, the distance from the point charge to the sphere has been chosen to be twice the radius of the sphere. The transition from small circles around the point charge to large circles around the whole system takes place via the equipotential curve that heads straight into the sphere and then splits in two, encircling the sphere. At the point where the curve splits, it changes direction abruptly. Since the electric field must be perpendicular to the equipotential surface at every point, this means that the electric field must point in two different directions is the zero vector. So this is a second way of seeing why there must be points on the surface of the sphere where the electric field is zero.





3.35. Surface density at a corner

Consider first the electric field due to an infinite strip of charge, with width b, infinite length, and negligible thickness. The surface charge density takes on the constant value σ . Let us find the electric field at a point P located a distance x from one edge of the strip, in the plane of the strip.

We can slice the strip into narrow rods, and then integrate over the rods. Consider a rod with width dr at a distance r from a given point P; see Fig. 67. The electric field at P due to the rod is $\lambda/2\pi\epsilon_0 r$, where $\lambda = \sigma dr$. The distance r runs from x to x + b,



Figure 67



Figure 68



Figure 69

so the total field at P due to the strip is

$$E(x) = \int_{x}^{x+b} \frac{\sigma \, dr}{2\pi\epsilon_0 r} = \frac{\sigma}{2\pi\epsilon_0} \ln\left(\frac{x+b}{x}\right). \tag{227}$$

If $x \to 0$ this result diverges (slowly, like $\ln x$). This divergence is what causes the electric field at the corner of the square tube to diverge, for the following reason.

If we treat the corner like an exact point, then a cross section is shown in Fig. 68. The given point P is near the edges of two different strips (the two adjacent faces of the tube). P doesn't lie exactly in the plane of each strip, but this doesn't matter. The field from each strip differs from the field in part (a) by a finite additive amount, so it still diverges as $x \to 0$. This is true because if we ignore the "rods" in the strip that are within a distance of, say, 5x from P, then P can be treated as essentially lying in the plane of the remaining part of the strip. The effective value of x is now 6x, but the factor of 6 doesn't matter; the field still diverges as $x \to 0$. (This reasoning holds for any location near the corner; P need not lie on the line of the angle bisector.) We are concerned only with the component of the field that lies along the angle bisector, so this brings in a factor of $\cos 45^\circ = 1/\sqrt{2}$ in the field from each strip. But this doesn't change the fact that the total field diverges.

If we treat the corner more realistically as curved (like a quarter circle), then the above reasoning still applies. Ignoring the nearby part of the charge distribution still leaves us with two strips that each produce an infinite field, in the limit where the radius of curvature of the quarter circle goes to zero (assuming that P is close to the quarter circle, on the order of the radius). If the radius does *not* go to zero, then the field certainly doesn't diverge. So the "corner" of the tube needs to be sharp in order for the field to diverge.

We have been treating the charge density σ as constant. But in a conducting tube, the density increases near the corners, because of the self-repulsion of the charges. This has the effect of making the field even larger than the above reasoning would imply, so the above conclusion of a diverging field is still valid. Since the conclusion is true for both conducting and nonconducting tubes, the word "conducting" in the statement of the problem could have been omitted.

In the case of a curved corner, if P is very close to the quarter circle (or whatever curve), then we can draw a tiny Gaussian pillbox Fig. 69 to say that the field at P equals σ/ϵ_0 . Since we just showed that the field diverges, this implies that the density also diverges at the corner. Intuitively, if it didn't diverge, then there wouldn't exist a sufficient force to keep the charges in the straight parts of Fig. 69 from flowing onto the curved part. So this would eventually lead to a very large density at the corner anyway.

All of the above reasoning still holds if the cross section of the tube is something other than a square. At any point where the direction of the surface changes abruptly, the field diverges. Even for a polygon with 100 sides, in which the surface bends by only a few degrees at each "corner," the field still diverges, because when taking the component along the angle bisector, the nonzero trig factor doesn't change the fact that the total field diverges.

If we kick things down a dimension and look at a kink in a wire, the field still diverges (even more quickly). This is true because the field near the end of a uniform stick diverges; Eq. (227) is replaced by

$$E(x) = \int_{x}^{x+b} \frac{\lambda \, dr}{4\pi\epsilon_0 r^2} = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{x} - \frac{1}{x+b}\right). \tag{228}$$

This diverges as $x \to 0$.

In the case of a cone, a slightly different calculation is required, but the field still diverges. See Problem 1.3.

3.36. Zero flow

The point is that although the field is weaker near the larger sphere, it has an appreciable size over a larger distance than does the field from the smaller sphere. The fields at the surfaces are proportional to $Q/R^2 \propto R/R^2 = 1/R$. And the fields fall off on a distance scale of R, because at a radius of 2R, the field has decreased by a factor $1/2^2$, etc. These two effects cancel.

We can be quantitative. As in Problem 3.10, the thin wire has essentially zero capacitance, so charge can't pile up on it. But a tiny bit can, so as mentioned in the solution to Problem 3.10, we effectively have a rigid stick of (a tiny bit of) charge extending from one sphere to the other. Assuming constant density λ , the repulsive force on the stick from the smaller sphere is $\int_{R_1}^{D-R_2} E_1 \lambda \, dr$, where D is the distance between the centers of the spheres, and where $E_1(r)$ is the field due to the smaller sphere. Likewise, the repulsive force from the larger sphere is $\int_{R_2}^{D-R_1} E_2 \lambda \, dr$. If the spheres are far apart, we can replace the upper limits of these integrations by ∞ ; the fields become negligible at large distances. If we set these two forces equal to each other and cancel the λ 's, we simply have the statement that the potentials of each shell, relative to infinity, are equal. In a sense, we can think of many weak people forcing the charge away from the larger sphere, with only a few strong people forcing it away from the small sphere. The two total forces exactly cancel, and no charge moves.

3.37. A charge between two plates

Consider the Gaussian surface shown in Fig. 70, which has very large extent in the x and z directions. Two of its faces lie inside the metal of the two plates. The field is zero inside the metal of the plates; and between the plates the field is negligible at points far away from the charge. So there is zero flux through the Gaussian surface. By Gauss's law, the total charge inside must be zero, which implies that the total charge on the inner surfaces of the plates must be -Q.

Dividing the charge Q into many smaller charges, all on the plane y = s, yields the same total charge on each plate, by superposition. We can take the continuum limit and smear out the charge Q uniformly onto a sheet with a large area A at y = b. The surface charge density on this sheet is $\sigma = Q/A$ (this is the sum of the densities on its two surfaces). If we can find the total charges on the two plates in this scenario, then we will have found the total charges on the two plates in the original scenario involving the point charge Q.

Call the charges on (the inner surface of) each plate Q_1 and Q_2 . The densities are then $\sigma_1 = Q_1/A$ and $\sigma_2 = Q_2/A$, so $Q_1/Q_2 = \sigma_1/\sigma_2$. The same area A applies to the two plates because we are assuming A is large. The edge effects will be negligible, so we can assume that σ_1 and σ_2 are essentially uniform over the area A. (And σ is uniform, by construction.)

From the standard argument using Gauss's law with one face of the Gaussian surface lying inside the metal of a conductor, the fields in the two different regions are $E_1 = \sigma_1/\epsilon_0$ and $E_2 = \sigma_2/\epsilon_0$. So $E_1/E_2 = \sigma_1/\sigma_2$. But the two plates are at the same potential, so we also know that $E_1b = E_2(s-b)$, because these are the differences in potential from the middle sheet. Hence $E_1/E_2 = (s-b)/b$. So we have

$$\frac{Q_1}{Q_2} = \frac{\sigma_1}{\sigma_2} = \frac{E_1}{E_2} = \frac{s-b}{b} \implies \frac{Q_1}{Q_2} = \frac{s-b}{b}.$$
 (229)



But we also have $Q_1 + Q_2 = -Q$. Solving these two equations for Q_1 and Q_2 gives

$$Q_1 = -Q \frac{s-b}{s}$$
 and $Q_2 = -Q \frac{b}{s}$. (230)

If $b \ll s$, then we have $Q_1 \approx -Q$ and $Q_2 \approx 0$, as expected. In general, the charges are in the inverse ratio of the distances from the plates to the intermediate sheet (or to the given point charge Q).

3.38. Two charges and a plane

First note that such a location must exist, due to a continuity argument: If the -Q charge is placed only slightly below the fixed Q charge, the upward attractive force from the Q charge will dominate. But if the -Q charge is placed only slightly above the conducting plane, the downward attractive force from the +Q image charge will dominate. So somewhere in between, the force on the -Q charge must be zero.

Let y be the distance from the -Q charge to the plane. The field above the plane due to the two given charges along with the induced charge on the plane is identical to the field due to the two given charges along with the two image charges below the plane shown in Fig. 71. The given -Q charge feels the fields due to the other three charges. Taking upward to be positive, the force on the given -Q charge is

$$F = \frac{Q^2}{4\pi\epsilon_0} \left(\frac{1}{(\ell-y)^2} - \frac{1}{(2y)^2} + \frac{1}{(\ell+y)^2} \right).$$
(231)

Setting this equal to zero yields

$$\frac{1}{4y^2} = \frac{2(\ell^2 + y^2)}{(\ell^2 - y^2)^2} \implies 7y^4 + 10\ell^2 y^2 - \ell^4 = 0.$$
(232)

This is a quadratic equation in y^2 . We are concerned with the positive root (since y^2 is positive), which is

$$y^{2} = \frac{(-5 + 4\sqrt{2})\ell^{2}}{7} \approx (0.0938)\ell^{2} \implies y = (0.306)\ell.$$
(233)

3.39. A wire above the earth

Let L = 200 m, h = 5 m, and $\lambda = 10^{-5} \text{ C/m}$. By superposition, the relevant image charge is an oppositely-charged wire of the same length below the surface of the earth. Because L is much larger than h, we can consider (except near the ends of the wire) the wires to be of infinite length, as far as finding the field goes. The field at the surface of the earth is due to both of the wires, so it points downward with magnitude

$$E_{\text{surface}} = 2 \cdot \frac{\lambda}{2\pi\epsilon_0 h} = \frac{10^{-5} \,\text{C/m}}{\pi \left(8.85 \cdot 10^{-12} \,\frac{\text{s}^2 \,\text{C}^2}{\text{kg m}^3}\right)(5 \,\text{m})} = 7.2 \cdot 10^4 \,\frac{\text{V}}{\text{m}} \,. \tag{234}$$

The electrical force on the given wire is the force due to the field arising from the imagecharge wire. The total charge on the given wire is $q = \lambda L = (10^{-5} \text{ C/m})(200 \text{ m}) = 2 \cdot 10^{-3} \text{ C}$. Over nearly the whole length of the wire, the field due to the imagecharge wire is essentially $\lambda/2\pi\epsilon_0(2h) = 1.8 \cdot 10^4 \text{ V/m}$, directed downward. This is a quarter of the field we found above, which involved two wires and half the distance. Neglecting the decrease in field near the ends of the wire, the force on the given wire is $qE = (2 \cdot 10^{-3} \text{ C})(1.8 \cdot 10^4 \text{ V/m}) = 36 \text{ N}$, directed downward. In terms of the various parameters, this force is $qE = (\lambda L)(\lambda/2\pi\epsilon_0(2h)) = (\lambda^2/4\pi\epsilon_0)(L/h)$.



o -Q

Figure 71
REMARK: The above result is a good approximation in the limit where $L \gg h$. Let's try to get a handle on the error involved. Since we assumed that the field was uniform over the length of the wire, what we actually calculated was the force on a finite wire due to an infinite wire (Fig. 72(a)). This is larger than the force between two finite wires (Fig. 72(b)) because of the two extra half-infinite wires (Fig. 72(c)) that need to be added to the lower finite wire to make the infinite wire.



As as exercise, you can show that the vertical component of the force between the two halfinfinite wires in Fig. 72(d) is $\lambda^2/4\pi\epsilon_0$, which is independent of the separation (this follows from a dimensional-analysis argument). To a good approximation, the original finite wire can be treated as half-infinite for this purpose. (The error is of order $(\lambda^2/4\pi\epsilon_0)(h/L)$.) So the force between the finite wire and the two half-infinite wires in Fig. 72(c) equals $2\lambda^2/4\pi\epsilon_0$. This is the correction we need to make to our original result. So the actual force between the two finite wires in Fig. 72(b) is

$$\frac{\lambda^2}{4\pi\epsilon_0}\frac{L}{h} - \frac{2\lambda^2}{4\pi\epsilon_0} = \frac{\lambda^2}{4\pi\epsilon_0}\frac{L}{h}\left(1 - \frac{2h}{L}\right),\tag{235}$$

up to corrections of higher order in h/L. In the present setup, 2h/L equals 1/20. So by ignoring the end effects, we over-estimated the force by about 5%.

3.40. Direction of the force

There are two negative image charges on the other side of the plane, at the mirrorimage locations. For very large z values of the top charge q, the lower q and its image charge -q look like a dipole from afar, which has a repulsive (upward) field that falls off like $1/z^3$. But the attractive (downward) field from the other image charge -q behaves like $1/(2z)^2$. This has a smaller power of z in the denominator, so it dominates for large z. The force on the top charge q is therefore downward for large z. So the answer to the stated question is "No."

Figure 73

3.41. Horizontal field line

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There are many different Gaussian surfaces that can be used to solve this problem. One is shown in Fig. 73 The bottom of the surface can be chosen to lie either inside the material of the conducting plane, or below it; in either case the field is zero there. The rest of the surface follows the field lines that start out horizontal. The exception is right near the point charge, where the surface takes the form of a tiny hemisphere. There is no flux through the surface except through the hemisphere, because everywhere else the field is either zero or parallel to the surface.

The flux into the surface is $(Q/2)/\epsilon_0$ because exactly half of the Q/ϵ_0 flux that passes outward through a tiny sphere around the charge Q (in a spherically symmetric manner very close to the charge) passes though the bottom half of the sphere. Since this flux passes *into* our Gaussian surface, Gauss's law tells us that there must be a charge of -Q/2 inside. The only place this charge can be is on the surface of the plane. So our task is to find the radius R of the circle that contains half of the total charge -Q on the plane.

Using the density σ given in Eq. (3.4), the requirement $-Q/2 = \int_0^R \sigma 2\pi r \, dr$ becomes

$$\frac{1}{2} = \int_0^R \frac{hr\,dr}{(r^2 + h^2)^{3/2}} = \frac{-h}{\sqrt{r^2 + h^2}} \Big|_0^R = 1 - \frac{h}{\sqrt{R^2 + h^2}} \,. \tag{236}$$

Therefore,

$$\frac{h}{\sqrt{R^2 + h^2}} = \frac{1}{2} \implies R = \sqrt{3} h.$$
(237)

REMARK: The above reasoning works for any starting angle of the field lines, not just horizontal. If we measure θ with respect to the upward vertical, then as an exercise you can shown that a field line that starts out at an angle θ meets the plane at a radius $R = h\sqrt{3} + 2\cos\theta - \cos^2\theta/(1-\cos\theta)$. (As a sub-problem, you will need to show that the fraction of the total surface area of a sphere that lies in a spherical cap subtended by the cone with half-angle θ is $(1 - \cos\theta)/2$. The remainder therefore subtends a fraction $(1 + \cos\theta)/2$.) For $\theta = \pi/2$, this correctly gives $R = \sqrt{3}h$. And for $\theta = 0$ and π it gives, respectively, $R = \infty$ and 0, as it should.

Interestingly, for θ close to π (call it $\pi - \epsilon$), that is, for field lines that start off pointing nearly straight downward, you can show that $R \approx h\epsilon/\sqrt{2}$. This is correctly smaller than $h\epsilon$, because that would be where the field line would hit the plane if the line were exactly straight, whereas we know that it must bend so that it ends up vertical when it meets the plane. But the $1/\sqrt{2}$ factor isn't obvious.

3.42. Point charge near a corner

The two equipotential surfaces are the planes shown in Fig. 74. The potential is zero at every point on these planes, because at any such point, $\sum (\pm q/r) = 0$ by symmetry. Equivalently, the electric field is perpendicular to the planes at every point. This can be seen by grouping the four charges into two dipoles (with charges on opposite sides of a given plane); the field from each dipole is perpendicular to the plane at every point on the plane. By superposition, the field from both dipoles is also perpendicular to the plane at every point on the plane at every point on the plane.



Figure 74

This setup satisfies the boundary conditions of the setup consisting of a point charge and a right-angled sheet. The field is shown (roughly) in Fig. 75. This field is similar to two copies of Fig. 3.10(b), tilted at right angles with respect to each other, but with the lines pushed away from the corner. The field at the center of the square of the original four charges is zero, so the field at the corner of the metal sheet is likewise zero. This implies that the surface density is zero at the corner.

In order for this method to work, we need to divide space into an *even* number of identical wedges (call it 2N), in order to have an alternating ring of charges. So it won't work unless the angle θ at the bend in the sheet is of the form $\theta = 2\pi/2N = \pi/N$, where N is an integer. It won't work, for example, with $\theta = 120^\circ = 2\pi/3$.

In the setup with the right-angled corner, note that the total charge on the two parts of the sheet must be -q (assuming the given charge is q). So the charge on each part must be -q/2. It is possible to verify this explicitly by writing down the field E(and hence density $\sigma = \epsilon_0 E$) as a function of the two coordinates associated with the sheet, and then performing (ideally with a computer) the double integral of σ over the half-infinite sheet.

If the given charge q is located a distance d from each sheet of the right-angled corner, then by looking at the forces from the three image charges, you can show that the force on the given charge q is $(q^2/32\pi\epsilon_0 d^2)(2\sqrt{2}-1)$, directed toward the corner.

3.43. Images from three planes

The required image charges are at the other seven corners of a cube of side 2d, as shown in Fig. 76. This configuration satisfies the condition of equipotential surfaces where the planes are; the potential is zero at every point on these planes, because at any such point, $\sum (\pm Q/r) = 0$ by symmetry. Equivalently, the total electric field is perpendicular to all of the planes at every point. This can be seen by grouping the eight charges into four dipoles (with charges on opposite sides of a given plane); the field from each dipole is perpendicular to the plane at every point on the plane. By superposition, the field from all four dipoles is also perpendicular to the plane at every point on the plane.

From the symmetry of the setup, the net force on Q is directed toward (or away) from the origin (the center of the cube of side 2d). So we need compute only the force components in that direction. There are three classes of charges:

• Three charges -Q at a distance 2d make an angle $\cos^{-1}(1/\sqrt{3})$ with the direction toward the origin. This follows from the fact that the diagonal of the cube has length $\sqrt{3}(2d)$. Alternatively, you can find the cosine by using the two standard expressions for the dot product. We therefore have (ignoring the $4\pi\epsilon_0$),

$$F_1 = 3\frac{Q^2}{(2d)^2} \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{4}\frac{Q^2}{d^2} \qquad \text{(toward origin)}.$$
 (238)

• Three charges Q at a distance $2\sqrt{2}\,d$ make an angle $\cos^{-1}(\sqrt{2}/\sqrt{3})$ with the direction toward the origin. So

$$F_2 = 3\frac{Q^2}{(2\sqrt{2}d)^2} \cdot \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{3}}{4\sqrt{2}}\frac{Q^2}{d^2} \qquad \text{(away from origin)}.$$
 (239)

• One charge -Q at a distance $2\sqrt{3} d$ is located in line with the origin. So

$$F_3 = \frac{Q^2}{(2\sqrt{3}d)^2} = \frac{Q^2}{12d^2}$$
. (toward origin) (240)







Figure 76

73

The total force on Q is therefore (bringing back in the $4\pi\epsilon_0$)

$$F = \frac{1}{4\pi\epsilon_0} \left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4\sqrt{2}} + \frac{1}{12} \right) \frac{Q^2}{d^2} \approx \frac{(0.210)}{4\pi\epsilon_0} \frac{Q^2}{d^2} \qquad \text{(toward origin)}.$$
 (241)

3.44. Force on a charge between two planes

If the charge is very close to the right plane, as in Fig. 12.53, we see that the force on the charge comes from the image charge nearby on its right, plus an infinite number of dipoles. Two these dipoles are at distances of approximately 2ℓ , two are at 4ℓ , and so on. The strength of the dipoles is p = q(2b). From Eq. (2.36), the field from a dipole, along the axis, is $E = 2p/4\pi\epsilon_0 r^3$. These fields all point to the left. The total force on the given charge is therefore approximately equal to

$$F = \frac{q^2}{4\pi\epsilon_0(2b)^2} - 2 \cdot q \frac{2(q(2b))}{4\pi\epsilon_0} \left(\frac{1}{(2\ell)^3} + \frac{1}{(4\ell)^3} + \frac{1}{(6\ell)^3} + \cdots \right)$$
$$= \frac{q^2}{16\pi\epsilon_0 b^2} - \frac{q^2 b}{4\pi\epsilon_0 \ell^3} \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \right).$$
(242)

The factor is parenthesis is approximately equal to 1.2. Note that the total force from the dipoles is smaller than the force from the closest image charge by a factor of order $(b/\ell)^3$. Two of these powers of b/ℓ come from the fact that the distances from the given charge to the dipoles are on the order of ℓ instead of b. And the third power comes from the dipole effect of taking the difference between nearly-canceling forces.

You can also calculate the total force by looking at the forces from the positive and negative image charges separately. From Fig. 12.53, the forces from the positive charges cancel, because they are symmetrically located with respect to the given charge. The force from the closest negative charge is $q^2/4\pi\epsilon_0(2b)^2$, directed to the right. The forces from the other negative charges nearly cancel in pairs. The sum of the forces from all these pairs points to the left, and its magnitude can be written as

$$F_{\text{neg}} = \frac{q^2}{4\pi\epsilon_0} \sum_{n=1}^{\infty} \left(\frac{1}{(2n\ell - 2b)^2} - \frac{1}{(2n\ell + 2b)^2} \right)$$

$$= \frac{q^2}{16\pi\epsilon_0\ell^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{(1 - b/n\ell)^2} - \frac{1}{(1 + b/n\ell)^2} \right)$$

$$\approx \frac{q^2}{16\pi\epsilon_0\ell^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\left(1 + \frac{2b}{n\ell} \right) - \left(1 - \frac{2b}{n\ell} \right) \right)$$

$$= \frac{q^2b}{4\pi\epsilon_0\ell^3} \sum_{n=1}^{\infty} \frac{1}{n^3}, \qquad (243)$$

in agreement with the dipole term in Eq. (242).

3.45. Charge on each plane

(a) At a given point P on the right plane in Fig. 12.53, we need to add up the x components of the fields due to the real charge and all the image charges. The two nearest charges on either side of the right plane (the real charge and image charge 1) are a distance $\sqrt{b^2 + r^2}$ away from P. So the magnitude of the field from each charge is $q/4\pi\epsilon_0(b^2 + r^2)$. Taking the x component brings in a factor

of $b/\sqrt{b^2 + r^2}$. Both charges produce a positive x component, so that brings in a factor of 2. Putting it all together gives the first term in Eq. (3.41).

Now consider images charges 2 and 4 in Fig. 12.53. They both are a distance $2\ell - b$ from the right plane, so we simply need to replace b with $2\ell - b$ in the above reasoning. And we also need to add on a minus sign since the x component is negative. So we obtain the first term in the sum in Eq. (3.41) with n = 1. Similarly, charges 3 and 5 both are a distance $2\ell + b$ from the right plane, so they yield the second term in the sum with n = 1.

In the same manner, charges 6 and 8 with distances $4\ell - b$, and charges 7 and 9 with distances $4\ell + b$, yield the n = 2 terms. And so on.

(b) As in Eq. (3.5), the integral of the first term in Eq. (3.41) (after dividing by -4π to obtain the density) is

$$-\frac{1}{4\pi} \int_0^R \frac{2qb}{(b^2 + r^2)^{3/2}} \, 2\pi r \, dr = \frac{qb}{(b^2 + r^2)^{1/2}} \Big|_0^R = -q, \tag{244}$$

where we have set $R = \infty$, which causes no problem with this term. In the same manner, the integral arising from the pair of terms with a particular value of n is

$$\left(-\frac{q(2n\ell-b)}{\left((2n\ell-b)^2+r^2\right)^{1/2}}+\frac{q(2n\ell+b)}{\left((2n\ell+b)^2+r^2\right)^{1/2}}\right)\Big|_0^R.$$
 (245)

As stated in the problem, if we set $R = \infty$ these two terms equal $\pm q$, so they cancel. We'll therefore let R be a large but finite distance. Then when the first term is evaluated at R it can be approximated as (dropping the b^2 term)

$$-\frac{q(2n\ell-b)}{\left(\left(4n^{2}\ell^{2}+R^{2}\right)-4n\ell b\right)^{1/2}}\approx-\frac{q(2n\ell-b)}{\left(4n^{2}\ell^{2}+R^{2}\right)^{1/2}}\left(1+\frac{1}{2}\frac{4n\ell b}{4n^{2}\ell^{2}+R^{2}}\right),\quad(246)$$

where we have used $1/(1-\epsilon)^{1/2} \approx 1+\epsilon/2$. The terms that don't involve b (or that involve b^2) will cancel with the corresponding terms in the second term in Eq. (245), which looks the same except for the overall minus sign and the replacement $b \to -b$. So we care only about the b terms. You can show that their sum for a given value of n is $2qbR^2/(4n^2\ell^2 + R^2)^{3/2}$. The factor of 2 out front comes from the fact that there are two terms in Eq. (245).

We must now sum this over n. For large R, the terms change slowly with n, so we can approximate the sum by an integral. Let's relabel n as z. In the original sum over n, we can multiply each term by dn, because dn simply equals 1 since n runs over the integers. We can then replace dn by dz. Integrating over z from 0 (although the exact starting point doesn't matter) to ∞ gives (you should verify this integral)

$$\int_0^\infty \frac{2qbR^2\,dz}{(4z^2\ell^2 + R^2)^{3/2}} = \frac{2qbz}{\sqrt{4\ell^2 z^2 + R^2}} \bigg|_0^\infty = \frac{qb}{\ell}\,.$$
 (247)

Remembering to include the -q charge in Eq. (244), the total charge on the right plane is $-q + qb/\ell = -q(\ell - b)/\ell$. This equals -q if b = 0, and 0 if $b = \ell$. These makes sense, because the left plane or right plane, respectively, is irrelevant in these two cases.

The charge on the left plane is obtained by letting $b \to \ell - b$ throughout the above calculation, which yields $-q(\ell - (\ell - b))/\ell = -qb/\ell$. Alternatively, we know that

the total charge on the inner surfaces of the two planes must be -q (by using a Gaussian surface with two faces lying inside the metal of the conducting planes, where the field is zero). So $-q + qb/\ell$ on the right plane implies $-qb/\ell$ on the left plane.

3.46. Sphere and plane image charges

Let the shell have radius R. Consider a point charge Q located at x = A, where A = R + h with $h \ll R$. Then from Problem 3.13, the image charge -q inside the shell equals $-QR/A = -QR/(R+h) \approx -Q$, and its location is

$$x = \frac{R^2}{A} = \frac{R^2}{R+h} = \frac{R}{1+h/R} \approx R(1-h/R) = R-h.$$
 (248)

Very close to a spherical shell, the shell looks locally like a plane. So what we essentially have here is two charges $\pm Q$ located on either side of a plane, a distance h from it. This is exactly what our image-charge setup looked like in the case of a plane. The same type of reasoning holds if the given charge lies inside the shell. The real and image charges have now simply switched sides of the plane.

3.47. Bump on a plane

The point charge Q is a distance A = 2R from the center of the hemisphere. So Problem 3.13 tells us that if an image charge of -q = -QR/A = -Q/2 is located a distance $a = R^2/A = R/2$ above the center of the hemisphere, the whole sphere will be at zero potential (even though we don't care about the bottom half). But we still need to make the whole *plane* an equipotential. We can do this by adding the opposites of the two existing charges (one real and one image), but now below the plane, as shown in Fig. 77. The two new image charges still have the sphere as an equipotential surface. And all four charges have the whole plane as an equipotential (at zero potential). This is clear if we group the charges into two pairs: the ones at $y = \pm R/2$, and the ones at $y = \pm 2R$. We have actually created an equipotential surface consisting of the union of the plane and the sphere, but the bottom half of the sphere is irrelevant, as is the equatorial disk inside the sphere.

3.48. Density at top of bump on a plane

From the reasoning in Exercise 3.47, we will put image charges of $\mp QR/A$ (which is much smaller than Q since $A \gg R$) at positions $\pm R^2/A$ (which is much smaller than R), along with an image charge of -Q at y = -A. Since the two small image charges are very close to each other, they effectively form a dipole with strength $p = (QR/A)(2R^2/A) = 2QR^3/A^2$. From Eq. (2.36), at the top of the hemisphere the field from this dipole points downward with magnitude $2p/4\pi\epsilon_0R^3 = 4Q/4\pi\epsilon_0A^2$. This is twice as strong as the sum of the downward fields due to the $\pm Q$ charges (one real and one image), which is essentially equal to $2Q/4\pi\epsilon_0A^2$ at the top of the hemisphere. The total downward field is therefore $6Q/4\pi\epsilon_0A^2$, which is three times as strong as the field in the case where we simply have a flat plane (because then only the $\pm Q$ charges are relevant). Since the electric field is zero below the conductor in any case, the surface density is proportional to the field. The surface density at the top of the hemisphere is therefore three times as large as the density in the case of the flat plane.

Using the above dipole approximation, you can also show that the field (and hence density) is zero at the corner where the hemisphere meets the plane. This is consistent with the fact that the hemisphere and plane form an equipotential surface, which the





electric field must be perpendicular to at all points. And the only vector that is perpendicular to two directions is the zero vector.

3.49. Positive or negative density

Let the point charge Q be at radius nR, where n is a numerical factor. (Working with this factor n makes the solution a little cleaner than it otherwise might be.) From Problem 3.13 there is an image charge -Q/n located at radius R/n. And then from Problem 3.16 there is an additional image charge (1 + 1/n)Q located at the center of the shell, to make the total charge on the shell be Q.

In the cutoff case where the surface charge density σ at the closest point on the shell changes from negative to positive, σ will be zero. But the field right outside the shell equals σ/ϵ_0 , so σ will be zero if the field is zero. The field equals the sum of the fields from the three charges (one real and two image). Being careful with the signs of the three fields, if we set the total field right outside the closest point equal to zero, we obtain (ignoring the $1/4\pi\epsilon_0$)

$$\frac{(1+1/n)Q}{R^2} = \frac{Q/n}{(1-1/n)^2 R^2} + \frac{Q}{(n-1)^2 R^2} \implies \frac{n+1}{n} = \frac{n}{(n-1)^2} + \frac{1}{(n-1)^2}$$

$$\implies \frac{1}{n} = \frac{1}{(n-1)^2} \implies n^2 - 3n + 1 = 0 \implies n = \frac{3 + \sqrt{5}}{2},$$
(249)

as desired. The other solution, $n = (3 - \sqrt{5})/2$, is smaller than 1 and hence not applicable, since we are assuming n > 1 (the given charge is outside the sphere).

REMARK: We can also ask the analogous question where we put a charge Q inside a nongrounded conducting spherical shell with radius R and net charge Q. A continuity argument again shows that there must be a cutoff radius, below which the charge density (the sum of the inner and outer surface densities on the shell) is positive everywhere on the shell. But the solution is slightly different, due to the fact that charge resides on both the inner and outer surfaces of the shell. The inner surface has a total charge -Q, nonuniformly distributed; the density is determined by considering an image charge located outside (see Problem 3.13). There is also a charge 2Q on the outer surface (to make the total charge on the shell equal to Q); this charge is uniformly distributed. You can show that the inner and outer densities cancel at the nearest point on the shell if the given charge Q is located at radius $R(5 - \sqrt{17})/4 \approx (0.219)R$.

3.50. Attractive or repulsive?

Let us write r as nR, where n is a numerical factor. (Working with this factor n makes the solution a little cleaner than it otherwise might be.) Our goal is to find the value of n for which the force on the point charge Q is zero. From Problem 3.13 there is an image charge -Q/n located at radius R/n. And then from Problem 3.16 there is an additional image charge of (1 + 1/n)Q located at the center of the shell, to make the total charge on the shell be Q. The net field at the location of the given charge Qfrom these two image charges is (ignoring the $1/4\pi\epsilon_0$)

$$E = \frac{(1+1/n)Q}{(nR)^2} + \frac{(-Q/n)}{(n-1/n)^2 R^2}.$$
(250)

Setting this equal to zero yields

$$\frac{1+1/n}{n^2} = \frac{1/n}{(n-1/n)^2}.$$
(251)

Simplifying gives

$$n+1 = \frac{n^4}{(n^2-1)^2} \implies n^5 - 2n^3 - 2n^2 + n + 1 = 0.$$
 (252)

This equation factors into

$$(n^2 - n - 1)(n^3 + n^2 - 1) = 0, (253)$$

and the roots of the quadratic are $n = (1 \pm \sqrt{5})/2$. We are concerned with the "+" root, since r > R. (Also, the real root of the cubic equation is less than 1.) So $r/R = n = (1 + \sqrt{5})/2 \approx 1.618$, as desired. Another point of interest is the location of the maximum repulsive force. You can show numerically that this location is $r \approx (2.074)R$.

3.51. Conducting sphere in a uniform field

(a) From the reasoning in the solution to Problem 1.27, the field due to the upper sphere, at a point at position \mathbf{r}_1 with respect to its center, is $\mathbf{E} = \rho \mathbf{r}_1/3\epsilon_0$. Likewise, the field due to the lower (negative) sphere, at a point at position \mathbf{r}_2 with respect to its center, is $\mathbf{E} = -\rho \mathbf{r}_2/3\epsilon_0$. The sum of these fields is (with the subscript "s" standing for "spheres")

$$\mathbf{E}_{\mathbf{s}} = \frac{\rho \mathbf{r}_1}{3\epsilon_0} - \frac{\rho \mathbf{r}_2}{3\epsilon_0} = \frac{\rho(\mathbf{r}_1 - \mathbf{r}_2)}{3\epsilon_0} = -\frac{\rho \mathbf{s}}{3\epsilon_0}, \qquad (254)$$

where **s** is the upward pointing vector between the centers. This result is independent of the position inside the cavity, so the field points downward with the uniform value of $\rho s/3\epsilon_0$ throughout the overlap region.

(b) The (radial) distance between the dashed circle and the bottom circle is s. And the radial displacement between the bottom circle and the top circle is $s \cos \theta$ (which is negative for $\theta > \pi/2$, where θ is measured down from the top of the circles), because the vertical distance between the two circles is always s, and the radial component of this distance brings in a factor of $\cos \theta$. So the total thickness of the shaded region (that is, the radial distance between the dashed circle and the top circle) is $\ell = s(1 + \cos \theta)$. This correctly equals 2s at the top of the circles and 0 at the bottom.

Consider the part of the shaded region that lies in a horizontal ring (that is, one that extends into and out of the page), all of whose points subtend an angle θ with respect to the vertical. Let the tangential thickness of the ring subtend an angle $d\theta$. The volume of the shaded region corresponding to this ring is

$$(2\pi R\sin\theta)(R\,d\theta)\ell = (2\pi R\sin\theta)(R\,d\theta)s(1+\cos\theta). \tag{255}$$

The charge in this ring is therefore $\rho 2\pi R^2 s \sin \theta (1 + \cos \theta) d\theta$. Since the ring is essentially located right on the surface of the bottom sphere (if s is small), each little piece dq feels a force with magnitude $Q dq/4\pi\epsilon_0 R^2$ due to the bottom sphere, where -Q is the charge of the sphere (which equals $4\pi R^3 \rho/3$, but we won't need this). Only the vertical component survives, and this brings in a factor of $\cos \theta$. So the vertical force due to the bottom sphere on the ring is directed downward with magnitude

$$\frac{Q(\rho 2\pi R^2 s \sin \theta (1+\cos \theta) \, d\theta)}{4\pi\epsilon_0 R^2} \cdot \cos \theta = \frac{Q\rho s \sin \theta \cos \theta (1+\cos \theta) \, d\theta}{2\epsilon_0} \,. \tag{256}$$

Integrating this from 0 to π gives the total downward force on the shaded region as

$$\frac{Q\rho s}{2\epsilon_0} \int_0^\pi \sin\theta \cos\theta (1+\cos\theta) \, d\theta = \frac{Q\rho s}{2\epsilon_0} \left(-\frac{\cos^2\theta}{2} - \frac{\cos^3\theta}{3} \right) \Big|_0^\pi \\ = \frac{Q\rho s}{2\epsilon_0} \cdot \frac{2}{3} = \frac{Q\rho s}{3\epsilon_0} \,, \tag{257}$$

where $Q = 4\pi R^3 \rho/3$.

(c) If the field points upward, then the force on the upper (positive) sphere is simply EQ pointing upward. This is equal to the downward force we found in part (b) if

$$EQ = \frac{Q\rho s}{3\epsilon_0} \implies s = \frac{3\epsilon_0 E}{\rho}.$$
 (258)

This distance will be small compared with R if ρ is sufficiently large. For the purposes of this problem we will assume this is the case.

- (d) The total field in the overlap region is the uniform field E plus the field due to the two spheres, which we found in part (a). This field points downward with magnitude $\rho s/3\epsilon_0$. Using the s we found in part (c), this equals $\rho(3\epsilon_0 E/\rho)/3\epsilon_0 = E$. This downward field therefore exactly cancels the upward uniform field E, so the total field is zero everywhere in the overlap region. And for small s this region is essentially the same as the interior of the spheres.
- (e) As we saw above in part (b), the thickness of the thin non-overlap regions in Fig. 3.34 is $s \cos \theta$. So the effective surface charge density is

$$\sigma = \rho(s\cos\theta) = \rho \frac{3\epsilon_0 E}{\rho} \cos\theta = 3\epsilon_0 E\cos\theta, \qquad (259)$$

as desired. (Note that the relevant thickness here is *not* the thickness of the shaded region in Fig. 3.35.) This $\sigma = 3\epsilon_0 E \cos \theta$ result implies that the magnitude of the field at the poles is three times the uniform field E. The conducting-sphere limit is obtained in the $\rho \to \infty$ and $s \to 0$ limit, with the product ρs equalling $3\epsilon_0 E$.

3.52. Aluminum capacitor

The capacitance is

$$C = \frac{\epsilon_0 A}{s} = \frac{\left(8.85 \cdot 10^{-12} \, \frac{\mathrm{s}^2 \, \mathrm{C}^2}{\mathrm{kg \, m}^3}\right) \left(\pi (0.075 \, \mathrm{m})^2\right)}{4 \cdot 10^{-5} \, \mathrm{m}} = 3.910 \cdot 10^{-9} \, \mathrm{F} = 3910 \, \mathrm{pF}. \tag{260}$$

3.53. Inserting a plate

Put charges Q and -Q on the two conductors in each of the two given capacitors. In the bottom capacitor in Fig. 78, one of the conductors consists of the two outer plates, because they are connected by a wire. The charge distributions on the various surfaces are shown. All the factors of 1/2 arise from symmetry. In the bottom capacitor, the potential difference (which is the difference between either of the outside plates and the inner plate) equals the field times the separation. The field is half of what it is in the top capacitor (because the density σ is half), and the separation is also half. So





Figure 78

the potential difference is (1/2)(1/2) = 1/4 of what it is in the top capacitor. Since the charge Q on each capacitor is the same, we have

$$Q = C_{\text{top}}\phi$$
, and $Q = C_{\text{bottom}}(\phi/4)$. (261)

These quickly give $C_{\text{bottom}} = 4C_{\text{top}}$. So our answer is 4C.

In the more general case where the middle plate is a fraction f of the distance from one of the outside plates to the other, you can show that the capacitance is C/[f(1-f)]. This correctly equals 4C when f = 1/2. It minimum when f = 1/2 and goes to infinity as f goes to 0 or 1.

3.54. Dividing the surface charge

If σ_1 is the surface density on the top face of the inner plate, and if σ_2 is the density on the bottom face, then the magnitudes of the electric fields in the top and bottom regions are $E_1 = \sigma_1/\epsilon_0$ and $E_2 = \sigma_2/\epsilon_0$. These follow from using Gauss's law with surfaces that pass through the interior of the middle plate where the field is zero. The difference in potential between the middle and top plates is $E_1(0.05 \text{ m})$, and the difference in potential between the middle and bottom plates is $E_2(0.08 \text{ m})$. Since the top and bottom plates are at the same potential, we must have $5E_1 = 8E_2 \implies$ $5\sigma_1 = 8\sigma_2$. Combining this with the given fact that $\sigma_1 + \sigma_2 = \sigma$, we quickly find $\sigma_1 = (8/13)\sigma$ and $\sigma_2 = (5/13)\sigma$.

REMARK: From similar reasoning involving Gaussian surfaces with one side lying inside a conductor, it follows that the density on the bottom face of the top plate is $-\sigma_1$, and the density on the top face of the bottom plate is $-\sigma_2$. Assuming that there is zero net charge on the outer two plates, this leaves at total of $\sigma_1 + \sigma_2 = \sigma$ for the outer surfaces of these plates. It must get divided evenly, because otherwise these two surfaces would create a nonzero field between them, which would change the above fields and make the outer plates not be at the same potential. If any additional charge is dumped on the outer plates, it simply gets divided evenly between their two outer surfaces.

3.55. Two pairs of plates

Since the top two plates are at the same potential, the field is zero between them. Likewise, the field is zero between the bottom two plates. This exercise is therefore basically the same as Problem 3.20, due to the fact that the field inside a conductor is zero, just as the field between the two pairs of plates is zero in the present exercise. The four sheets here are equivalent to the four surfaces of the two plates in Problem 3.20. The solution is therefore basically the same.

Consider the Gaussian surface indicated by the dotted box in Fig. 79. Since there is no flux out of the top or bottom, the net charge enclosed must be zero. Hence there are equal and opposite charges on the inner two plates.

We now claim that the charges on the outer two plates must be equal. This is true because the two inner plates create zero net field in the E = 0 regions (because these two plates are on the *same* side of each of the E = 0 regions and have *opposite* charges, so their fields cancel). The outer two sheets must therefore also create zero net field in the E = 0 regions. Since these sheets are on *opposite* sides of a given E = 0 region, their charges must be the *same* if the fields are to cancel.

We can therefore describe the charges on the four plates, from top to bottom, as q_1 , q_2 , $-q_2$, and q_1 . The given information tells us that $q_1 + q_2 = Q_1$ and $q_1 - q_2 = Q_2$.





Solving for q_1 and q_2 gives $q_1 = (Q_1 + Q_2)/2$ and $q_2 = (Q_1 - Q_2)/2$. So from top to bottom, the charges on the four plates are

$$\frac{Q_1+Q_2}{2}, \quad \frac{Q_1-Q_2}{2}, \quad \frac{Q_2-Q_1}{2}, \quad \frac{Q_1+Q_2}{2}$$
 (262)

We can also state which four of the eight surfaces (top and bottom of the each of the four plates) these charges lie on. None of the charges can border the E = 0 regions, because otherwise the standard argument involving a Gaussian surface with one face lying inside the metal of the conductor would imply a nonzero field in these regions. So the four charges lie on the top of the top plate, the bottom of the next, the top of the next, and finally the bottom of the bottom plate.

3.56. Field just outside a capacitor

If the disks were infinitely large, the desired field would be zero. But with a finite R, the repulsive field from the positive disk (which acts like an infinite plane, for points infinitesimally close to it) is slightly larger than the attractive field from the negative disk, which does't quite act like an infinite plane.

Let's find the field due to a disk with radius R and surface density σ , at a general point a distance z from the center of the disk along the axis. This can be found by slicing up the disk into rings and finding the z component of the field due to the charge in each ring. We obtain (the $z/\sqrt{r^2 + z^2}$ factor here gives the z component):

$$E = \frac{1}{4\pi\epsilon_0} \int_0^R \frac{(2\pi r \, dr)\sigma}{r^2 + z^2} \cdot \frac{z}{\sqrt{r^2 + z^2}} = \frac{\sigma z}{2\epsilon_0} \int_0^R \frac{r \, dr}{(r^2 + z^2)^{3/2}}$$
$$= -\frac{\sigma z}{2\epsilon_0 \sqrt{r^2 + z^2}} \Big|_0^R = \frac{\sigma}{2\epsilon_0} - \frac{\sigma z}{2\epsilon_0 \sqrt{R^2 + z^2}} \,.$$
(263)

As expected, if $z \ll R$ we obtain the standard $\sigma/2\epsilon_0$ field from an infinite plane. In the case of the negative disk in this problem, z equals the separation s. So the difference in the magnitudes of the (oppositely pointing) fields from the two disks, at a point just outside the positive disk, is $\sigma s/2\epsilon_0\sqrt{R^2+s^2}$. The net field therefore has this magnitude and is directed away from the positive disk. In the (usual) case at hand where $s \ll R$, the net field is essentially equal to $\sigma s/2\epsilon_0 R$, which is s/R times the $\sigma/2\epsilon_0$ field from an infinite plane.

3.57. A 2N-plate capacitor

This solution requires only a slight modification of the solution to Problem 3.21. Let the charge densities on the first N plates be $\sigma_1, -\sigma_2, \sigma_3, \ldots$ Then by left-right symmetry, the charge densities on the second N plates are $\ldots, -\sigma_3, \sigma_2, -\sigma_1$.

The total charge is zero, so there is no field outside the plates. Hence the field between the 1st and 2nd plates is σ_1/ϵ_0 . The potential difference between these plates is therefore $\phi = \sigma_1 s/\epsilon_0$. The magnitude of the potential difference is the same between all pairs of adjacent plates, because all of the odd-numbered plates have the same potential due to the connecting wires, as do all of the even-numbered plates. So the field between any two adjacent plates is σ_1/ϵ_0 , with the direction alternating as shown in Fig. 80.

A Gaussian surface spanning any of the interior plates tells us that all of these plates have charge density $\pm 2\sigma_1$. The total charge on the N positive plates is therefore $Q = (\sigma_1 + (N-1)2\sigma_1)A$, which gives $\sigma_1 = Q/(2N-1)A$. The potential difference



Figure 80

between the positive and negative (sets of N) plates in the capacitor can then be written as

$$\phi = \frac{\sigma_1 s}{\epsilon_0} = \frac{Qs}{(2N-1)A\epsilon_0} \implies Q = \left(\frac{(2N-1)A\epsilon_0}{s}\right)\phi \implies C = \frac{(2N-1)A\epsilon_0}{s}.$$
(264)

As $N \to \infty$, this is essentially equal to $2NA\epsilon_0/s$. The capacitance in Eq. (264) is larger than the capacitance we would obtain if we juxtapose the two pairs of N plates to create two plates with area NA. If we keep the separation s, then the capacitance of the resulting standard two-plate capacitor would be $C = \epsilon_0(NA)/s$. As mentioned in the solution to Problem 3.21, the reason why the capacitance in Eq. (264) is larger than $C = \epsilon_0(NA)/s$ is because we effectively have 2N - 1 identical area-A capacitors of alternating orientation lined up next to each other, instead of N area-A capacitors (with the same orientation).

3.58. Capacitor paradox

The second reasoning is correct. Plate 3 is indeed at a lower potential than plate 1, so charge will flow. The error in the first reasoning is encompassed in the word, "So." Although it is true that the potential differences of the two capacitors are the same, this does *not* imply that the potentials of the two positive plates are equal. If we arbitrarily assign zero potential to plate 1, and if the common potential difference is ϕ , then the potentials of the four plates are, from left to right, $0, -\phi, -\phi$, and -2ϕ . No matter where we define the zero of potential, the potential of the leftmost plate is ϕ larger than the potential of the third plate, and 2ϕ larger than the potential of the rightmost plate.

3.59. Coaxial capacitor

Neglecting end effects, we can assume that the charge $\pm Q$ is uniformly distributed along each cylinder. The field between the cylinders is that of a line charge with density $\lambda = Q/L$, so $E = \lambda/2\pi\epsilon_0 r = Q/2\pi\epsilon_0 Lr$. The magnitude of the potential difference between the cylinders is then

$$|\Delta\phi| = \int_{b}^{a} E \, dr = \int_{b}^{a} \frac{Q \, dr}{2\pi\epsilon_0 Lr} = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{b}\right). \tag{265}$$

Since $C = Q/|\Delta \phi|$, the capacitance is given by $C = 2\pi\epsilon_0 L/\ln(a/b)$. If $a - b \ll b$, then we can use the Taylor series $\ln(1 + \epsilon) \approx \epsilon$ to write

$$\ln\left(\frac{a}{b}\right) = \ln\left(1 + \frac{a-b}{b}\right) \approx \frac{a-b}{b}.$$
(266)

So the capacitance becomes $C \approx 2\pi\epsilon_0 bL/(a-b)$. But $2\pi bL$ is the area A of the inner cylinder, and a-b is the separation s between the cylinders. So the capacitance can be written as $C = \epsilon_0 A/s$, which agrees with the standard result for the parallel-plate capacitor.

3.60. A three-shell capacitor

(a) Let Q_1 and Q_3 be the final charges on the inner and outer shells, respectively. The outward-pointing field between the inner and middle shells is due only to the inner shell, and it equals (ignoring the $1/4\pi\epsilon_0$ since it will cancel) Q_1/r^2 . So the potential difference between the inner and middle shells is $Q_1(1/R - 1/2R)$, with the inner shell at the higher potential. If the inner and outer shells are at the same potential, then $Q_1(1/R - 1/2R)$ must also be the potential difference between the outer and middle shells, with the outer shell at the higher potential. The field between the middle and outer shells must therefore point inward. This field is due to the inner two shells, so it points inward with magnitude $(Q - Q_1)/r^2$, given that -Q is the charge on the middle shell. Note that Q must be larger than Q_1 . The potential difference between the outer two shells is then $(Q - Q_1)(1/2R - 1/3R)$, with the outer shell at the higher potential.

Equating the inner-middle and outer-middle potential differences gives

$$Q_1\left(\frac{1}{R} - \frac{1}{2R}\right) = (Q - Q_1)\left(\frac{1}{2R} - \frac{1}{3R}\right) \implies \frac{Q_1}{2} = \frac{Q - Q_1}{6} \implies Q_1 = \frac{Q}{4}.$$
(267)

And then $Q_3 = 3Q/4$, to make the total charge on the inner and outer shells be equal to Q.

(b) The potential difference between the inner and middle shells, which is the same as the difference between the outer and middle shells, is (bringing the $1/4\pi\epsilon_0$ back in, and using $Q_1 = Q/4$)

$$\phi = \frac{Q_1}{4\pi\epsilon_0} \left(\frac{1}{R} - \frac{1}{2R}\right) = \frac{Q}{32\pi\epsilon_0 R}.$$
(268)

Therefore $Q = (32\pi\epsilon_0 R)\phi$, so the capacitance is $32\pi\epsilon_0 R$.

(c) Note that Q_3 didn't appear anywhere in the calculation in part (a). It can therefore take on any value, and the inner-middle and outer-middle potential differences will still be equal, provided that $Q_1 = Q/4$. So if we add charge qto the outer shell, it will simply stay there, uniformly distributed on the outside surface of the shell. It will raise the potential everywhere inside by $q/4\pi\epsilon_0(3R)$, but since this change is uniform inside, all differences remain the same. If any charge flowed across the wire from the outer shell to the inner shell, the final charge Q_1 on the inner shell would violate the $Q_1 = Q/4$ result we found above (because the middle shell is now isolated, so its charge of -Q doesn't change).

3.61. Capacitance of a spheroid

If $b \approx a$, then $\epsilon \ll 1$, and we can use the Taylor approximation, $\ln(1 \pm \epsilon) \approx \pm \epsilon$. The capacitance then becomes

$$C = \frac{8\pi\epsilon_0 a\epsilon}{\ln(1+\epsilon) - \ln(1-\epsilon)} \approx \frac{8\pi\epsilon_0 a\epsilon}{2\epsilon} = 4\pi\epsilon_0 a,$$
(269)

in agreement with the capacitance of a sphere given in Eq. (3.10).

The stored energy equals $Q^2/2C$. Since Q is held constant, we need only determine how C changes as the sphere is deformed. We will find that C increases, which means that the stored energy decreases.

Let C_0 be the capacitance of a sphere of unit radius, a = b = 1. And let C be the capacitance of a prolate spheroid of equal volume. This spheroid has $b = 1/\sqrt{a}$, since the volume equals $(4/3)\pi ab^2$. (We'll ignore the units of a and b. Equivalently, a and b are defined as the dimensionless ratios of the new lengths to the original radius of the sphere.) We have

$$\frac{C}{C_0} = \frac{2a\epsilon}{\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)}, \quad \text{where } \epsilon = \sqrt{1-\frac{1}{a^3}}.$$
(270)



Figure 81



Figure 82

A plot of C/C_0 is shown in Fig. 81. (The expression for the capacitance given in the statement of the problem is valid only for a prolate spheroid, that is, one with a > 1.) As promised, C is larger than C_0 .

In the limit where a is very large (so we have a long stick-like object), you can show with a Taylor series that $C/C_0 \approx 2a/\ln(4a^3) \approx 2a/(3\ln a)$, to leading order. This grows with a, so a long spheroid-shaped stick can have a capacitance much greater than that of a sphere of equal volume. If you want to write things more generally in terms of both a and b, the capacitance given in the problem equals $C \approx 4\pi\epsilon_0 a/\ln(2a/b)$ in the $a \gg b$ limit. As an example, consider a spheroid with a = 1 km and b = 1 mm. Its volume is that of a sphere with radius $(ab^2)^{1/3} = 0.1$ m, but its capacitance is that of a sphere with radius r = 69 meters, because $C \approx 4\pi\epsilon_0(10^3 \text{ m})/\ln(2 \cdot 10^6) = 4\pi\epsilon_0(69 \text{ m})$.

3.62. Deriving C for a spheroid

We'll assume here the validity of the result from Exercise 2.44, namely that the potential due to a stick with uniform charge density is constant over an ellipsoid that has the ends of the stick as its foci. Such an ellipse is shown in Fig. 82. The axes have lengths 2a and 2b, and the stick has length 2h. From the properties of an ellipse, we have $a = \sqrt{h^2 + b^2}$. To find the (constant) potential over this ellipse, we may conveniently pick points at the ends of either axis. An end of the major axis yields a slightly simpler integral (you can check that an end of the minor axis yields the same result). We have

$$\phi = \frac{1}{4\pi\epsilon_0} \int_{-h}^{h} \frac{\lambda \, dy}{a - y} = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{a + h}{a - h}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}}\right)$$
$$= \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{1 + \epsilon}{1 - \epsilon}\right), \quad (271)$$

where

$$\equiv \sqrt{1 - \frac{b^2}{a^2}} \,. \tag{272}$$

The charge contained within the ellipsoid is the charge on the stick,

F

$$Q = (2h)\lambda = 2\lambda\sqrt{a^2 - b^2} = 2\lambda a\epsilon.$$
(273)

From the reasoning near the end of Section 3.4, in the region of space exterior to the conducting ellipsoid, the field due to the ellipsoid with charge Q is identical to the field due to the uniform stick with charge Q. This is true because the latter field satisfies the boundary conditions for the ellipsoid (the field is perpendicular to the surface, and the total flux is Q/ϵ_0), so the uniqueness theorem tells us that this solution must be *the* solution for the ellipsoid.

Using $Q = 2\lambda a\epsilon$ in the $Q = C\phi$ relation for the ellipsoid gives

$$C = \frac{Q}{\phi} = \frac{2\lambda a\epsilon}{\frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{1+\epsilon}{1-\epsilon}\right)} = \frac{8\pi\epsilon_0 a\epsilon}{\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)},$$
(274)

as desired.

3.63. Capacitance coefficients for shells

We can write in general,

$$Q_1 = C_{11}\phi_1 + C_{12}\phi_2,$$

$$Q_2 = C_{21}\phi_1 + C_{22}\phi_2.$$
(275)

With charge Q_1 on the inner shell, the field between the shells equals $Q_1/4\pi\epsilon_0 r^2$, so the potential difference is

$$\phi_2 - \phi_1 = -\int_b^a E \, dr = -\int_b^a \frac{Q_1 \, dr}{4\pi\epsilon_0 r^2} = \frac{Q_1}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b}\right). \tag{276}$$

Hence,

$$Q_1 = \frac{4\pi\epsilon_0 ab}{a-b}(\phi_1 - \phi_2).$$
 (277)

Comparing this with Eq. (275) gives

$$C_{11} = \frac{4\pi\epsilon_0 ab}{a-b} \qquad \text{and} \qquad C_{12} = -\frac{4\pi\epsilon_0 ab}{a-b} \,. \tag{278}$$

In the region external to the outer shell of radius a, both shells look like point charges. So the potential at radius a is simply $\phi_2 = (Q_1 + Q_2)/4\pi\epsilon_0 a$, which yields $Q_2 = 4\pi\epsilon_0 a\phi_2 - Q_1$. Using the Q_1 from Eq. (277) allows us to write Q_2 in terms of ϕ_1 and ϕ_2 :

$$Q_2 = 4\pi\epsilon_0 a\phi_2 - \frac{4\pi\epsilon_0 ab}{a-b}(\phi_1 - \phi_2) = -\frac{4\pi\epsilon_0 ab}{a-b}\phi_1 + \frac{4\pi\epsilon_0 a^2}{a-b}\phi_2.$$
 (279)

Comparing this with Eq. (275) gives

$$C_{21} = -\frac{4\pi\epsilon_0 ab}{a-b}$$
 and $C_{22} = \frac{4\pi\epsilon_0 a^2}{a-b}$. (280)

As expected, $C_{12} = C_{21}$. In the event that we have a standard capacitor with charge Q on the inner shell and -Q on the outer, the field is zero outside the outer shell. So $\phi_2 = 0$, and ϕ_1 equals the $\Delta \phi$ between the shells. Both of the equations in Eq. (275) then reduce to $Q = [4\pi\epsilon_0 ab/(a-b)]\Delta\phi$, which agrees with the result in Eq. (3.18) for a two-sphere capacitor.

3.64. Capacitance-coefficient symmetry

(a) We can write in general,

$$Q_1 = C_{11}\phi_1 + C_{12}\phi_2,$$

$$Q_2 = C_{21}\phi_1 + C_{22}\phi_2.$$
(281)

STEP 1: Let us add charge to conductor 1, while holding ϕ_2 constant at zero. During this process, we will have to remove charge from conductor 2 to maintain $\phi_2 = 0$. (Equivalently, charge will naturally flow off conductor 2 if it is grounded, because charge will be repelled from the charge added to conductor 1.) So conductor 2 will become negatively charged. But this process doesn't involve any work, as we will see. From above, if $\phi_2 = 0$ then $dQ_1 = C_{11} d\phi_1$ and $dQ_2 = C_{21} d\phi_1$ (so evidently C_{21} is negative). The total work done in raising ϕ_1 from 0 to ϕ_{1f} equals the work done in adding charge to conductor 1 and removing charge from conductor 2. Therefore,

$$W_{\text{step 1}} = \int (\phi_1 \, dQ_1 + \phi_2 \, dQ_2)$$

=
$$\int_0^{\phi_{1f}} \left[\phi_1(C_{11} \, d\phi_1) + (0)(C_{21} \, d\phi_1) \right] = \frac{1}{2} C_{11} \phi_{1f}^2, \qquad (282)$$

where we have used $\phi_2 = 0$. As promised, no work is involved in the flow of charge off conductor 2.

STEP 2: Now let us add charge to conductor 2, while holding ϕ_1 constant at ϕ_{1f} . If ϕ_1 is constant then $dQ_1 = C_{12} d\phi_2$ and $dQ_2 = C_{22} d\phi_2$. Therefore,

$$W_{\text{step 2}} = \int (\phi_1 \, dQ_1 + \phi_2 \, dQ_2)$$
(283)
=
$$\int_0^{\phi_{2f}} \left[\phi_{1f}(C_{12} \, d\phi_2) + \phi_2(C_{22} \, d\phi_2) \right] = C_{12} \phi_{1f} \phi_{2f} + \frac{1}{2} C_{22} \phi_{2f}^2.$$

The total work done is

$$W_{\text{total}} = \frac{1}{2}C_{11}\phi_{1f}^2 + \frac{1}{2}C_{22}\phi_{2f}^2 + C_{12}\phi_{1f}\phi_{2f}.$$
 (284)

(b) In this process, the roles of 1 and 2 are interchanged, so we simply need to switch the 1's and 2's in the result in part (a). Therefore, the total work done is

$$W_{\text{total}} = \frac{1}{2}C_{22}\phi_{2f}^2 + \frac{1}{2}C_{11}\phi_{1f}^2 + C_{21}\phi_{2f}\phi_{1f}.$$
 (285)

Since the final state is the same, and since there is no dissipation of energy in the charging process, the total work done must be the same. Hence $C_{12} = C_{21}$.

3.65. Capacitor energy

As usual, the charges on the two conductors (label them as C_1 and C_2) of the capacitor are Q and -Q. If the potential difference between the conductors is ϕ , then we can write the potentials in the general forms of $\phi_0 + \phi$ and ϕ_0 , for some value of ϕ_0 (which may be zero). The integral in Eq. (2.32) breaks up into two separate integrals, one for each conductor (which are the only places where ρ is nonzero; more precisely, the surface density σ is nonzero). Since the potential takes on a constant value on each conductor, these potentials can be taken out of the integrals, yielding

$$U = \frac{1}{2}(\phi_0 + \phi) \int_{C_1} \rho \, dv + \frac{1}{2}\phi_0 \int_{C_2} \rho \, dv = \frac{1}{2}(\phi_0 + \phi)Q + \frac{1}{2}\phi_0(-Q) = \frac{1}{2}Q\phi, \quad (286)$$

as desired.

3.66. Adding a capacitor

Let the two capacitors be labeled 1 and 2. If the initial charge on capacitor 1 is Q, then

$$Q = C_1 V_i, \tag{287}$$

where $C_1 = 100$ pF and $V_i = 100$ volts. So $Q = (10^{-10} \text{ F})(100 \text{ V}) = 10^{-8} \text{ C}$. When capacitor 2 is connected in parallel, the charge Q is shared between the two capacitors, that is, $Q = Q_1 + Q_2$. But the voltages across the two capacitors are equal because they are connected in parallel. This voltage is $V_{\rm f} = 30$ volts. So we have $Q_1 = C_1 V_{\rm f}$ and $Q_2 = C_2 V_{\rm f}$. Adding these relations gives

$$Q = (C_1 + C_2)V_{\rm f},\tag{288}$$

which is the statement that capacitances in parallel simply add. Equating the righthand sides of Eqs. (287) and (288) gives

$$C_1(100 \text{ V}) = (C_1 + C_2)(30 \text{ V}) \implies C_2 = C_1 \cdot \frac{70}{30} = 233 \text{ pF}.$$
 (289)

The initial energy stored is

$$\frac{1}{2}QV_{\rm i} = \frac{1}{2}(10^{-8}\,{\rm C})(100\,{\rm V}) = 5\cdot 10^{-7}\,{\rm J}.$$
(290)

The final energy stored is

$$\frac{1}{2}Q_1V_f + \frac{1}{2}Q_2V_f = \frac{1}{2}QV_f = \frac{1}{2}(10^{-8}\,\mathrm{C})(30\,\mathrm{V}) = 1.5\cdot10^{-7}\,\mathrm{J}.$$
(291)

(The final energy is smaller than the initial energy by the factor $V_{\rm f}/V_{\rm i}$.) Therefore, $3.5 \cdot 10^{-7}$ J of energy is lost. This much energy has to go *somewhere* before the system can settle down to static equilibrium. If it is not stored anywhere else (for instance, in a weight lifted by a motor driven by the current from C_1 to C_2) it will eventually be dissipated in circuit resistance, no matter how small that resistance may be. (If the circuit is superconducting, the current will keep sloshing back and forth. We'll talk about LC circuits in Chapter 8.)

3.67. Energy in coaxial tubes

We'll solve this exercise first by using the energy density in the electric field, and then by using the capacitance. If λ is the charge per unit length on the inner cylinder (with $-\lambda$ on the outer cylinder), then the field between the cylinders (ignoring end effects) is $\lambda/2\pi\epsilon_0 r$. The energy stored in the field is therefore (with $\ell = 0.3$ m being the length)

$$U = \frac{\epsilon_0}{2} \int E^2 \, dv = \frac{\epsilon_0}{2} \int_{r_1}^{r_2} \left(\frac{\lambda}{2\pi\epsilon_0 r}\right)^2 2\pi r\ell \, dr = \frac{\lambda^2\ell}{4\pi\epsilon_0} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\lambda^2\ell}{4\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right). \tag{292}$$

To write this in terms of the (magnitude of the) potential difference ϕ between the tubes, instead of in terms of λ , note that

$$\phi = \int_{r_1}^{r_2} E \, dr = \int_{r_1}^{r_2} \frac{\lambda \, dr}{2\pi\epsilon_0 r} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right). \tag{293}$$

Solving for λ and plugging the result into Eq. (292) gives

$$U = \left(\frac{2\pi\epsilon_0\phi}{\ln(r_2/r_1)}\right)^2 \frac{\ell}{4\pi\epsilon_0} \ln(r_2/r_1) = \frac{\pi\epsilon_0\ell\phi^2}{\ln(r_2/r_1)}$$
$$= \frac{\pi\left(8.85 \cdot 10^{-12} \frac{s^2 C^2}{\text{kg m}^3}\right)(0.3 \text{ m})(45 \text{ V})^2}{\ln(4/3)} = 5.9 \cdot 10^{-8} \text{ J}.$$
(294)

ALTERNATIVELY: We can solve the problem using capacitance. Using the value of ϕ we found above, the capacitance of the tubes is given by

$$C = \frac{Q}{\phi} = \frac{\lambda \ell}{(\lambda/2\pi\epsilon_0)\ln(r_2/r_1)} = \frac{2\pi\epsilon_0\ell}{\ln(r_2/r_1)},$$
(295)

which is independent of λ , as it should be. (The capacitance depends only on the geometry of the system, and not on the charge that is placed on the conductors.) The energy stored is then

$$\frac{1}{2}C\phi^2 = \frac{1}{2}\left(\frac{2\pi\epsilon_0\ell}{\ln(r_2/r_1)}\right)\phi^2 = \frac{\pi\epsilon_0\ell\phi^2}{\ln(r_2/r_1)},$$
(296)

in agreement with the first solution.

3.68. Maximum energy storage between cylinders

First, note that the stored energy should indeed achieve a maximum for some value of b between 0 and a, due to the following reasoning. The energy is zero when b = a, because there is zero volume containing a nonzero field (a nonzero field exists only between the cylinders). And the energy is essentially zero when $b \approx 0$, because in that case the charge per unit length on the inner cylinder will have to be very small (otherwise the field at the surface, which is proportional to 1/b, would exceed E_0). This then means that the field is very small in the region between the cylinders (except very close to the inner cylinder, where it is E_0). Therefore, since the stored energy is zero at both b = a and $b \approx 0$, it must achieve a maximum at some intermediate value.

We'll solve this exercise first by using the energy density in the electric field, and then by using the capacitance. For convenience, let $b \equiv ka$, where k is a numerical factor to be determined. If E_0 is the field at radius ka, then since $E \propto 1/r$ for a cylinder, the field equals $E_0(ka/r)$ at larger values of r (but less than a). The energy stored in the field in a length ℓ of the capacitor is then

$$U = \frac{\epsilon_0}{2} \int E^2 \, dv = \frac{\epsilon_0}{2} \int_{ka}^a \left(E_0 \frac{ka}{r} \right)^2 2\pi r \ell \, dr$$

= $\pi \epsilon_0 \ell k^2 a^2 E_0^2 \int_{ka}^a \frac{dr}{r} = \pi \epsilon_0 \ell a^2 E_0^2 k^2 \ln(1/k).$ (297)

As noted above, this equals zero when k = 0 or k = 1. (At k = 0, the smallness of k^2 wins out over the largeness of $\ln(1/k)$.) Taking the derivative of $-k^2 \ln k$ to find the maximum gives $-k^2(1/k) - 2k \ln k = 0 \implies \ln k = -1/2 \implies k = e^{-1/2}$. Hence $b = e^{-1/2}a \approx (0.607)a$. The stored energy is then

$$U = \pi \epsilon_0 \ell a^2 E_0^2 \left(e^{-1/2} \right)^2 (1/2) = \frac{\pi \epsilon_0 \ell a^2 E_0^2}{2e} \,. \tag{298}$$

The energy per unit length is obtained by erasing the ℓ .

ALTERNATIVELY: We can solve the problem using capacitance. The (magnitude of the) potential difference ϕ between the tubes is

$$\phi = \int_{b}^{a} E \, dr = \int_{b}^{a} \frac{\lambda \, dr}{2\pi\epsilon_0 r} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{a}{b}\right). \tag{299}$$

The capacitance is then

$$C = \frac{Q}{\phi} = \frac{\lambda \ell}{(\lambda/2\pi\epsilon_0)\ln(a/b)} = \frac{2\pi\epsilon_0\ell}{\ln(a/b)},$$
(300)

which is independent of λ , as it should be. (The capacitance depends only on the geometry of the system, and not on the charge that is placed on the conductors.) With $b \equiv ka$ this becomes $C = 2\pi\epsilon_0 \ell / \ln(1/k)$. If λ is the charge per unit length on the inner cylinder, then $E_0 = \lambda/2\pi\epsilon_0(ka) \Longrightarrow \lambda = 2\pi\epsilon_0(ka)E_0$. The stored energy is then

$$U = \frac{Q^2}{2C} = \frac{(\lambda \ell)^2}{2C} = \frac{(2\pi\epsilon_0 (ka)E_0\ell)^2}{2 \cdot 2\pi\epsilon_0\ell/\ln(1/k)} = \pi\epsilon_0\ell a^2 E_0^2 k^2 \ln(1/k).$$
(301)

in agreement with Eq. (297). The solution proceeds as above.

3.69. Force, and potential squared

(a) In Gaussian units,

$$(\text{potential})^2 \sim \left(\frac{\text{charge}}{\text{distance}}\right)^2 \sim \frac{\text{charge}^2}{\text{distance}^2} \sim \text{force}.$$
 (302)

In the case of 1 statvolt, we have

$$(1 \text{ statvolt})^2 = \frac{(1 \text{ esu})^2}{(1 \text{ cm})^2} = 1 \text{ dyne.}$$
 (303)

(b) 1 volt equals 1/300 statvolts, so

$$(1 \text{ megavolt})^2 = \left(\frac{10^6}{300} \text{ statvolt}\right)^2 \sim 10^7 \text{ dynes} = 100 \text{ newtons.}$$
 (304)

But 9.8 newtons equals 2.2 pounds (these are both the weight of 1 kilogram). So the desired force is $(100 \text{ N})(2.2 \text{ pounds}/9.8 \text{ N}) \approx 20 \text{ pounds}.$

3.70. Force and energy for two plates

From Eq. (1.49), the force per unit area on one of the plates is σ times the average of the fields on either side of the plate. (Equivalently, it is σ times the field from the other plate.) This average field is E/2, where E is the field between the plates. But E equals σ/ϵ_0 , so $\sigma = \epsilon_0 E$ (it will be more useful to write the field in terms of E than σ). The force per unit area is therefore

$$\frac{F}{A} = \sigma \frac{E}{2} = (\epsilon_0 E) \frac{E}{2} \implies F = A \frac{\epsilon_0 E^2}{2}.$$
(305)

Since E is given by ϕ/s , we can write F in terms of the potential as

$$F = \frac{A\epsilon_0 \phi^2}{2s^2} = \frac{(0.2 \,\mathrm{m})^2 \left(8.85 \cdot 10^{-12} \,\frac{s^2 \,\mathrm{C}^2}{\mathrm{kg \,m^3}}\right) (10 \,\mathrm{V})^2}{2(0.03 \,\mathrm{m})^2} = 2.0 \cdot 10^{-8} \,\mathrm{N}. \tag{306}$$

If the charge is held constant as the plates come together, then the electric field is independent of the separation, so we see from Eq. (305) that the force is also independent of the separation. (Equivalently, ϕ is proportional to s in Eq. (306), so F is independent of s.) The total work done by the electric force (which could be used to lift an external object, etc.) is then $W = F \cdot s = (2.0 \cdot 10^{-8} \text{ N})(0.03 \text{ m}) = 6 \cdot 10^{-10} \text{ J}$. Note that the work can be written symbolically as

$$W = F \cdot s = \frac{A\epsilon_0 E^2}{2} \cdot s = (As)\frac{\epsilon_0 E^2}{2} = (\text{volume})\frac{\epsilon_0 E^2}{2}.$$
 (307)

Since $\epsilon_0 E^2/2$ is the energy density, the work does indeed equal the energy initially stored in the field. Alternatively, the work can be written in terms of ϕ as (using $C = \epsilon_0 A/s$ for a parallel-plate capacitor)

$$W = F \cdot s = \frac{A\epsilon_0 \phi^2}{2s^2} \cdot s = \frac{1}{2} \frac{\epsilon_0 A}{s} \phi^2 = \frac{1}{2} C \phi^2,$$
(308)

which is the energy stored in the capacitor.

What is the work done if the plates remain connected to the 10 volt battery? In this case, since ϕ is constant, the force of $A\epsilon_0\phi^2/2s^2$ in Eq. (306) grows like $1/s^2$ as s goes to zero. The integral of this diverges near zero, so the work is theoretically infinite. However, eventually the battery won't be able to supply the necessary charge to the plates, so ϕ will inevitably decrease.

3.71. Conductor in a capacitor

(a) The initial energy stored in the capacitor is $\epsilon_0 E^2/2$ times the volume As. The field is $E = \sigma/\epsilon_0$, so the initial energy is

$$U_{\rm i} = \frac{\epsilon_0}{2} \left(\frac{\sigma}{\epsilon_0}\right)^2 (As) = \frac{\sigma^2 As}{2\epsilon_0} \,. \tag{309}$$

Alternatively, we get the same result if we use $U_i = Q^2/2C$, with $Q = A\sigma$ and $C = \epsilon_0 A/s$.

Now consider the moment when the conducting slab is completely inside the capacitor. Since the charge on each plate remains fixed, the final charge densities are still $\pm \sigma$. (The charges shift around at intermediate times, but this doesn't concern us.) The field in the vacuum half of the capacitor is therefore still σ/ϵ_0 . And the field inside the conductor is zero, of course. What happens is that the conductor basically becomes two sheets of charge: $-\sigma$ on top, and σ on bottom. The σ bottom sheet neutralizes the $-\sigma$ bottom plate of the capacitor, so there is effectively no charge in the bottom half of the original capacitor, which means we can ignore that part. The stored energy U therefore decreases, simply because the volume of nonzero field decreases. The final energy is

$$U_{\rm f} = \frac{\epsilon_0}{2} \left(\frac{\sigma}{\epsilon_0}\right)^2 \left(A \cdot s/2\right) = \frac{\sigma^2 A s}{4\epsilon_0} \,. \tag{310}$$

 $U_{\rm f}$ is correctly smaller than $U_{\rm i}$. (If $U_{\rm f}$ turned out to be larger than $U_{\rm i}$ in an isolated system, we would have a problem.) The decrease in U shows up as kinetic energy, so $K = \sigma^2 As/4\epsilon_0$.

Note that when the slab is completely inside, since the effective width of the capacitor decreases to s/2, the voltage decreases to $(\sigma/\epsilon_0)(s/2) = \sigma s/2\epsilon_0$. But we didn't need this fact when finding K.

(b) As in part (a), the initial energy stored in the capacitor is $U_i = \sigma^2 A s / 2\epsilon_0$.

Now consider the moment when the conducting slab is completely inside the capacitor. The field inside the conductor is zero, so if the final charge densities on the capacitor are $\pm \sigma'$, we must have the opposite densities on the two faces of the slab, as shown in Fig. 83. This creates zero field inside the slab.

The field in the vacuum half of the capacitor is σ'/ϵ_0 , so the voltage difference between the plates is $(\sigma'/\epsilon_0)(s/2)$. But we are told that the voltage is held constant by the battery, so it must still take on the original value of $\phi = (\sigma/\epsilon_0)s$. Hence $\sigma' = 2\sigma$. (In short, half the separation means twice the field.) The final energy stored in the capacitor is therefore

$$U_{\rm f} = \frac{\epsilon_0}{2} \left(\frac{2\sigma}{\epsilon_0}\right)^2 \left(A \cdot s/2\right) = \frac{\sigma^2 A s}{\epsilon_0} \,. \tag{311}$$

This is *larger* than the initial energy, in contrast with the situation in part (a). So if there weren't anything else going on, we would have a violation of energy conservation. But there is indeed something else going on; the battery is doing work. It must dump more charge onto the plates to increase the density from σ to $\sigma' = 2\sigma$. It does this by moving charges from the negative plate to the positive plate, through the constant potential difference $\phi = \sigma s/\epsilon_0$. The amount of charge moved is $q = (\sigma' - \sigma)A = \sigma A$, so the work done is $W = q\phi = (\sigma A)(\sigma s/\epsilon_0) =$



 $\sigma^2 As/\epsilon_0.$ Conservation of energy therefore gives the final kinetic energy of the slab as

$$W + U_{\rm i} = U_{\rm f} + K \implies \frac{\sigma^2 A s}{\epsilon_0} + \frac{\sigma^2 A s}{2\epsilon_0} = \frac{\sigma^2 A s}{\epsilon_0} + K \implies K = \frac{\sigma^2 A s}{2\epsilon_0} . \tag{312}$$

Basically, of the $\sigma^2 As/\epsilon_0$ work done, half goes into increasing U, and half goes into K. It makes sense that the K here is larger than the K in part (a), because the present case ends up with more charge on the plates, so the forces involved are larger.

3.72. Force on a capacitor sheet

We will neglect the edge fields on the assumption that the gap s is much smaller than y and b. The charge Q on sheet A will be split between the area yb on each side of the sheet; so the surface density on each side is $\sigma = (Q/2)/yb$. Likewise the charge -Q on sheet B will be split between the area yb on each "wing" of the bent sheet. The electric field on either side of sheet A is therefore

$$E = \frac{\sigma}{\epsilon_0} = \frac{(Q/2)/yb}{\epsilon_0} = \frac{Q}{2\epsilon_0 yb}.$$
(313)

The voltage difference between the sheets is $V = Es = Qs/2\epsilon_0 yb$, so the capacitance is $C = Q/V = 2\epsilon_0 yb/s$. (This is just the standard parallel-plate expression $C = \epsilon_0 A/s$ with area A = 2yb.) Equation 3.32 then gives¹

$$F = \frac{Q^2}{2} \frac{d}{dy} \left(\frac{1}{C}\right) = \frac{Q^2}{2} \frac{d}{dy} \left(\frac{s}{2\epsilon_0 y b}\right) = -\frac{Q^2 s}{4\epsilon_0 b y^2}.$$
(314)

The negative sign indicates that the energy of the capacitor decreases as y increases. So the direction of the force on A is downward, because the decrease in energy will show up as kinetic energy, or work done on some other object. (Equivalently, the F in Eq. (3.32) was defined as the force that some other object must apply to A to keep it at rest. This force is upward, in the direction of decreasing y; hence the negative sign.) We can write y in terms of V via the above relation, $V = Qs/2\epsilon_0 yb \Longrightarrow y = Qs/2\epsilon_0 Vb$. Hence, the magnitude of the force is

$$F = \frac{Q^2 s}{4\epsilon_0 b} \left(\frac{2\epsilon_0 V b}{Q s}\right)^2 = \frac{\epsilon_0 V^2 b}{s} \implies V = \sqrt{\frac{F s}{\epsilon_0 b}}.$$
(315)

Note that the force F is independent of y. This is consistent with the discussion at the end of the solution to Problem 3.26. Although we (justifiably) ignored the edge effects in computing the capacitance, it is in fact these edge effects that produce the force. These edge effects depend on the charge *density* on the plates, and for a given voltage V, this density is independent of y.

ALTERNATIVELY: We can find the force by directly calculating how the stored energy changes with y. The stored energy is

$$U = \frac{1}{2}QV = \frac{Q^2s}{4\epsilon_0 yb}.$$
(316)

¹The parameter y need not represent the distance *between* the plates, for Eq. (3.32) to hold. That equation is valid for any y describing the relative position of the conductors in a capacitor; it gives the force in the direction corresponding to the parameter y.

This U can also be found via:

$$U = (\text{volume})\frac{\epsilon_0 E^2}{2} = (2syb)\frac{\epsilon_0}{2} \left(\frac{Q}{2\epsilon_0 yb}\right)^2 = \frac{Q^2 s}{4\epsilon_0 yb}.$$
(317)

This decreases as y increases, that is, as sheet A moves downward. (The volume increases with y, but E decreases with y, and E is squared in the expression for U). Taking the differential of Eq. (316) gives the decrease in U as

$$dU = -\frac{Q^2 s}{4\epsilon_0 b} \frac{dy}{y^2}.$$
(318)

If A is attached to some other object D, this decrease in energy equals the increase in the energy of D, due to the work F dy that sheet A does on D. Therefore, $F = Q^2 s/4\epsilon_0 by^2$. (We're now defining F to be the force that A applies to D, instead of the other way around, as it is defined in Eq. (3.32).)

Since the sheets are isolated, Q remains constant as y increases, whereas V does not. On the other hand, if the sheets are connected to a constant voltage source, Q will increase with increasing y, and U will also increase. But in this case the voltage source will supply energy for both the increase in U and the external work. The expressions for the force F (given in Eqs. (314) and (315)) will be exactly the same; so F is now constant. See the solution to Problem 3.26 for a more complete discussion of this point.

3.73. Force on a coaxial capacitor

We'll need to know the capacitance of the cylinders. Let ℓ be the distance of overlap. And let $\pm \lambda = \pm Q/\ell$ be the charge densities per unit length on the overlap region of the cylinders. The (magnitude of the) potential difference ϕ between the cylinders is

$$\phi = \int_{r_1}^{r_2} E \, dr = \int_{r_1}^{r_2} \frac{\lambda \, dr}{2\pi\epsilon_0 r} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right). \tag{319}$$

The capacitance is then

$$C = \frac{Q}{\phi} = \frac{\lambda \ell}{(\lambda/2\pi\epsilon_0)\ln(r_2/r_1)} = \frac{2\pi\epsilon_0\ell}{\ln(r_2/r_1)},$$
 (320)

which is independent of λ , as it should be. (The capacitance depends only on the geometry of the system, and not on the charge that is placed on the conductors.)

Consider how the energy changes when there is a downward displacement of the inner cylinder, so that the overlap distance increases by $\Delta \ell$. The capacitance increases by $\Delta C = 2\pi\epsilon_0 \Delta \ell / \ln(r_2/r_1)$. With constant potential difference, the stored electrical energy $C\phi^2/2$ increases by $(\Delta C)\phi^2/2$. At the same time, an amount of charge $\Delta Q = (\Delta C)\phi$ flows onto the capacitor. The battery thereby does work, in amount $(\Delta Q)\phi = (\Delta C)\phi^2$. This is *twice* the increase in stored energy in the field. The difference is the work done against the external force F that balances the electrical attraction of the cylinders. That is, the work done by the battery shows up as energy in the capacitor plus energy of an external object:

$$(\Delta C)\phi^2 = \frac{1}{2}(\Delta C)\phi^2 + F\Delta\ell.$$
(321)

Hence $F\Delta\ell = (\Delta C)\phi^2/2$, from which we obtain $F = (1/2)\phi^2\Delta C/\Delta\ell$. This is a quite general formula. In the case at hand, $\Delta C/\Delta\ell = 2\pi\epsilon_0/\ln(r_2/r_1)$. With $r_2/r_1 = 3/2$

and $\phi = 5000$ volts, we find

$$F = \frac{1}{2}\phi^2 \frac{2\pi\epsilon_0}{\ln(r_2/r_1)} = \frac{1}{2}(5000 \text{ V})^2 \frac{2\pi\left(8.85 \cdot 10^{-12} \frac{\text{s}^2 \text{ C}^2}{\text{kg m}^3}\right)}{\ln(3/2)} = 1.71 \cdot 10^{-3} \text{ N}.$$
 (322)

3.74. Equipotentials for two pipes

Consider the potential ϕ due to a single line charge with density λ . If ϕ is chosen to be zero at a point a distance r_0 from the line, then ϕ is zero on the entire circle (or cylinder) of radius r_0 around the line. To find the potential at a point at a different radius r_1 , we need only find the change in potential along a radial line from r_0 to r_1 . This change is

$$\phi(r_1) = -\int_{r_0}^{r_1} \frac{\lambda}{2\pi\epsilon_0 r} \, dr = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_1}{r_0}\right). \tag{323}$$

If $r_1 > r_0$ then $\phi(r_1)$ is negative (assuming λ is positive), which is correct.

With two line charges with densities $\pm \lambda$ located at positions $(\pm x_0, 0)$, the potential (relative to the origin) at an arbitrary point located r_1 from the positive line and r_2 from the negative line is

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_1}{x_0}\right) - \frac{(-\lambda)}{2\pi\epsilon_0} \ln\left(\frac{r_2}{x_0}\right) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right).$$
(324)

If $r_1 < r_2$ (so the point is closer to the positive line) then ϕ is positive, which is correct. We see that the potential is constant on a curve for which $r_2/r_1 = k$, for some constant k. So our goal is to show that the curve defined by $r_2/r_1 = k$ is a circle. Since $r_1^2 = (x - x_0)^2 + y^2$ and $r_2^2 = (x + x_0)^2 + y^2$, we can rewrite the relation $r_2^2 = k^2 r_1^2$ as

$$(x+x_0)^2 + y^2 = k^2 [(x-x_0)^2 + y^2]$$

$$\implies (k^2 - 1)(x^2 + y^2 + x_0^2) - 2(k^2 + 1)x_0 x = 0$$

$$\implies \left(x^2 - 2\frac{k^2 + 1}{k^2 - 1}x_0 x\right) + y^2 = -x_0^2.$$
(325)

The fact that the coefficients of x^2 and y^2 here are the same means that the curve is a circle. But let's finish the calculation anyway. Letting $b \equiv (k^2 + 1)x_0/(k^2 - 1)$ and completing the square yields

$$(x-b)^2 + y^2 = b^2 - x_0^2.$$
(326)

This describes a circle with radius $r = \sqrt{b^2 - x_0^2}$ centered at the point (b, 0). b is positive if k > 1 (relevant to the right half-plane), and b is negative if k < 1 (relevant to the left half-plane). Note that since $b > x_0$, r is indeed real. If $k \to \infty$, then $b \to x_0$ and $r \to 0$, as expected (we have a small circle around the positive line). And if $k \to 1$ from the positive side, then $b \to \infty$ and $r \to b \to \infty$, also as expected (we have a large circle with its leftmost point at the origin).

You can show that $r = 2k/|k^2 - 1|$. If k is replaced with 1/k (that is, r_1 and r_2 interchange their values), then b becomes -b, and r remains the same, as expected.

If you also want to demonstrate that the field lines are circles, you can show that any circle described by $x^2 + (y-h)^2 = x_0^2 + h^2$ intersects any circle described by Eq. (326) at right angles, for any values of b and h. (The parameter h gives the y value of the center of the field-line circle; any such circle must pass through the pipes and have

its center on the y axis.) There are various ways to show this. One is to take the differential of each circle equation to find the slopes of the curves, and to then show that these slopes are the negative reciprocals of each other, by using the difference of the circle equations, namely $bx - hy = x_0^2$.

3.75. Average of six points

The Taylor expansions are

$$\phi(x_0 + \delta, y_0, z_0) = \phi(x_0, y_0, z_0) + \delta \frac{\partial \phi}{\partial x} + \frac{\delta^2}{2!} \frac{\partial^2 \phi}{\partial x^2} + \frac{\delta^3}{3!} \frac{\partial^3 \phi}{\partial x^3} + \cdots,$$

$$\phi(x_0 - \delta, y_0, z_0) = \phi(x_0, y_0, z_0) - \delta \frac{\partial \phi}{\partial x} + \frac{\delta^2}{2!} \frac{\partial^2 \phi}{\partial x^2} - \frac{\delta^3}{3!} \frac{\partial^3 \phi}{\partial x^3} + \cdots, \qquad (327)$$

and likewise for the $y_0 \pm \delta$ and $z_0 \pm \delta$ points. When we add up all six terms and divide by 6 to take the average, the terms with odd powers of δ cancel in pairs, and we are left with

$$\phi_{\text{avg}} = \frac{1}{6} \left[6\phi(x_0, y_0, z_0) + \delta^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + \mathcal{O}(\delta^4) \right].$$
(328)

If $\nabla^2 \phi = 0$, then the δ^2 term here is zero, so we have

$$\phi_{\text{avg}} = \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4), \tag{329}$$

which equals $\phi(x_0, y_0, z_0)$ through terms of order δ^3 , as desired.

3.76. The relaxation method

After four iterations, the values of a through g, as they would appear in the array, are (at least for the order of my path running through the array):

 $\begin{array}{cccc} 29.2 & 61.9 \\ 26.2 & 56.5 \\ 18.4 & 37.5 \\ 9.2 \end{array}$

These values are close to the true values (which settle down after about 25 iterations), obtained from the computer program in Exercise 3.77:

29.213561.797825.842756.179817.977537.07878.98876

Using either of the above sets of ϕ values, we see that the $\phi = 25$ and $\phi = 50$ equipotentials will look something like the ones shown in Fig. 84.

3.77. Relaxation method, numerical

Here is a *Mathematica* program that gets the job done for an 18×18 array instead of the 9×9 array that appeared in Exercise 3.76:

```
ITS=150; (* number of iterations *)
n=6; (* width of box in middle; 1/3 size of whole array *)
NN=3n+1; (* number of lattice points in each direction *)
T=Table[100, {i,1,NN},{j,1,NN}]; (* create matrix with all entries = 100 *)
(* create an initial array with equipotential squares varying linearly from 0
on the outer boundary to 100 on the inner square: *)
```



Figure 84

```
Do[Do[T[[i,j]]=(100./n)*(k-1), {i,k,NN-(k-1)}, {j,k,NN-(k-1)}], {k,1,n+1}];
(* now do the iterative averaging: *)
Do[
 (* averaging for rows above middle box: *)
Do[Do[T[[i,j]]=(T[[i,j-1]]+T[[i,j+1]]+T[[i-1,j]]+T[[i+1,j]])/4.,
       {j,2,NN-1}],{i,2,n}];
 (* averaging for rows left of middle box: *)
 Do[Do[T[[i,j]]=(T[[i,j-1]]+T[[i,j+1]]+T[[i-1,j]]+T[[i+1,j]])/4.,
       {j,2,n}],{i,n+1,2n+1}];
 (* averaging for rows right of middle box: *)
 Do[Do[T[[i,j]]=(T[[i,j-1]]+T[[i,j+1]]+T[[i-1,j]]+T[[i+1,j]])/4.,
       {j,2n+2,NN-1}],{i,n+1,2n+1}];
 (* averaging for rows below middle box: *)
 Do[Do[T[[i,j]]=(T[[i,j-1]]+T[[i,j+1]]+T[[i-1,j]]+T[[i+1,j]])/4.,
       {j,2,NN-1}],{i,2n+2,NN-1}],
{it,1,ITS}]
               (* repeat averaging process ITS times *)
PaddedForm[MatrixForm[T], {6, 3}]
                                   (* print matrix *)
```

The results for the "triangle" of entries analogous to those in Exercise 3.76 are:

14.357	29.179	44.917	61.945	80.423
14.124	28.722	44.271	61.221	79.872
13.416	27.313	42.224	58.796	77.846
12.227	24.892	38.515	53.892	72.717
10.601	21.512	33.052	45.541	59.129
8.666	17.503	26.641	36.091	
6.561	13.192	19.917		
4.386	8.789			
2.193				

The bold entries correspond to the seven entries in Exercise 3.76. The agreement is reasonably good. If the middle box is 48×48 instead of the above 6×6 or the 3×3 we had in Exercise 3.76, then, for example, the 36.091 entry becomes 35.2961. By

looking at how this number changes with the size of the box, it appears to converge to approximately 35.2 in the continuum limit involving an infinite number of lattice points.

Interestingly, the computing time doesn't appear to be helped much by our choice of initial equipotentials that varied linearly from the outer boundary to the central square. If we had instead picked $\phi = 0$ on the outer boundary and $\phi = 100$ at every other point (*all* the other points, not just the ones in the middle square), then the computing time would be only slightly longer. The computing time increases by a factor of 16 for every doubling of the array's width, because there are 4 times as many points that each iteration needs to run through, and also it turns out that we need to do 4 times as many iterations to achieve a given accuracy.

Chapter 4

Electric currents

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

4.19. Synchrotron current

The number of round trips that each electron makes per second is $(3 \cdot 10^8 \text{ m/s})/(240 \text{ m}) = 1.25 \cdot 10^6 \text{ s}^{-1}$. The charge per second passing any given point is therefore

$$(1.25 \cdot 10^6 \,\mathrm{s}^{-1})Ne = (1.25 \cdot 10^6 \,\mathrm{s}^{-1})(10^{11})(1.6 \cdot 10^{-19} \,\mathrm{C}) = 0.02 \,\mathrm{A}.$$
 (330)

4.20. Combining the current densities

The current density is given by $\mathbf{J} = qN\mathbf{u}$. So the magnitudes of the two \mathbf{J} 's are

$$J_{\rm ion} = (2e)N_{\rm ion}u_{\rm ion} = (2 \cdot 1.6 \cdot 10^{-19} \,\mathrm{C})(5 \cdot 10^{16} \,\mathrm{m}^{-3})(10^5 \,\mathrm{m/s}) = 1.6 \cdot 10^3 \,\frac{\mathrm{A}}{\mathrm{m}^2} \,,$$

$$J_{\rm e} = eN_{\rm e}u_{\rm e} = (1.6 \cdot 10^{-19} \,\mathrm{C})(10^{17} \,\mathrm{m}^{-3})(10^6 \,\mathrm{m/s}) = 1.6 \cdot 10^4 \,\frac{\mathrm{A}}{\mathrm{m}^2} \,.$$
(331)

 \mathbf{J}_{ion} points west, and \mathbf{J}_{e} points southwest due to the negative charge of the electron; see Fig. 85. The components of the total current density \mathbf{J} are therefore

$$\mathbf{J}_{\text{west}} = J_{\text{ion}} + \frac{J_{\text{e}}}{\sqrt{2}} = \left(1.6 \cdot 10^3 + \frac{1.6 \cdot 10^4}{\sqrt{2}}\right) \frac{A}{\text{m}^2} = 1.29 \cdot 10^4 \frac{A}{\text{m}^2},$$

$$\mathbf{J}_{\text{south}} = \frac{J_{\text{e}}}{\sqrt{2}} = 1.13 \cdot 10^4 \frac{A}{\text{m}^2}.$$
 (33)

Paris J_{total} J_e J_{ion} S (2) Figure 85

41.2

Since $\tan^{-1}(1.13/1.29) = 41.2^{\circ}$, we see that **J** points in a direction 41.2° south of west. The magnitude of **J** is $\sqrt{1.29^2 + 1.13^2} \cdot 10^4 \text{ A/m}^2 = 1.71 \cdot 10^4 \text{ A/m}^2$.

4.21. Current pulse from an alpha particle

(a) Let x be the distance from the left plate, and let Q_r be the charge on the right plate. From Exercise 3.37 we have $Q_r = -(2e)x/\ell$, where $\ell = 2$ mm is the distance between the plates. The current flowing out of the right plate is therefore

$$I = -\frac{dQ_{\rm r}}{dt} = \frac{2e}{\ell}\frac{dx}{dt} = \frac{2ev}{\ell} = \frac{2(1.6 \cdot 10^{-19}\,{\rm C})(10^6\,{\rm m/s})}{0.002\,{\rm m}} = 1.6 \cdot 10^{-10}\,{\rm A}.$$
 (333)





This current lasts for a time $t = \ell/v = (0.002 \text{ m})/(10^6 \text{ m/s}) = 2 \cdot 10^{-9} \text{ s}$, which is 2 nanoseconds. The current is constant during this time, so we have the bold line shown in Fig. 86. The total charge that flows during this time is It, which equals 2e as expected.

If the path slopes upward at 45°, then $dx/dt = v \cos 45^\circ$. From above, the current pulse is therefore reduced in amplitude by a factor $1/\sqrt{2}$ and stretched out in time by a factor $\sqrt{2}$; see the dotted line in Fig. 86. Again the total charge transferred is It = 2e.

(b) Following the strategy of the solution to Exercise 3.37, we know that if Q_1 and Q_2 are the charges on the inner and outer electrodes (with radii a and b, respectively), then $Q_1 + Q_2 = -2e$. How is the charge of -2e distributed between Q_1 and Q_2 when the alpha particle is at radius r? As in Exercise 3.37, the key points are that (1) we can smear out the alpha particle into a cylinder of charge, and (2) the potentials of the two electrodes are the same, which means that the line integrals of the electric field from radius r to the two electrodes must be equal. The field inside radius r is proportional to Q_1/r (this points inward since Q_1 is negative), and the field outside radius r is proportional to $(2e+Q_1)/r = -Q_2/r$ (this points outward since Q_2 is negative). Equating the two line integrals gives (note that both sides of the following equation are positive since dr is negative in the left integral)

$$\int_{r}^{a} \frac{Q_{1}}{r} dr = \int_{r}^{b} \frac{-Q_{2}}{r} dr \implies Q_{1} \ln(a/r) = -Q_{2} \ln(b/r)$$
$$\implies Q_{1} \ln(r/a) = Q_{2} \ln(b/r).$$
(334)

Combining this equation with $Q_1 + Q_2 = -2e$ and solving for Q_1 and Q_2 gives

$$Q_1 = \frac{-(2e)\ln(b/r)}{\ln(b/a)}$$
 and $Q_2 = \frac{-(2e)\ln(r/a)}{\ln(b/a)}$. (335)

The current flowing out of the outer cylinder is then

$$I = -\frac{dQ_2}{dt} = \frac{2e}{\ln(b/a)}\frac{d(\ln r)}{dt} = \frac{2e}{\ln(b/a)}\frac{1}{r}\frac{dr}{dt} = \frac{2ev}{\ln(b/a)}\frac{1}{a+vt},$$
(336)

where we have used r = a + vt. We see that I(t) is not constant. A plot of the general shape of I(t) is shown in Fig. 87 (with b chosen to equal 4a). For a given value of b, if a is very small then the current starts out very large, because at t = 0 the smallness of a in the denominator in Eq. (336) wins out over the largeness of $\ln(b/a)$.

In the case of a 45° angle of the path, the same modifications that applied in part (a) also apply here. That is, the curve is stretched horizontally by a factor of $\sqrt{2}$, and squashed vertically by a factor of $1/\sqrt{2}$.

4.22. Transatlantic cable

(a) The resistance of the seven wires together is

$$R = \frac{\rho L}{A} = \frac{(3 \cdot 10^{-8} \,\Omega \,\mathrm{m})(3 \cdot 10^{6} \,\mathrm{m})}{7 \cdot \pi (3.65 \cdot 10^{-4} \,\mathrm{m})^{2}} = 3.1 \cdot 10^{4} \,\Omega.$$
(337)

Adding seven resistors in parallel would give the same answer.





(b) Our goal is to obtain a rough upper bound on the resistance of the ocean path, and then see if this is smaller than the answer to part (a). As stated, we'll take the electrodes to be spheres with a radius of 0.1 m. (Any factors of order 1 will be irrelevant.) Let's imagine the path of the current to be roughly a hemisphere expanding out from one sphere, then a tube with a large cross section, and then a hemisphere tapering down to the other sphere. (Even if the end parts are more conical than hemispherical, we'll still get the same order of magnitude.) For the tubular middle part, it doesn't matter much what cross section we pick, if our only goal is to obtain an upper bound on the resistance. Even if the cross section is just a kilometer in diameter, the resistance is (using a length of 3000 km)

$$R = \frac{\rho L}{A} = \frac{(0.25\,\Omega\,\mathrm{m})(3\cdot10^6\,\mathrm{m})}{\pi(10^3\,\mathrm{m})^2/4} \approx 1\,\Omega,\tag{338}$$

which is negligible compared with the resistance of the cable.

What is the resistance of the hemispheres? If $d \approx 0.1 \,\mathrm{m}$ is their radius, then from dimensional analysis we expect the resistance to be roughly ρ/d , in order of magnitude. And indeed, from Problem 4.4 the resistance is proportional to ρ/d , assuming that the other radius involved is large. In the present case, ρ/d equals $(0.25 \,\Omega \,\mathrm{m})/(0.1 \,\mathrm{m}) = 2.5 \,\Omega$. This is negligible compared with the cable's resistance. Even if the electrode's size was on the order of a millimeter, the hemisphere's resistance would still only be 250 ohms, which is again negligible compared with the cable's resistance.

4.23. Intervals between independent events

(a) If we divide the time t into a very large number, N, of equal small intervals, then the length of each interval is dt = t/N. The probability that an event happens in a particular one of these small intervals is therefore pdt = p(t/N). So the probability that an event happens in *none* of these N intervals is $(1 - pt/N)^N$. Multiplying this by the probability pdt that an event *does* happen in the next dtinterval tells us that the probability of the next event happening between t and t + dt is (in the $N \to \infty$ limit)

$$(1 - pt/N)^N p \, dt = e^{-pt} p \, dt, \tag{339}$$

as desired. The integral of this probability must be 1, because the next event must happen at *some* time. And indeed,

$$\int_{0}^{\infty} e^{-pt} p \, dt = -e^{-pt} \Big|_{0}^{\infty} = 1.$$
(340)

We are free to pick the t = 0 point as the time of an event, so the probability in Eq. (339) is also the probability that an interval between events has length between t and t + dt. That is, if we look at a million successive intervals, then approximately $(10^6)(e^{-pt}p dt)$ of them will have length between t and t + dt.

(b) To find the average value (or expectation value) of a quantity, we must multiply the probability of a value occurring times the value itself, and then integrate over all the values. Equivalently, we can look at a million waiting times and calculate their average by adding up all the times and dividing by a million. So the average waiting time (starting at any given time, not necessarily the time of an event) until the next event is

$$\bar{t}_{\text{wait}} = \int_0^\infty t \cdot e^{-pt} p \, dt = -e^{-pt} (t+1/p) \Big|_0^\infty = 1/p.$$
(341)

(You should check this integral by differentiating it.) It makes sense that this time decreases with p; if p is large, then the events happen more frequently, so the waiting time is shorter.

Since this 1/p result holds for any arbitrary starting time, we are free to choose the starting time as the time of an event. A special case of this result is therefore the statement that the average waiting time between events is 1/p. This is consistent with the fact that pt is the average number of events that occur during a (not necessarily infinitesimal) time t.

- (c) If we pick a random point in time, then the average waiting time until the next event is 1/p, from part (b). And the average time since the previous event is also 1/p, because we can use the same reasoning that we used in part (a), going backward in time, to calculate the probability that the most recent event occurred at a time between t and t + dt earlier. The direction of time is irrelevant; the process is completely described by saying that p dt is the probability of an event happening in an infinitesimal time dt, and this makes no reference to a direction of time. The average length of the interval surrounding a randomly chosen point in time is therefore 1/p + 1/p = 2/p.
- (d) From part (a), an event-to-event interval with length between t and t + dt occurs with probability $e^{-pt}p dt$ (in the sense that out of a million successive intervals, $(10^6)(e^{-pt}p dt)$ of them will have this length). But if you pick a random point in time, $e^{-pt}p dt$ is not the probability that you will end up in an interval with length between t and t + dt, because you are more likely to end up in an interval that is longer.

Consider the simple case where there are only two possible lengths of intervals, 1 and 100, and these occur with equal probabilities of 1/2. If you look at 1000 successive intervals, then about 500 will have length 100. But if you pick a random point in time, you are of course 100 times more likely to end up in one of the large intervals. The probability of landing in each type of interval is *not* 1/2. The probability of landing in an interval of a given length (1/101 and 100/101 in the present example) does not equal the probability of that given length occurring in a list of the lengths (1/2 and 1/2 here). In this example, the average time between events is 50.5, while the average time surrounding a randomly chosen point in time is, as you can show, 99.02. (These results don't have anything to do with the above results involving p, because the present example isn't a random process described by a given probability per unit time. But it illustrates the basic point.)

In short, the probability of landing in an interval with length between t and t+dt is proportional both to $e^{-pt}p dt$ (because the more intervals there are of a certain length, the more likely you are to land in one of them), and to the length t of the intervals (because the longer they are, the more likely you are to land in one of them).

(e) Consider a large number N of intervals. The number of intervals with length between t and t + dt is $N(e^{-pt}p dt)$. The total length of these intervals with length between t and t + dt is therefore $N(e^{-pt}p dt)t$. The total length of all of the N intervals is the integral of this, which you can quickly show equals N/p, as it should.

The probability of picking a point in time that lands in one of the intervals with length between t and t + dt equals the total length associated with these intervals, divided by the total length of all of the N intervals, which gives $(Ne^{-pt}pt dt)/(N/p) = e^{-pt}p^2t dt$. As mentioned in part (d), this probability

is proportional to both $e^{-pt}p dt$ and t. The expectation value of the length of the interval that the given point lands in is obtained by multiplying this probability by the interval length t and integrating. This gives

$$\int_0^\infty e^{-pt} p^2 t^2 \, dt = -\frac{e^{-pt}}{p} \left(2 + 2pt + p^2 t^2\right)\Big|_0^\infty = \frac{2}{p}, \qquad (342)$$

as desired. (Again, you should check this integral by differentiating it.) To sum up, there are two different probabilities in this problem: (1) the probability that a randomly chosen interval has length between t and t + dt (this equals $e^{-pt}p dt$), and (2) the probability that a randomly chosen point in time falls in an interval with length between t and t + dt (this equals $e^{-pt}p^2t dt$). In the first case, by "randomly" we mean that we label each interval with a number and then pick a random number. The length of each interval is irrelevant in this case, whereas it is quite relevant in the second case.

4.24. Mean free time in water

From Eq. (4.23), the conductivity is $\sigma = 2Ne^2\tau/m$, where the factor of 2 comes from the fact that in pure water there are both positive and negative ions. The mass of the OH⁻ and OH₃⁺ ions is essentially the mass of 17 or 19 nucleons, which is about $3 \cdot 10^{-26}$ kg. So we have

$$\tau = \frac{m\sigma}{2Ne^2} = \frac{(3 \cdot 10^{-26} \,\mathrm{kg}) \left(4 \cdot 10^{-6} \,(\Omega \,\mathrm{m})^{-1}\right)}{2(6 \cdot 10^{19} \,\mathrm{m}^{-3}) (1.6 \cdot 10^{-19} \,\mathrm{C})^2} \approx 4 \cdot 10^{-14} \,\mathrm{s.}$$
(343)

The distance traveled in this time is $v\tau = (500 \text{ m/s})(4 \cdot 10^{-14} \text{ s}) = 2 \cdot 10^{-11} \text{ m}$. As expected, this is smaller than the size of a water molecule, which is on the order of a couple angstroms (10^{-10} m) .

4.25. Drift velocity in seawater

We know that the current density is $J = Nev \implies v = J/Ne$. So our goal is to determine J. It is given by

$$J = \sigma E = \frac{1}{\rho} \frac{V}{L} \,. \tag{344}$$

Alternatively, we can find J from

$$J = \frac{I}{A} = \frac{V/R}{A} = \frac{V}{A} \frac{1}{\rho L/A} = \frac{V}{\rho L}.$$
 (345)

The drift velocity is therefore (the factor of 2 in the N here comes from the fact that there are both Na^+ and Cl^- ions)

$$v = \frac{J}{Ne} = \frac{V}{\rho L Ne} = \frac{12 \,\mathrm{V}}{(0.25 \,\Omega \,\mathrm{m})(2 \,\mathrm{m})(2 \cdot 3 \cdot 10^{26} \,\mathrm{m^{-3}})(1.6 \cdot 10^{-19} \,\mathrm{C})}$$

= 2.5 \cdot 10^{-7} \, m/s. (346)

4.26. Silicon junction diode

We know that the densities of the slabs are equal and opposite, because ϕ is constant is each region outside the junction. It turns out that the slab in the *n*-type material is actually the positive slab (which is the opposite of what you might think), due to the fact that the slabs are brought about by the diffusion of holes from the *p*-type to *n*-type material, and the diffusion of electrons from the *n*-type to *p*-type material. But for the purposes of this exercise, it isn't critical which is which.

Poisson's equation, $\nabla^2 \phi = -\rho/\epsilon_0$, tells us that

$$\frac{d^2\phi}{dx^2} = -\frac{\rho}{\epsilon_0} \implies \phi = A + Bx - \frac{\rho}{2\epsilon_0}x^2.$$
(347)

This can also be written in the alternative general form, $\phi = -(\rho/2\epsilon_0)(x+C)^2 + D$. We see that ϕ varies quadratically with x if ρ is constant. The curvature is positive in the region where ρ is negative, and negative in the region where ρ is positive.

We are told that the slope of ϕ is zero outside the charge layers. It must therefore also be zero just inside the boundaries. If there were a discontinuity in the slope of ϕ , then the second derivative would be infinite. Poisson's equation would then imply an infinite ρ , such as that arising from a surface charge. But there are no surface charges in this setup.

The zero slope at the boundaries implies that ϕ must look like the curve shown in Fig. 88. Let us define $\phi = 0$ at the left boundary of the left slab, and let x = 0 be the location of the midplane. Let $\ell \equiv 10^{-4}$ m. The density is $-\rho$ in the left (*p*-type) region, and $+\rho$ in the right (*n*-type) region. In the left region we have half of a rightside-up parabola centered at $x = -\ell$, and in the right region we have half of an upside-down parabola centered at $x = \ell$. So if $\mp \rho$ are the charge densities in the two regions, we quickly see that the potential ϕ in Eq. (347) (or rather the alternate form listed right after) must take the forms,

$$\phi_p = (\rho/2\epsilon_0)(x+\ell)^2,
\phi_n = (0.3 \text{ V}) - (\rho/2\epsilon_0)(x-\ell)^2.$$
(348)



Figure 88

These two forms must agree at x = 0, so we have

$$\frac{\rho \ell^2}{2\epsilon_0} = (0.3 \,\mathrm{V}) - \frac{\rho \ell^2}{2\epsilon_0}$$

$$\implies \rho = \frac{\epsilon_0 (0.3 \,\mathrm{V})}{\ell^2} = \frac{\left(8.85 \cdot 10^{-12} \frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg \,m^3}}\right) (0.3 \,\mathrm{V})}{(10^{-4} \,\mathrm{m})^2} = 2.7 \cdot 10^{-4} \,\frac{\mathrm{C}}{\mathrm{m^3}} \,. \quad (349)$$

The electric field at x = 0 is obtained via $E_x = -d\phi/dx$. We can use either of the forms of ϕ in Eq. (348) for this, and we obtain $E_x = -\rho \ell/\epsilon_0$. Rather that plugging in

the various numbers, we can be a little more economical by writing

$$E_x = -\frac{\rho\ell}{\epsilon_0} = -\frac{(\epsilon_0(0.3\,\mathrm{V})/\ell^2)\ell}{\epsilon_0} = -\frac{0.3\,\mathrm{V}}{\ell} = -\frac{0.3\,\mathrm{V}}{10^{-4}\,\mathrm{m}} = -3000\,\frac{\mathrm{V}}{\mathrm{m}}\,.$$
 (350)

Note that this is twice the average field in the slabs, which is $-(0.3 \text{ V})/(2\ell)$.

4.27. Unbalanced current

The capacitance of the isolated box must be roughly that of a sphere intermediate between the inscribed and circumscribed spheres. So let's use a sphere with radius 6 cm. The capacitance is then $C = 4\pi\epsilon_0 r = 4\pi (8.85 \cdot 10^{-12} \,\mathrm{s^2 C^2/kg \,m^3})(0.06 \,\mathrm{m}) \approx 7 \cdot 10^{-12} \,\mathrm{F}$. The box is acquiring charge at a rate of $10^{-6} \,\mathrm{A}$, so the charge after time t is $Q(t) = (10^{-6} \,\mathrm{C/s})t$. The potential is then V(t) = Q(t)/C. Setting this equal to 1000 volts gives

$$\frac{(10^{-6} \,\mathrm{C/s})t}{7 \cdot 10^{-12} \,\mathrm{F}} = 1000 \,\mathrm{V} \implies t = 7 \cdot 10^{-3} \,\mathrm{s.}$$
(351)

7 milliseconds is rather quick on an everyday timescale.

4.28. Parallel resistors

The two loop equations are

$$\mathcal{E} - (I_1 - I_2)R_1 = 0,$$

-(I_2 - I_1)R_1 - I_2R_2 = 0. (352)

Adding these two equations quickly gives $I_2 = \mathcal{E}/R_2$ (which corresponds to the loop around the whole circuit). Either equation then gives

$$I_1 = \frac{\mathcal{E}(R_1 + R_2)}{R_1 R_2} \equiv \frac{\mathcal{E}}{R_{\text{eff}}}, \quad \text{where} \quad R_{\text{eff}} \equiv \frac{R_1 R_2}{R_1 + R_2}.$$
(353)

The current through the battery is I_1 , so $\mathcal{E} = I_1 R_{\text{eff}}$ tells us that the effective resistance is R_{eff} .

4.29. Keeping the same resistance

We have an R_1 resistor in series with the parallel combination of R_1 and $(R_1 + R_0)$. So we want

$$R_{1} + \frac{R_{1}(R_{1} + R_{0})}{R_{1} + (R_{1} + R_{0})} = R_{0} \implies (2R_{1}^{2} + R_{1}R_{0}) + (R_{1}^{2} + R_{1}R_{0}) = 2R_{1}R_{0} + R_{0}^{2}$$
$$\implies 3R_{1}^{2} = R_{0}^{2} \implies R_{1} = \frac{R_{0}}{\sqrt{3}}.$$
(354)

4.30. Automobile battery

If the voltage drop across the $0.5\,\Omega$ resistor is $9.8\,\mathrm{V}$, then the current in the circuit is $I=V/R=(9.8\,\mathrm{V})/(0.5\,\Omega)=19.6\,\mathrm{A}$. The voltage drop across the internal resistor is then $R_\mathrm{i}(19.6\,\mathrm{A})$. But we know that this voltage drop is $12.3\,\mathrm{V}-9.8\,\mathrm{V}=2.5\,\mathrm{V}$. Therefore, $2.5\,\mathrm{V}=R_\mathrm{i}(19.6\,\mathrm{A})\Longrightarrow R_\mathrm{i}=0.128\,\Omega$.

4.31. Equivalent boxes

The resistances in the first box are simply $R_{ab} = 10 + 20 = 30$, $R_{ac} = 10 + 50 = 60$, and $R_{bc} = 20 + 50 = 70$, as desired. In the second box, in each case we have one

resistor in parallel with the series combination of the other two resistors. So the the resistances in the second box are

$$R_{ab} = \frac{34(170+85)}{289} = 30,$$

$$R_{ac} = \frac{85(34+170)}{289} = 60,$$

$$R_{bc} = \frac{170(85+34)}{289} = 70,$$
(355)

as desired.

For the two configurations pictured, the above resistances are the only possibilities that lead to the given resistances between the terminals. (Each configuration yields three equations in the three resistances, so they are uniquely determined.) However, there are other configurations that also work, for example, the one pictured in Fig. 89 (as you can check).

The potential assumed by a free terminal when the potentials at the other two terminals are fixed is the same for the two boxes. For example, if the potentials at b and cin the first box are fixed at ϕ_b and ϕ_c , then the potential at a divides the difference $\phi_b - \phi_c$ in the ratio of 20 to 50. And in the second box the ratio is 34 to 85, which is the same. The two boxes are therefore indistinguishable by external measurements (using direct currents). You can show that the configuration in Fig.89 also yields the same ratio of 2 to 5.

4.32. Tapered rod

Let the length of the rods be ℓ . Then the resistance of the cylindrical rod is $\rho L/A = \rho \ell / \pi a^2$.

Now consider the tapered cone. The resistance of a cross-sectional disk is $\rho L/A = \rho dx/\pi r^2$, where $r = a + (b - a)(x/\ell)$ from similar triangles in Fig. 90. So $dx = dr \ell/(b-a)$, and we have

$$R = \int_0^\ell \frac{\rho \, dx}{\pi r^2} = \frac{\rho}{\pi} \frac{\ell}{b-a} \int_a^b \frac{dr}{r^2} = \frac{\rho}{\pi} \frac{\ell}{b-a} \left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\rho\ell}{\pi ab} \,. \tag{356}$$

This is a/b times the resistance of the cylindrical rod, as desired. If b < a (or b > a) then this fraction is larger (or smaller) than 1, which makes sense. If $b \to 0$ then $R \to \infty$. Note that the conical rod has the same resistance as a cylindrical rod with radius \sqrt{ab} . See Problem 4.6 for a discussion of the approximate nature of this solution.

As a check on this result, consider three objects, all with the same length: a cylinder with radius a, a tapered cone with radii a and b, and another cylinder with radius b. For concreteness, assume a < b, as in Fig. 90. Then Eq. (356) tells us that the resistance of the second of these objects is a/b times that of the first, and also that the resistance of the third is a/b times that of the second. So the first and third resistances are in the ratio of b^2 to a^2 , or equivalently $1/a^2$ to $1/b^2$. This is correct, because the resistances of the two cylinders are inversely proportional to their cross-sectional areas, which are proportional to length squared.

4.33. Laminated conductor extremum

In the solution to Problem 4.5, if we replace the number 7.2 with n, and the fraction 1/3 with f, then we can repeat the same reasoning and derive the general result,

$$\frac{\sigma_{\perp}}{\sigma_{\parallel}} = \frac{1}{\left[f + n(1-f)\right] \left[f + (1/n)(1-f)\right]} \,. \tag{357}$$









(Or you can just look at Eqs. (12.204) and (12.206) and make the above replacements.) When the denominator is multiplied out, it equals

$$-f^{2}(n+1/n-2) + f(n+1/n-2) + 1.$$
(358)

Alternatively, if we don't expand things as far, we can write the denominator as $f^2 + (1-f)^2 + f(1-f)(n+1/n)$. Since $n + 1/n \ge 2$ (by the arithmetic-geometric-mean inequality, or by taking a derivative), the denominator is greater than or equal to $f^2 + (1-f)^2 + 2f(1-f) = (f + (1-f))^2 = 1$. This means that σ_{\perp} is always less than or equal to σ_{\parallel} , for any values of f (between 0 and 1) and n.

(a) For a given n, taking the derivative of the denominator given in Eq. (358), with respect to f, shows that it achieves a maximum when f = 1/2, which means that σ_⊥/σ_{||} achieves a minimum when f = 1/2, that is, when the two thicknesses are equal. This result is independent of n. So no matter what the ratio of conductivities is, σ_⊥/σ_{||} is minimum when the materials have the same thickness. The minimum value of σ_⊥/σ_{||} is quickly found to be 4/(2 + n + 1/n). It makes sense that there should be an extremum of σ_⊥/σ_{||} for an intermediate value of f (between 0 and 1), because if f = 0 or f = 1, the material consists of

value of f (between 0 and 1), because if f = 0 or f = 1, the material consists of only one substance, so $\sigma_{\perp}/\sigma_{\parallel} = 1$. Therefore, unless $\sigma_{\perp}/\sigma_{\parallel}$ is a flat curve (which it undoubtedly isn't), it must reach a maximum or minimum for some f between 0 and 1.

(b) For a given f (between 0 and 1), taking the derivative of the denominator given in Eq. (358), with respect to n, shows that it achieves a minimum when n = 1, which means that $\sigma_{\perp}/\sigma_{\parallel}$ achieves a maximum when n = 1. This result is independent of f. So no matter what the ratio of thicknesses is, $\sigma_{\perp}/\sigma_{\parallel}$ is maximum when the materials have the same σ . And the maximum value of $\sigma_{\perp}/\sigma_{\parallel}$ is 1, of course, because the two materials are the same.

It makes sense that there should be an extremum of $\sigma_{\perp}/\sigma_{\parallel}$ for an intermediate value of n (between 0 and ∞), because setting one of the σ 's equal to zero makes σ_{\perp} (but not σ_{\parallel}) equal to zero; and setting one of the σ 's equal to infinity makes σ_{\parallel} (but not σ_{\perp}) equal to infinity. So $\sigma_{\perp}/\sigma_{\parallel}$ is zero at both extremes. Therefore, unless $\sigma_{\perp}/\sigma_{\parallel}$ is a flat curve (which, again, it undoubtedly isn't), it must reach a maximum or minimum for some n between 0 and ∞ .

4.34. Effective resistances in lattices

As in Problem 4.8, we will superpose two setups, one with a current of 1 A entering at one node and heading out to infinity, and the other with a current of 1 A heading in from infinity and exiting at an adjacent node. The only modification we need to make to the solution to Problem 4.8 is the number of resistors connected to a given node. If there are n resistors connected to each node (n was 4 in Problem 4.8), then each of the above two setups has a current of 1/n A heading from one node to the other. (All of the setups have the necessary symmetry for the current to divide equally.) So the current between the two nodes is 2/n A when the setups are superposed. The voltage drop across the $1\,\Omega$ resistor is then 2/n V, so $V = IR_{\rm eff}$ gives 2/n V = $(1\,{\rm A})R_{\rm eff}$. The effective resistance is therefore 2/n Ω . Hence:

- (a) The 3-D cubic lattice has n = 6, so $R_{\text{eff}} = 2/6 = 1/3 \Omega$.
- (b) The 2-D triangular lattice also has n = 6, so $R_{\text{eff}} = 1/3 \Omega$.
- (c) The 2-D hexagonal lattice has n = 3, so $R_{\text{eff}} = 2/3 \Omega$.



Figure 91

(d) The 1-D lattice has n = 2, so $R_{\text{eff}} = 2/2 = 1 \Omega$. This makes sense, because there is only one path between two adjacent nodes on a line, namely across the 1Ω resistor connecting them. All the other resistors outside the two nodes are irrelevant.

4.35. Resistances in a cube

(a) In Fig. 91 the three vertices adjacent to A (which are labeled as "a") are all at the same potential (by symmetry under rotations around the AB diagonal), so we can collapse them to one point. (Equivalently, if we connect them with resistance-less wires, no current will flow in these wires.) Likewise for the three vertices adjacent to B (which are labeled as "b"). So the circuit is equivalent to the second setup shown in Fig. 91 (the number of lines is still 12), which can be simplified as indicated. The equivalent resistance is therefore 5R/6.

Alternatively, we can work in terms of currents. The input current I_0 gets divided evenly, by symmetry, into three $I_0/3$ currents. It then divides into six $I_0/6$ currents, and then converges to three $I_0/3$ currents. The total potential drop across any of the possible paths from A to B is given by $V = (I_0/3)R + (I_0/6)R + (I_0/3)R = (5/6)I_0R$. The effective resistance is then $V/I_0 = 5R/6$.

- (b) In Fig. 92 there are four vertices (labeled as "c") that lie in the plane that is equidistant from A and B. These vertices are all at the same potential (halfway between V_A and V_B), so we can collapse them to a point. (In the second setup shown, there are only 10 lines because 2 of the original 12 lines were collapsed). The circuit can then be simplified as shown, and the equivalent resistance is 3R/4.
- (c) From symmetry, the two points marked as a in Fig. 93 are at the same potential, so we can collapse them to a point. Likewise for the two b's. The circuit can then be simplified as shown, and the equivalent resistance is 7R/12. As expected, this is smaller than the answer to part (b), which in turn is smaller than the answer to part (a).










Figure 93



Figure 94



Figure 95

Note that the sum of the effective resistances across all 12 resistors is 12(7R/12) = 7R = (8-1)R, where the 8 here is the number of corners in the cube. This is a special case of the general result in Problem 4.9.

4.36. Attenuator chain

If R is the effective resistance of the infinite chain, then the chain is equivalent to the circuit shown in Fig. 94. So we have

$$R = R_1 + \frac{R_2 R}{R_2 + R} \implies R(R_2 + R) = R_1(R_2 + R) + R_2 R$$

$$\implies R^2 - R_1 R - R_1 R_2 = 0$$

$$\implies R = \frac{R_1 + \sqrt{R_1^2 + 4R_1 R_2}}{2}, \qquad (359)$$

where we have chosen the positive root.

To demonstrate the stated geometric-series result, consider four points A, A', B, B' that form a square somewhere within the circuit, as shown in Fig.95. Given the voltage V' between A' and B' (so $V' = V_0$ if A' and B' are at the left end of the chain), what is the voltage V between A and B? Let the current flowing toward point A' be I. This current splits into the currents I_1 and I_2 . The circuit to the right of A' and B' is equivalent to a resistance R, so currents of I_1 and I_2 pass through resistances R and R_2 , respectively. These currents are in the ratio of R_2/R ; hence $I_1 = IR_2/(R_2 + R)$.

Now, the voltage V' between A' and B' is proportional to the current I flowing into A' (by dimensional analysis). Likewise, the voltage V between A and B is proportional to the current I_1 flowing into A, with the same constant of proportionality (because we have the same infinite circuit, independent of where it starts). Therefore, the ratio of the voltage between A and B to the voltage between A' and B' is

$$\frac{V}{V'} = \frac{I_1}{I} = \frac{R_2}{R_2 + R} \,. \tag{360}$$

This result is independent of where along the chain we pick the adjacent nodes, so the voltages across successive nodes decrease in a geometric series.

If we want V/V' = 1/2, we must have $R = R_2$. Equation (359) then gives

$$2R_2 = R_1 + \sqrt{R_1^2 + 4R_1R_2} \implies (2R_2 - R_1)^2 = R_1^2 + 4R_1R_2$$

$$\implies 4R_2^2 = 8R_1R_2 \implies R_2 = 2R_1.$$
 (361)

From Eqs. (359) and (360), we see that if we want $V/V' \approx 1$ (that is, the voltage hardly decreases), then we need $R \ll R_2$, which implies $R_1 \ll R_2$. On the other hand, if we want $V/V' \ll 1$ (that is, the voltage decreases quickly), then we need $R \gg R_2$, which implies $R_1 \gg R_2$. These results make intuitive sense.

To terminate the ladder after any section, without changing its resistance from that of the infinite chain, we can simply connect a single resistor R given by Eq. (359) in parallel with the last R_2 , because this R mimics the rest of the infinite chain. For example, after one step of the ladder, we would have the setup in Fig. 94.

109 (a)

4.37. Some golden ratios

(a) For simplicity, let's set R = 1. Let the desired resistance be r. Following the strategy from Exercise 4.36, the circuit to the right of C and D in Fig. 96(a) is identical to the original infinite chain. So the infinite chain consists of a resistance of 1 connected in series with the parallel combination of 1 and r. The net result is r, so

$$1 + \frac{1 \cdot r}{1 + r} = r \implies \frac{1 + 2r}{1 + r} = r \implies r^2 - r - 1 = 0$$
$$\implies r = \frac{1 + \sqrt{5}}{2} = 1.618 \equiv r_1. \tag{362}$$

This number is the golden ratio.

(b) Again set R = 1, and let the desired resistance be r. The circuit to the right of C and D in Fig. 96(b) is identical to the original infinite chain. So the infinite chain consists of a resistance of 1 connected in parallel with the series combination of 1 and r. The net result is r, so

$$\frac{1 \cdot (1+r)}{1+(1+r)} = r \implies r^2 + r - 1 = 0$$

$$\implies r = \frac{-1+\sqrt{5}}{2} = 0.618 \equiv r_2.$$
(363)

This number is the inverse of the golden ratio. You should convince yourself why this setup is actually the same as the setup in Problem 4.7(b).

As a double check, the only difference between the two given circuits is the extra vertical resistor connecting A to B in the second circuit. So r_2 should equal the parallel combination of 1 and r_1 . And indeed,

$$\frac{1 \cdot r_1}{1 + r_1} = r_2, \tag{364}$$

as you can verify. This works out due to the various properties of the golden ratio.

4.38. Two light bulbs

- (a) The power dissipated takes the form of V^2/R . Both bulbs have the same voltage drop V, so if Bulb 1 is twice as bright as Bulb 2, it must have half the R. Bulb 2's resistance is therefore larger by a factor of 2. (The larger resistor is dimmer.)
- (b) The power dissipated also takes the form of I^2R . Both bulbs now have the same current I, so if Bulb 2 has twice the resistance, as we found in part (a), then it is twice as bright the opposite of the case in part (a). (The larger resistor is brighter.) Note that in part (a) we used the expression $P = V^2/R$ because both bulbs (in parallel) had the same V, whereas now we are using the expression $P = I^2R$ because both bulbs (in series) have the same I.

We can also compare the total power dissipated in each case. If the resistances are R and 2R, then in part (a) the total power dissipated is $V^2/R+V^2/2R = 3V^2/2R$. In part (b) the total power is $I^2R + I^2(2R) = 3I^2R$, where I = V/3R. So the power is $V^2/3R$. This is 2/9 of the power in part (a). In units of V^2/R , the powers in part (a) are 1 and 1/2, while in part (b) they are 1/9 and 2/9.



Figure 96

4.39. Maximum power

The R_i and R resistors are in series, so the current in the circuit is $I = \mathcal{E}/(R + R_i)$. The power dissipated in the R resistor is therefore $P = I^2 R = \mathcal{E}^2 R/(R + R_i)^2$. Taking the derivative with respect to R and setting the result equal to zero gives

$$0 = \frac{(R+R_{\rm i})^2 \cdot 1 - R \cdot 2(R+R_{\rm i})}{(R+R_{\rm i})^4} = \frac{R_{\rm i} - R}{(R+R_{\rm i})^3} \implies R = R_{\rm i}.$$
 (365)

This is indeed a maximum, because dP/dR > 0 for $R < R_i$, and dP/dR < 0 for $R > R_i$. Equivalently, the second derivative is negative at $R = R_i$, as you can check.

It makes sense that a maximum exists for some finite value of R, because P = 0 both at R = 0 (because $P = I^2 R$, with I finite and R zero) and at $R = \infty$ (because $P = V^2/R$, with V finite and R infinite).

Consider a different question, "Given a fixed external resistance R, what value of the internal resistance R_i yields the maximum power delivered to the external resistor R?" In view of the above expression for the power, the answer is simply $R_i = 0$. This makes sense; we want the largest possible current passing through the given external resistor.

4.40. Minimum power dissipation

The power dissipated in the two resistors is

$$P = I_1^2 R_1 + I_2^2 R_2 = I_1^2 R_1 + (I_0 - I_1)^2 R_2 = I_1^2 (R_1 + R_2) - 2I_0 R_2 I_1 + R_2 I_0^2.$$
 (366)

Minimizing P by taking the derivative with respect to I_1 gives

$$0 = \frac{dP}{dI_1} = 2I_1(R_1 + R_2) - 2I_0R_2 \implies I_1 = \frac{I_0R_2}{R_1 + R_2}, \qquad (367)$$

which is also what we obtain from Ohm's law (that is, equating the voltage drops I_1R_1 and I_2R_2 , and using $I_1 + I_2 = I_0$). Note that P is indeed a minimum, and not a maximum, because dP/dI_1 is less than (or greater than) 0 if I_1 is less than (or greater than) $I_0R_2/(R_1 + R_2)$. Equivalently, the second derivative of P equals $2(R_1 + R_2)$, which is positive.

Alternatively, we can set the differential of $P = I_1^2 R_1 + I_2^2 R_2$ equal to zero, which gives $dP = 2R_1I_1 dI_1 + 2R_2I_2 dI_2 = 0$. But $I_1 + I_2 = I_0$ tells us that $dI_2 = -dI_1$, so we obtain $R_1I_1 = R_2I_2$. This is simply the statement of equal voltage drops, as given by Ohm's law. Combining this with $I_1 + I_2 = I_0$ yields the above value of I_1 .

4.41. **D-cell**

(a) The total charge produced by 0.1 A flowing for 30 hours is $(0.1 \text{ C/s})(30 \cdot 3600 \text{ s}) = 10,800 \text{ C}$. This charge passes through a potential difference of 1.5 V, so the total energy output is E = (10,800 C)(1.5 J/C) = 16,200 J. Since the mass of the battery is 0.09 kg, the energy storage in J/kg is $(16,200 \text{ J})/(0.09 \text{ kg}) = 1.8 \cdot 10^5 \text{ J/kg}$.

In the example in Section 4.9, the 10 kg battery had an energy output of $8.6 \cdot 10^5$ J, which implies $8.6 \cdot 10^4$ J/kg. This is about half as much as the D cell.

(b) Lifting a 70 kg person 1 m requires an energy of $mgh = (70 \text{ kg})(9.8 \text{ m/s}^2)(1 \text{ m}) \approx 700 \text{ J}$. The D cell with 50% efficiency can supply 8,100 J. This corresponds to about 11.5 m.

4.42. Making an ohmmeter

If adding 15Ω to the circuit between the leads cuts the current through the ammeter in half (compared with the R = 0 case where the leads are connected together), then the current I in the external (to the ammeter) circuit must be in order of magnitude $(1.5 V)/(15 \Omega) = 0.1 A$. This is *much* larger than the maximum current $50 \,\mu A = 5 \cdot 10^{-5} A$ in the ammeter. This implies that R_1 must be much smaller than the resistance $R_a = 20 \,\Omega$ of the ammeter's coil, so that nearly all of the current I is shunted through R_1 .

Said in a different way, assume that R_1 is not much smaller than R_a . Then with the leads connected together, the current through R_1 will be no larger (in order of magnitude) than the current through R_a (that is, $5 \cdot 10^{-5}$ A), because R_1 and R_a are in parallel. But by looking at the bottom loop in the circuit, this means that R_2 must be very large, roughly $(1.5 \text{ V})/(5 \cdot 10^{-5} \text{ A}) = 3 \cdot 10^4 \Omega$ in order of magnitude. This implies that the insertion of an $R = 15 \Omega$ resistor in series with R_2 will hardly change the current in the circuit. In particular, there is no way it can cut the current in half. Our initial assumption (that R_1 is not much smaller than R_a) must therefore have been incorrect.

Having shown that $R_1 \ll R_a$, we see that the resistance of the whole circuit is essentially equal to $R_2 + R$. Therefore, if I is to be half as large for $R = 15 \Omega$ as for R = 0, then we must have $R_2 = 15 \Omega$.

This $R_2 = 15 \Omega$ result then implies that the current I shown in Fig. 4.53 is 0.1 A when R = 0. Now, the fraction of I that goes through the ammeter's coil is given by $I_{\rm a}/I = R_1/(R_1 + R_{\rm a}) \approx R_1/R_{\rm a}$. The stated conditions tell us that I = 0.1 A (which corresponds to R = 0) must yield $I_{\rm a} = 5 \cdot 10^{-5}$ A. So $I_{\rm a}/I = R_1/R_{\rm a}$ gives $(5 \cdot 10^{-5} \text{ A})/(0.1 \text{ A}) = R_1/(20 \Omega) \Longrightarrow R_1 = 0.01 \Omega$.

If $R = 5 \Omega$, all currents are reduced by a factor of essentially 15/(15+5) = 3/4, compared with the R = 0 case. So we have 3/4 of full deflection, which corresponds to the $37.5 \,\mu$ A mark. If $R = 50 \,\Omega$, we have 15/(15+50) = 23% of full deflection, which corresponds to the $11.5 \,\mu$ A mark.

We made two approximations above, namely $R_1 \ll R_2$ and $R_1 \ll R_a$. If you want to solve the problem exactly, you can show that R_2 should be 14.99 Ω instead of 15 Ω . And R_1 should be $(0.1 \Omega)/(1 - 0.0005) \approx 0.010005 \Omega$ instead of 0.01Ω . Neither of these refinements would ordinarily be necessary.

4.43. Using symmetry

The symmetry is evident in Fig. 97. The first missing term in the numerator must be $R_1R_3R_4$ (the mirror-image of the $R_1R_2R_3$ term). The second missing term in the numerator must be R_1R_4 (the mirror-image of the R_2R_3 term). The missing term in the denominator must be R_2R_3 (the mirror-image of the R_1R_4 term). So we have

$$R_{\rm eq} = \frac{R_1 R_2 R_3 + R_1 R_2 R_4 + R_1 R_3 R_4 + R_2 R_3 R_4 + R_5 (R_1 R_3 + R_2 R_3 + R_1 R_4 + R_2 R_4)}{R_1 R_2 + R_1 R_4 + R_2 R_3 + R_3 R_4 + R_5 (R_1 + R_2 + R_3 + R_4)}.$$
(368)

(a) If $R_5 = 0$ we equivalently have the circuit shown in Fig. 98(a), where we have collapsed R_5 to a point since it is short circuited. The parallel combination of R_1 and R_3 is in series with the parallel combination of R_2 and R_4 . So the resistance is

$$\frac{R_1R_3}{R_1+R_3} + \frac{R_2R_4}{R_2+R_4} = \frac{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4 + R_2R_3R_4}{R_1R_2 + R_1R_4 + R_2R_3 + R_3R_4}, \quad (369)$$







Figure 98

which agrees with the formula in Eq. (368) when $R_5 = 0$.

(b) If $R_5 = \infty$ we equivalently have the circuit shown in Fig. 98(b). The series combination of R_1 and R_2 is in parallel with the series combination of R_3 and R_4 . So the resistance is

$$\frac{(R_1 + R_2)(R_3 + R_4)}{R_1 + R_2 + R_3 + R_4},\tag{370}$$

which agrees with the formula when $R_5 = \infty$.

- (c) If $R_1 = R_3 = 0$ we equivalently have the circuit shown in Fig. 98(c). The R_5 resistor doesn't matter, since it is short circuited via R_1 and R_3 . So we just have R_2 and R_4 in parallel, and the resistance is $R_2R_4/(R_2 + R_4)$. This agrees with the formula when all terms containing R_1 or R_3 are dropped.
- (d) If R_1 through R_4 have the common value R, then by symmetry no current flows through R_5 , so we can collapse that resistor to a point. We then have the circuit shown in Fig. 98(d). R_1 and R_3 are in parallel, so the effective resistance across them is R/2. And likewise for R_2 and R_4 . So the total resistance is R. This agrees with the formula; the numerator is $4R^3 + 4R_5R^2$, and the denominator is $4R^2 + 4R_5R$, so the whole fraction equals R.

More generally, if $R_1 = R_3 \equiv a$ and $R_2 = R_4 \equiv b$, then no current flows through R_5 , so we again have two sets of parallel resistors. You should check that the formula agrees with what you calculate directly. Likewise for $R_1 = R_2 \equiv a$ and $R_3 = R_4 \equiv b$.

4.44. Using the loop equations

The three loop equations are

$$\mathcal{E} - (I_3 - I_1)R_3 - (I_3 - I_2)R_4 = 0,$$

$$-I_1R_1 - (I_1 - I_2)R_5 - (I_1 - I_3)R_3 = 0,$$

$$-I_2R_2 - (I_2 - I_3)R_4 - (I_2 - I_1)R_5 = 0.$$
(371)

Solving these equations via *Mathematica* (for the unknowns I_1 , I_2 , I_3) yields $I_3 = \mathcal{E}/R_{\rm eq}$, where $R_{\rm eq}$ is given in Eq. (4.48). The equivalent resistance, \mathcal{E}/I_3 , is then $R_{\rm eq}$, as desired.

4.45. Battery/resistor loop

If the circuit is open, we can ignore the leads to A and B, so we just have the given loop. The current around this loop is $(3 \cdot 1.5 \text{ V})/(5 \cdot 100 \Omega) = 9 \cdot 10^{-3} \text{ A}$. The opencircuit voltage is then (using the upper part of the loop, with two cells and three resistors)

$$V_B - V_A = 2(1.5 \,\mathrm{V}) - 3(9 \cdot 10^{-3} \,\mathrm{A})(100 \,\Omega) = 0.3 \,\mathrm{V}.$$
 (372)

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The lower part of the loop gives the same result: $V_A - V_B = 1.5 \text{ V} - 2(9 \cdot 10^{-3} \text{ A})(100 \Omega) = -0.3 \text{ V}.$

To find the short-circuit current, we can draw the circuit in the manner shown in Fig. 99. We quickly find that the two loop currents are $I_1 = (3 \text{ V})/(300 \Omega) = 0.01 \text{ A}$ counterclockwise, and $I_2 = (1.5 \text{ V})/(200 \Omega) = 0.0075 \text{ A}$ counterclockwise. The net short-circuit current from B to A is therefore $I_{\text{sc}} = 0.0025 \text{ A}$. (You should verify that you obtain the same result if you simply connect A and B in Fig. 4.57 and use the two resulting loops.)

In the Thevenin equivalent circuit, \mathcal{E}_{eq} is simply the open-circuit voltage, 0.3 V. And R_{eq} is given by

$$R_{\rm eq} = \frac{\mathcal{E}_{\rm eq}}{I_{\rm sc}} = \frac{0.3\,\rm V}{0.0025\,\rm A} = 120\,\Omega. \tag{373}$$

As a double check, we can calculate R_{eq} by setting all the emf's equal to zero and finding the equivalent resistance of the resulting circuit. A and B are connected by the parallel combination of 200Ω and 300Ω , which quickly yields 120Ω , as desired.

4.46. Maximum power via Thevenin

The open circuit (with the bottom resistor R absent) has current flowing only in the top loop. This current is $(120 \text{ V})/(10 \Omega + 10 \Omega) = 6 \text{ A}$. The open-circuit voltage, which equals \mathcal{E}_{eq} , is the same as the voltage across the lower of the 10 Ω resistors (because no current flows in the 15 Ω resistor when the circuit is open), so we have $\mathcal{E}_{eq} = (6 \text{ A})(10 \Omega) = 60 \text{ V}$.

The resistance $R_{\rm eq}$ can be found by ignoring (that is, shorting) the emf. As far as A and B are concerned, we then have a circuit with 15Ω in series with the parallel combination of 10Ω and 10Ω . So $R_{\rm eq} = 15 \Omega + 5 \Omega = 20 \Omega$.

Alternatively, we can find $R_{\rm eq}$ by finding the short-circuit current $I_{\rm sc}$, and then using $R_{\rm eq} = \mathcal{E}_{\rm eq}/I_{\rm sc}$. To find $I_{\rm sc}$, note that the short-circuit setup consists of 10 Ω in series with the parallel combination of 10 Ω and 15 Ω (which is equivalent to 6 Ω). The whole circuit therefore has a resistance of 16 Ω . The total current is then $(120 \text{ V})/(16 \Omega) = 7.5 \text{ A}$. A fraction 10/(10 + 15) = 2/5 of this goes through the bottom branch of the circuit. So $I_{\rm sc} = (2/5)(7.5 \text{ A}) = 3 \text{ A}$. Hence $R_{\rm eq} = \mathcal{E}_{\rm eq}/I_{\rm sc} = (60 \text{ V})/(3 \text{ A}) = 20 \Omega$, as above. You can also find $I_{\rm sc}$ by using Kirchhoff's rules with two loops.

From Exercise 4.39, the power dissipated in R will be greatest when $R = R_{eq} = 20 \Omega$. The current through R is then $I = \mathcal{E}_{eq}/(R_{eq} + R) = (60 \text{ V})/(40 \Omega) = 1.5 \text{ A}$. So the power dissipated in R is $P = I^2 R = (1.5 \text{ A})^2 (20 \Omega) = 45 \text{ J/s}$.

Alternatively, we can solve the problem from scratch. A two-loop exercise (which is quicker than using the series/parallel rules) shows that for a general value of R, the current through R is 60/(20 + R) (ignoring the units). So we want to maximize $I^2R \propto R/(20 + R)^2$. You can quickly show that this is maximized when R = 20.

4.47. Discharging a capacitor

From Eq. (4.44) the current is $I(t) = (V_0/R)e^{-t/RC}$. The power dissipated in the resistor is $P = I^2 R = (V_0^2/R)e^{-2t/RC}$, so the total energy dissipated is

$$E = \int_0^\infty P \, dt = \int_0^\infty \frac{V_0^2}{R} e^{-2t/RC} \, dt = -\frac{V_0^2}{R} \frac{RC}{2} e^{-2t/RC} \Big|_0^\infty = \frac{1}{2} C V_0^2, \tag{374}$$

which is the initial energy in the capacitor, as desired.

Suppose we have a 1 microfarad capacitor charged to 100 volts. The initial charge is $Q = CV_0 = (10^{-6} \text{ F})(100 \text{ V}) = 10^{-4} \text{ C}$. From Eq. (4.43) the charge decreases



Figure 99

according to $Q(t) = Q_0 e^{-t/t_0}$, where $t_0 = RC$ is the time constant. Since the charge of an electron is $1.6 \cdot 10^{-19}$ C, we will have roughly one electron left when $1.6 \cdot 10^{-19}$ C = $(10^{-4} \text{ C})e^{-t/t_0} \Longrightarrow t = -t_0 \ln(1.6 \cdot 10^{-15}) = 34t_0$. So if the time constant were, say, 1 second (which would mean $R = 10^6 \Omega$ here), we would be down to roughly one electron in a little over half a minute. For a $1 \text{ k}\Omega$ resistor, the time would be 0.034 s.

4.48. Charging a capacitor

The total work done by the battery is $Q_{\rm f}\mathcal{E}$, where $Q_{\rm f}$ is the final charge on the capacitor. This is true because the battery transfers a charge $Q_{\rm f}$ through a constant potential difference of \mathcal{E} .

The final energy of the capacitor is $Q_{\rm f}\phi/2 = Q_{\rm f}\mathcal{E}/2$, because the final potential ϕ across the capacitor equals the voltage \mathcal{E} across the battery. (There is no current flowing after a long time, so there is no voltage drop across the resistor.)

The energy dissipated in the resistor is the integral of the power, that is, $\int RI^2 dt$. From the solution to Problem 4.17, we have $I(t) = (\mathcal{E}/R)e^{-t/RC}$. Therefore,

$$\int_{0}^{\infty} RI^{2} dt = R \frac{\mathcal{E}^{2}}{R^{2}} \int_{0}^{\infty} e^{-2t/RC} dt = -\frac{\mathcal{E}^{2}}{R} \frac{RC}{2} e^{-2t/RC} \Big|_{0}^{\infty} = \frac{\mathcal{E}^{2}}{R} \frac{RC}{2} = \frac{C\mathcal{E}^{2}}{2} = \frac{Q_{f}\mathcal{E}}{2},$$
(375)

where we have used $Q_{\rm f} = C\mathcal{E}$. The conservation-of-energy statement is then

$$W_{\text{battery}} = U_{\text{capacitor}} + E_{\text{resistor}} \implies Q_{\text{f}}\mathcal{E} = \frac{Q_{\text{f}}\mathcal{E}}{2} + \frac{Q_{\text{f}}\mathcal{E}}{2},$$
 (376)

which is indeed true.

REMARK: It is also possible to use the general formulas for Q(t) and I(t) from Problem 4.17 to show that energy is conserved at all times (not just $t \to \infty$), as we know it must be. But we can show this in a quicker manner by demonstrating that the conservation-of-energy statement is equivalent to the Kirchhoff loop equation, $\mathcal{E} - Q/C - RI = 0$. We can do this either by differentiating the former to obtain the latter, or by integrating the latter to obtain the former. Let's take the first of these routes. The conservation-of-energy statement at any intermediate time is

$$\mathcal{E}Q(t) = \frac{Q(t)^2}{2C} + \int_0^t RI^2 \, dt.$$
(377)

Differentiating with respect to t gives (using dQ/dt = I and canceling a factor of I)

$$\mathcal{E}\frac{dQ}{dt} = \frac{Q\,dQ}{C\,dt} + RI^2 \implies \mathcal{E} = \frac{Q}{C} + RI, \tag{378}$$

which is the Kirchhoff loop equation, as desired. If you want to go in the reverse direction, just multiply by I and then integrate with respect to t (using the fact that Q = 0 at t = 0).

4.49. Displacing the electron cloud

The charge density on each of the sheets is $\sigma = \epsilon_0 E$. The effective number of electrons per unit area on the sheets is Nd, where d is the displacement of the electron cloud. The field will be neutralized if $(Nd)e = \sigma$. Hence,

$$(Nd)e = \epsilon_0 E \implies d = \frac{\epsilon_0 E}{Ne} = \frac{\left(8.85 \cdot 10^{-12} \frac{s^2 C^2}{\text{kg m}^3}\right)(3 \cdot 10^4 \text{ V/m})}{(10^{21} \text{ m}^{-3})(1.6 \cdot 10^{-19} \text{ C})} = 1.7 \cdot 10^{-9} \text{ m.}$$
(379)

Chapter 5

The fields of moving charges

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

5.10. Capacitor plates in two frames

The electric field in the lab frame is $E_0 = V_0/d = (300 \text{ V})/(.02 \text{ m}) = 15,000 \text{ V/m}$. The charge density is $\sigma_0 = \epsilon_0 E_0$, so the charge on the plates is $Q_0 = \sigma_0 A_0 = \epsilon_0 E_0 A_0$. The number of excess electrons on the negative plate in therefore $N = Q_0/e = \epsilon_0 E_0 A_0/e$, which yields

$$N = \frac{\left(8.85 \cdot 10^{-12} \, \frac{\mathrm{s}^2 \, \mathrm{C}^2}{\mathrm{kg \, m^3}}\right) (15,000 \, \mathrm{V/m}) (0.02 \, \mathrm{m^2})}{1.6 \cdot 10^{-19} \, \mathrm{C}} = 1.66 \cdot 10^{10}. \tag{380}$$

In the frame F_1 moving east at v = 0.6c, the plates are moving west at 0.6c. The γ factor associated with 0.6c is 5/4. So the EW dimension is shrunk to $(20 \text{ cm})/\gamma = 16 \text{ cm}$. The NS dimension is unchanged, as is the vertical separation. The number of electrons on the negative plate is the same. But the area of the plates is smaller by $1/\gamma$, so σ_1 (and hence E_1) is larger by a factor γ . Therefore, $E_1 = \gamma E_0 = 18,750 \text{ V/m}$. In the frame F_2 moving upward at v = 0.6c, the plates are moving downward at 0.6c. The plate dimensions are still 20 cm and 10 cm, but the vertical separation is shrunk to $(2 \text{ cm})/\gamma = 1.6 \text{ cm}$. The number of electrons is the same. And the density is the same also, so $E_2 = E_0 = 15,000 \text{ V/m}$.

5.11. Electron beam

(a) $0.05 \,\mu\text{A}$ equals $5 \cdot 10^{-8} \,\text{C/s}$, so the number of electrons passing a given point per second is $n = (5 \cdot 10^{-8} \,\text{C/s})/(1.6 \cdot 10^{-19} \,\text{C}) = 3.1 \cdot 10^{11} \,\text{s}^{-1}$. They move at essentially speed c, so the average distance between them is $d = c/n = (3 \cdot 10^8 \,\text{m/s})/(3.1 \cdot 10^{11} \,\text{s}^{-1}) \approx 0.001 \,\text{m} = 1 \,\text{mm}$. The linear charge density is then $\lambda = e/d = (1.6 \cdot 10^{-19} \,\text{C})/(0.001 \,\text{m}) = 1.6 \cdot 10^{-16} \,\text{C/m}$. The electric field 1 cm from the beam is therefore

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{1.6 \cdot 10^{-16} \,\mathrm{C/m}}{2\pi \left(8.85 \cdot 10^{-12} \,\frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg \,m^3}}\right) (0.01 \,\mathrm{m})} = 2.88 \cdot 10^{-4} \,\mathrm{V/m}. \tag{381}$$

Note that the distance between the electrons (1 mm) is small compared with the distance from the beam (1 cm), so the beam looks roughly like a uniform distribution of charge.

(b) In the electron rest frame, the electrons are "uncontracted" compared with the lab frame, so their average separation is $d' = \gamma d = (20)(0.001 \text{ m}) = 0.02 \text{ m} = 2 \text{ cm}$. The linear charge density, and hence electric field, is therefore decreased by a factor $1/\gamma$. So the field 1 cm from the beam is

$$E' = \frac{E}{\gamma} = \frac{2.88 \cdot 10^{-4} \,\mathrm{V/m}}{20} = 1.44 \cdot 10^{-5} \,\mathrm{V/m}.$$
 (382)

This is the *average* (along a line parallel to the beam) of the radial component of the field. Because the electrons are relatively far apart (2 cm, compared with the 1 cm distance from the beam), there is a large variation in the field as the position varies along the line parallel to the beam.

5.12. Tilted sheet

In the notation of the example in Section 5.5, the distance between A' and B' is $\ell\sqrt{1+(1/\gamma)^2}$. So the same charge-invariance argument gives

$$\sigma'\sqrt{1+(1/\gamma)^2} = \sigma\sqrt{2} \implies \sigma' = \frac{\sqrt{2}\gamma}{\sqrt{1+\gamma^2}}\sigma.$$
(383)

The factor here correctly equals 1.1043 if $\gamma = 5/4$ (that is, if v = 3c/5).

The same reasoning also gives the magnitude of the field in F' as $E' = (E/\sqrt{2})\sqrt{1 + \gamma^2}$. To find the normal component, E'_n , we must multiply this by $\cos(2\theta - 90^\circ)$, where $\tan \theta = \gamma$. This trig factor can alternatively be written as $\sin 2\theta$, which in turn can be written as $2\sin\theta\cos\theta$. So we have

$$E'_{n} = E' \cos(2\theta - 90^{\circ}) = \frac{E}{\sqrt{2}} \sqrt{1 + \gamma^{2}} \cdot 2\sin\theta\cos\theta$$
$$= \frac{E}{\sqrt{2}} \sqrt{1 + \gamma^{2}} \cdot 2\frac{\gamma}{\sqrt{1 + \gamma^{2}}} \frac{1}{\sqrt{1 + \gamma^{2}}} = \frac{\sqrt{2}\gamma}{\sqrt{1 + \gamma^{2}}} E.$$
(384)

This is the same factor that relates σ' to σ . So if $E = \sigma/2\epsilon_0$ is true (which it is), then $E'_n = \sigma'/2\epsilon_0$ is also true. That is, Gauss's law holds in F'.

Limits: If $\gamma = 1$ (that is, v = 0), we obtain $\sigma' = \sigma$ and $E'_n = E$, as expected. And if $\gamma \to \infty$ (that is, $v \to c$), we obtain $\sigma' = \sqrt{2}\sigma$ and $E'_n = \sqrt{2}E$; the sheet is vertical, so a given amount of charge in F' is now located within a span that is $1/\sqrt{2}$ times as large as it was in F.

5.13. Adding the fields

All fields in this problem involve the factor $e/4\pi\epsilon_0 a^2$, so let's ignore that for now. At the point P = (a, 0, 0), the field from the proton points down to the right in Fig. 100 with magnitude $(1/\sqrt{2})^2$. The components of this are $E_x^{\rm p} = 1/(2\sqrt{2})$ and $E_z^{\rm p} = -1/(2\sqrt{2})$.

We'll assume that the negative muon is moving in the positive x direction, although this doesn't affect the answer; the field is the same in front or behind. The field directly in front of the muon is obtained by setting $\theta = 0$ in Eq. (5.15). After stripping off the $e/4\pi\epsilon_0 a^2$ we have $E_x^{\mu} = -(1 - \beta^2) = -0.36$.

Adding the fields from the two particles gives a total field with $E_x = -0.00645$ and $E_z = -0.354$ (times $e/4\pi\epsilon_0 a^2$). Note that E_x is very close to zero. A β value of about 0.80402 would make E_x exactly equal to zero. By continuity, such a speed must exist,



Figure 100

because if $\beta = 0$ then the muon's field dominates and the net E_x is negative, whereas if $\beta \to 1$ then the x component of the muon's field is zero, so the net E_x (due only to the proton) is positive.

5.14. Forgetting relativity

Assuming $\beta \neq 0$, the factor $(1 - \beta^2)/(1 - \beta^2 \sin^2 \theta)^{3/2}$ in Eq. (5.15) is smaller than 1 when $\theta = 0$ and larger than 1 when $\theta = 90^{\circ}$. So there must be an angle in between where the factor equals 1. This occurs when

$$(1 - \beta^2 \sin^2 \theta) = (1 - \beta^2)^{2/3} \implies \sin^2 \theta = \frac{1 - (1 - \beta^2)^{2/3}}{\beta^2}.$$
 (385)

If $\beta \approx 1$ then $\sin^2 \theta \approx (1-0)/1 = 1$. So $\theta \approx 90^\circ$. (The field drops off very quickly from its maximum value at $\theta = 90^\circ$.) If $\beta \approx 0$ then we need to use the Taylor series, $(1-\epsilon)^n \approx 1-n\epsilon$. This gives $\sin^2 \theta \approx (1-(1-2\beta^2/3))/\beta^2 = 2/3$. So $\sin \theta = \sqrt{2/3} \Longrightarrow \theta = 54.7^\circ$. Interestingly, this is the same result as for the $\beta \approx 0$ limit in Problem 5.2.

5.15. Gauss's law for a moving charge

Consider the sphere in Fig. 5.14, where the charge moves in the x direction (we'll drop the primes on the coordinates). The angle θ in Eq. (5.15) is measured with respect to the x axis. If we slice the sphere into circular strips with constant θ , the radius of a strip at angle θ is $2\pi r \sin \theta$. So if the strip subtends an angle $d\theta$, its area is $dA = (2\pi r \sin \theta)(r d\theta)$. The field points radially, so the total flux through the sphere is

$$\int E \, da = \int_0^\pi \frac{q}{4\pi\epsilon_0 r^2} \frac{1-\beta^2}{(1-\beta^2 \sin^2 \theta)^{3/2}} \, 2\pi r^2 \sin \theta \, d\theta$$
$$= \frac{q(1-\beta^2)}{2\epsilon_0} \int_0^\pi \frac{\sin \theta \, d\theta}{(1-\beta^2 \sin^2 \theta)^{3/2}} \,. \tag{386}$$

Using the integral table in Appendix K, this becomes

$$\frac{q(1-\beta^2)}{2\epsilon_0} \cdot \frac{-\cos\theta}{(1-\beta^2)\sqrt{1-\beta^2\sin^2\theta}} \Big|_0^\pi = \frac{q(1-\beta^2)}{2\epsilon_0} \cdot \frac{2}{(1-\beta^2)} = \frac{q}{\epsilon_0}, \quad (387)$$

as desired.

Note: having shown that Gauss's law holds for a sphere, the same reasoning as in Section 1.10 can be used to show that it holds for any shape. The critical property of the field is the $1/r^2$ dependence on r.

5.16. Cosmic rays

The maximum value of the field in Eq. (5.15) is achieved when $\theta' = 90^{\circ}$, in which case the value is $(Q/4\pi\epsilon_0 r'^2)/\sqrt{1-\beta^2} = \gamma Q/4\pi\epsilon_0 r'^2$. Therefore,

$$E_{\text{max}} = \frac{1}{4\pi\epsilon_0} \frac{\gamma e}{r'^2}$$

$$\implies r'^2 = \frac{1}{4\pi\epsilon_0} \frac{\gamma e}{E} = \left(9 \cdot 10^9 \,\frac{\text{kg m}^3}{\text{s}^2 \,\text{C}^2}\right) \frac{(10^{10})(1.6 \cdot 10^{-19} \,\text{C})}{1 \,\text{V/m}} = 14.4 \,\text{m}^2, \,(388)$$

so $r=3.8\,\mathrm{m}$. The thickness of the pancake at this distance is $r\,\Delta\theta\approx r/\gamma\approx (4\,\mathrm{m})/10^{10}=4\cdot10^{-10}\,\mathrm{m}$. Very thin!



Figure 101

5.17. Reversing the motion

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Let t = 0 be the time of the collision at the origin. Outside a sphere of radius ct centered at the origin, the field at time t is that of a uniformly moving proton that would have been located at x = -(c/2)t. See Fig. 101; we have drawn just the top half of the picture. Inside a sphere of radius ct, the field at time t is that of the actual uniformly moving proton that is located at x = (c/2)t. At $t = 10^{-10}$ s, the radius of the sphere is $ct = (3 \cdot 10^8 \text{ m/s})(10^{-10} \text{ s}) = 0.03 \text{ m}$, or 3 cm.

Note that the interior and exterior field lines are indeed connected in the manner shown, with lines of equal slope being connected by a (nearly) tangential line. This follows from the fact that in Fig. 5.19 the flux through the spherical cap FD equals the flux through the spherical cap EA.

5.18. A nonuniformly moving electron

- (a) The electron had been traveling in the positive direction along the negative x axis toward the origin, where it rather suddenly stopped. Since the speed of light is $c = 3 \cdot 10^{10}$ cm/s, and since the "transition" sphere surrounding the static E field has a radius of about r = 15 cm, the stopping must have taken place at $t = -r/c = -5 \cdot 10^{-10}$ s. (From the figure, it looks like the stopping commenced at this time, and then lasted for perhaps $0.5 \cdot 10^{-10}$ s.) If the electron hadn't stopped, it would be located at x = 12 cm, because that is where the external field lines point. So its speed was (12/15)c = (0.8)c.
- (b) At $t = -7.5 \cdot 10^{-10}$ s, which was $\Delta t = 2.5 \cdot 10^{-10}$ s before the electron stopped, it was located at $x = -v \Delta t = -(0.8)(3 \cdot 10^{10} \text{ m/s})(2.5 \cdot 10^{-10} \text{ s}) = -6 \text{ cm}$, with respect to the origin.
- (c) To find the field strength at the origin when the electron was at x = -6 cm, we can use Eq. (5.15) with $\theta' = 0$. This gives

$$E = \frac{1}{4\pi\epsilon_0} \frac{e(1-\beta^2)}{r^2} = \left(9 \cdot 10^9 \,\frac{\text{kg m}^3}{\text{s}^2 \,\text{C}^2}\right) \frac{(1.6 \cdot 10^{-19} \,\text{C})(1-0.8^2)}{(0.06 \,\text{m})^2} = 1.44 \cdot 10^{-7} \,\text{V/m.}$$
(389)

5.19. Colliding particles

No charge remains after the collision, so the field is zero within a sphere of radius ct centered at the origin. Outside this sphere, the field is what the field would be if the two charges had kept moving. The positive charge would be on the right, at position x = vt, and the negative charge would be on the left, at position x = -vt. The field lines can't end, so the incoming lines on the left side must connect (via lines along the surface of the sphere with radius ct) with the outgoing lines on the right side; see Fig. 102. The thickness of the shell containing the connecting lines is determined by the duration of the deceleration period. This thickness remains constant. This radiation is called Bremsstrahlung.

As the spherical shell expands, it turns out that the E field in the shell decreases like 1/r instead of the usual $1/r^2$. This is demonstrated in Appendix H, but we can also understand it in another more qualitative way. We know that the density of radial field lines falls off like $1/r^2$. Equivalently, if we paint a collection of dots on a balloon (which represent the intersection of the field lines with the balloon) and then expand the balloon, the density of dots falls off like $1/r^2$. However, if we paint a collection of lines on the balloon (which represent the tangential field lines in the present setup), then the density of these lines falls off only like 1/r. This is true because there is





now effectively only one dimension that is expanding (at least with regard to the spacing between the lines), namely the dimension lying on the surface of the sphere and perpendicular to the lines. (We have used the fact that the thickness of the shell remains constant.) Since the density of field lines is proportional to the field strength, we arrive at the desired result that the tangential field falls off like 1/r.

5.20. Relating the angles

The flux through the inner cap is (ignoring the $1/4\pi\epsilon_0$)

$$\int_{0}^{\theta_{0}} \frac{Q}{r^{2}} 2\pi r^{2} \sin \theta \, d\theta = -2\pi Q \cos \theta \Big|_{0}^{\theta_{0}} = 2\pi Q (1 - \cos \theta_{0}). \tag{390}$$

The flux through the outer cap is (ignoring the $1/4\pi\epsilon_0$)

$$\int_{0}^{\phi_{0}} \frac{Q}{r^{2}} \frac{1-\beta^{2}}{(1-\beta^{2}\sin^{2}\phi)^{3/2}} 2\pi r^{2} \sin\phi \, d\phi = 2\pi Q(1-\beta^{2}) \int_{0}^{\phi_{0}} \frac{\sin\phi \, d\phi}{(1-\beta^{2}+\beta^{2}\cos^{2}\phi)^{3/2}} \,. \tag{391}$$

With $x \equiv \cos \phi \implies dx = -\sin \phi \, d\phi$, this becomes (using the integral given in the exercise)

$$-\frac{2\pi Q(1-\beta^2)}{\beta^3} \int_1^{\cos\phi_0} \frac{dx}{\left(\frac{1-\beta^2}{\beta^2}+x^2\right)^{3/2}} = -\frac{2\pi Q(1-\beta^2)}{\beta^3} \frac{x}{\frac{1-\beta^2}{\beta^2} \left(\frac{1-\beta^2}{\beta^2}+x^2\right)^{1/2}} \Big|_1^{\cos\phi_0} \\ = \frac{2\pi Q}{\beta} \left(\beta - \frac{\beta\cos\phi_0}{(1-\beta^2+\beta^2\cos^2\phi_0)^{1/2}}\right) \\ = 2\pi Q \left(1 - \frac{\cos\phi_0}{(1-\beta^2\sin^2\phi_0)^{1/2}}\right).$$
(392)

Alternatively, we could have obtained this result in a quicker manner by applying Eq. (K.15) in Appendix K to the integral on the left-hand side of Eq. (391). Equating this flux with the flux in Eq. (390) gives

$$\cos \theta_0 = \frac{\cos \phi_0}{(1 - \beta^2 \sin^2 \phi_0)^{1/2}} \,. \tag{393}$$

Now let's show that this is equivalent to $\tan \phi_0 = \gamma \tan \theta_0$. If $\tan \theta_0 = \tan \phi_0 / \gamma$, then

$$\cos \theta_0 = \frac{1}{\left(1 + \tan^2 \theta_0\right)^{1/2}} = \frac{1}{\left(1 + \frac{\tan^2 \phi_0}{\gamma^2}\right)^{1/2}}$$
(394)
$$= \frac{1}{\left(1 + (1 - \beta^2) \frac{\sin^2 \phi_0}{\cos^2 \phi_0}\right)^{1/2}} = \frac{\cos \phi_0}{\left((\cos^2 \phi_0 + \sin^2 \phi_0) - \beta^2 \sin^2 \phi_0\right)^{1/2}},$$

which equals the righthand side of Eq. (393), as desired.

5.21. Half of the flux

From Eq. (392) in the solution to Exercise 5.20, the flux through a cap that subtends an angle θ is (bringing the $1/4\pi\epsilon_0$ back in)

$$\frac{Q}{2\epsilon_0} \left(1 - \frac{\cos\theta}{(1 - \beta^2 \sin^2\theta)^{1/2}} \right). \tag{395}$$

With $\theta \equiv \pi/2 - \delta$ this becomes

$$\frac{Q}{2\epsilon_0} \left(1 - \frac{\sin\delta}{(1 - \beta^2 \cos^2\delta)^{1/2}} \right). \tag{396}$$

If this flux equals $Q/4\epsilon_0$, then the mirror-image cap will also contain a flux of $Q/4\epsilon_0$, which will leave $Q/2\epsilon_0$ for the region between the cones, which is half of the total flux Q/ϵ_0 . So we want

$$\frac{\sin\delta}{(1-\beta^2\cos^2\delta)^{1/2}} = \frac{1}{2} \implies 4\sin^2\delta = 1-\beta^2(1-\sin^2\delta)$$
$$\implies \sin^2\delta = \frac{1-\beta^2}{4-\beta^2}.$$
(397)

For $\beta = 0$ we have $\sin \delta = 1/2 \Longrightarrow \delta = 30^{\circ}$, which is the correct result for a spherically symmetric field, as you can verify. (A cap with a half-angle of 60° has half the area of a hemisphere.) For $\beta \to 1$ (and $\gamma \to \infty$) we have $\sin^2 \delta \approx (1 - \beta^2)/3$, so $\delta \approx \sin \delta \approx 1/(\sqrt{3}\gamma)$. The angle between the two cones is then $2\delta \approx 2/(\sqrt{3}\gamma)$.

5.22. Electron in an oscilloscope

- (a) The kinetic energy of the electron is 250 keV. The total energy, including the rest energy, $mc^2 = 500$ keV, is therefore 750 keV. But the total energy is also given by $\gamma mc^2 = \gamma(500 \text{ keV})$. So $\gamma = 750/500 = 1.5$. The β factor is given by $\beta = \sqrt{1 1/\gamma^2} = \sqrt{5}/3 = 0.745$.
 - The momentum is $p_x = \gamma m(\beta c) = (1.5)(0.745)mc = (1.12)mc$.
 - The time spent between the plates is $t = (0.04 \text{ m})/(\beta c) = 1.79 \cdot 10^{-10} \text{ s}.$
 - The transverse force, which has the constant value of Ee, equals the rate of change of the transverse momentum. So $p_y = (Ee)t$. But the electric field is E = V/s, where s is the separation between the plates. So $p_y = Vet/s$. We could plug in the numbers, but since we want to write the result in units of mc anyway, it is easier to take the following route. We have $p_y/mc = Vet/smc$. Multiply by c/c to obtain

$$\frac{p_y}{mc} = \frac{eV}{mc^2} \frac{tc}{s} = \frac{6 \text{ keV}}{500 \text{ keV}} \cdot \frac{(1.79 \cdot 10^{-10} \text{ s})(3 \cdot 10^8 \text{ m/s})}{0.008 \text{ m}} = 0.0805.$$
 (398)

(Remember that an eV is the change in energy of an electron moving through a potential difference of 1 volt. So to be precise, "eV" should actually be written as "eV." That is, it is the charge of one electron, e, multiplied by 1 volt. And the V in Eq. (398) is 6 kV.)

• The transverse velocity at exit is given by

$$p_y = \gamma m v_y \implies v_y = \frac{p_y}{\gamma m} = \frac{(0.0805)mc}{\gamma m} = 1.61 \cdot 10^7 \,\mathrm{m/s.}$$
 (399)

- Since the transverse force (and hence transverse acceleration) is constant, the average transverse velocity is half of the v_y we just found, that is, $\overline{v}_y = 8.05 \cdot 10^6 \text{ m/s.}$ (v_y is small enough so that nonrelativistic kinematics works fine here.) The transverse distance traveled is then $y = \overline{v}_y t =$ $(8.05 \cdot 10^6 \text{ m/s})(1.79 \cdot 10^{-10} \text{ s}) = 1.44 \cdot 10^{-3} \text{ m}$, which is 1.44 mm.
- The angle of the trajectory at exit is (using $\tan \theta \approx \theta$ for small θ)

$$\theta = \frac{v_y}{v_x} = \frac{p_y}{p_x} = \frac{0.0805}{1.12} = 0.072 \text{ rad} = 4.1^\circ.$$
(400)

(b) In the frame in which the electron is initially at rest, the plates are moving to the left with speed $\beta c = (0.745)c$, and their length is $(0.04 \text{ m})/\gamma = 0.0267 \text{ m}$. (The other two dimensions are unchanged.) The plates are above and below the electron for a time $t' = (0.0267 \text{ m})/(0.745c) = 1.19 \cdot 10^{-10} \text{ s}$ (which can also be obtained from time dilation), during which time the electron is accelerated upward in the field $E' = \gamma E = \gamma V/s$. The upward momentum acquired is $E'et' = (\gamma E)e(t/\gamma) = Eet = (0.0805)mc$. This is the same as in the lab frame, which is consistent with the fact that transverse momenta are unaffected by a Lorentz transformation.

In this frame the electron is non-relativistic, to a good approximation, so the final v'_y is $v'_y \approx (0.0805)c = 2.42 \cdot 10^7 \,\mathrm{m/s.}$ (If you want, you can show that taking relativity into account would decrease the speed by only about 0.3%.) The average transverse velocity is then $\overline{v}'_y = v'_y/2 = 1.21 \cdot 10^7 \,\mathrm{m/s.}$ The total transverse distance is therefore $y' = \overline{v}'_y t' = (1.21 \cdot 10^7 \,\mathrm{m/s})(1.19 \cdot 10^{-10} \,\mathrm{s}) = 1.44 \cdot 10^{-3} \,\mathrm{m}$, in agreement with the result in the lab frame, which is consistent with the fact that transverse distances are unaffected by a Lorentz transformation. In short, the transverse distances are the same in the two frames because in the electron's frame v'_y is larger by a factor γ , but t' is smaller by a factor γ .

5.23. Two views of an oscilloscope

The various answers are:

125,000 eV [125 keV by construction, just *e* times $\Delta \phi$] 5/4 [$\gamma = (500 + 125)/500$] (3/5)*c* [$\beta = \sqrt{1 - 1/\gamma^2}$] 2.0 · 10⁻²² kg m/s [$p_x = \gamma m(\beta c)$] 3 · 10⁴ V/m [E = V/d] 4.8 · 10⁻¹⁵ newtons [F = Ee] 2.78 · 10⁻¹⁰ s [$t = \ell/(\beta c)$] 1.33 · 10⁻²⁴ kg m/s [$p_y = Ft$] 0.0066 radians [$\theta = p_y/p_x$]

 $\begin{array}{ll} 1.8 \cdot 10^8 \ \mathrm{m/s} & \left[\mathrm{same} \ \beta \ \mathrm{as} \ \mathrm{above} \right] \\ 0.04 \ \mathrm{m} & \left[\ell' = \ell/\gamma \right] \\ 2.22 \cdot 10^{-10} \ \mathrm{s} & \left[t' = \ell'/(\beta c); \ \mathrm{equivalently}, \ t' = t/\gamma \right] \\ 3.75 \cdot 10^4 \ \mathrm{V/m} & \left[E' = \gamma E, \ \mathrm{since} \ \mathrm{the} \ \mathrm{charge} \ \mathrm{density} \ \mathrm{is} \ \mathrm{larger} \ \mathrm{by} \ \gamma \right] \\ 1.46 \cdot 10^6 \ \mathrm{m/s} & \left[v'_y = a'_y t = (E'e/m)t', \ \mathrm{non-relativistic} \ \mathrm{is} \ \mathrm{fine} \ \mathrm{here} \right] \\ 1.33 \cdot 10^{-24} \ \mathrm{kg} \ \mathrm{m/s} & \left[p'_y = mv'_y \ \mathrm{for} \ \mathrm{non-relativistic} \ \mathrm{speeds} \right] \end{array}$

The y components of the momenta in the two frames are equal, as dictated by the Lorentz transformations. In short, this is due to the fact that in the electron-neutron frame the transverse field E' (and hence transverse force F'_y) is larger by a factor γ , but the time t' is smaller by a factor γ . So the product F'_yt' is the same as in the lab frame.

5.24. Acquiring transverse momentum

(a) Let E_{1y} be the y component of the electric field due to q_1 , at the location of q_2 ; see Fig. 103. Let P_{2y} be the y component of the momentum of particle q_2 . Then $dP_{2y}/dt = q_2 E_{1y} \Longrightarrow \Delta P_{2y} = \int q_2 E_{1y} dt$. We can exchange the dt here for dx



via dt = dx/v. Therefore, since $P_{2y} = 0$ at $x = -\infty$, the value of P_{2y} at $x = +\infty$ is $P_{2y} = (q_2/v) \int_{-\infty}^{\infty} E_{1y} dx$.

By Gauss's law, the surface integral of \mathbf{E}_1 over the infinitely long cylinder of radius *b* with q_1 lying on the axis (represented by the dotted lines in Fig. 103) is q_1/ϵ_0 . So $\int_{-\infty}^{\infty} E_{1y} 2\pi b \, dx = q_1/\epsilon_0 \Longrightarrow \int_{-\infty}^{\infty} E_{1y} \, dx = q_1/2\pi\epsilon_0 b$. Therefore, $P_{2y} = q_1 q_2/2\pi\epsilon_0 v b$, as desired. Also, $P_{2x} = 0$ by symmetry (\mathbf{E}_1 points the right just as much as it points to the left along the cylinder). So P_2 points in the *y* direction, that is, it is perpendicular to \mathbf{v} .

By Newton's third law, the momentum gained by q_1 is equal and opposite to the momentum gained by q_2 . But we can also calculate this in the same manner as above, by considering a cylinder with radius b, drawn with the path of q_2 as its axis; see Fig. 104. At any given instant, the flux through this cylinder is q_2/ϵ_0 (Gauss's law still holds for relativistic speeds). So as above, $\int_{-\infty}^{\infty} E_{2y} dx =$ $q_2/2\pi\epsilon_0 b$. (The E_{2y} here is the magnitude of the y component.) In this integral, we are considering a snapshot in time, with x running from $-\infty$ to ∞ . But we are free to write the integral in terms of the time t, for which dt = dx/v. If we multiply both sides of the previous equation by q_1 and substitute v dt for dx, we obtain $q_1 \int_{-\infty}^{\infty} E_{2y} dt = q_1 q_2/2\pi\epsilon_0 v b$. But the left side here is simply the expression for the magnitude of the final P_{1y} . The direction is downward, so the final P_{1y} equals $-q_1 q_2/2\pi\epsilon_0 v b$. In short, the surface integral over the cylinder is the same whether you calculate it for a stationary cylinder, or stand at a fixed position (at q_1) and have the cylinder (which you can imagine as attached to q_2) slide past you, adding up the integrals over the circular strips as they pass by.

(b) We want the transverse distances that the particles move to be much smaller than b. Consider q_2 . Its final y speed is $P_y/\gamma m$, so in order of magnitude, its y speed throughout the main part of the interaction (when the particles are a distance of order b apart) is also $P_y/\gamma m$. The time of the main part of the interaction is on the order of b/v, since the force is negligible when the particles are far apart. The rough transverse distance traveled by q_2 is the product of the rate and time, so we want

$$\frac{P_y}{\gamma m} \cdot \frac{b}{v} \ll b \implies \frac{q_1 q_2}{2\pi\epsilon_0 v b \cdot \gamma m} \cdot \frac{b}{v} \ll b \implies \frac{q_1 q_2}{2\pi\epsilon_0 \gamma v^2 b} \ll m.$$
(401)

This is the desired condition on m. Note, interestingly, that the transverse distance $(P/\gamma m)(b/v)$ is independent of b. This is due to the facts that the force decreases like $1/b^2$, but the distance (and hence time) of the interaction increases like b. And the latter matters twice in the nonrelativistic expression for distance, $y = a_y t^2/2$.

To do the same calculation for q_1 , we must remember that since the x speed of q_2 is relativistic, we need to take into account the fact that q_2 's field lines are squashed into a "pancake." The result from Problem 5.6 or Exercise 5.21 shows that the angular width of the pancake is on the order of $1/\gamma$. So the time of the main part of the interaction (as far as the force on q_1 is concerned) is only on the order of $b/\gamma v$. But since q_1 is moving slowly, we now have $v_y \approx P_y/m$, with no need for a γ factor in the denominator. We therefore end up with the condition $(P_y/m)(b/\gamma m) \ll b \implies q_1q_2/2\pi\epsilon_0\gamma v^2b \ll m$, in agreement with the above result.

Note that this condition can be written as $q_1q_2/4\pi\epsilon_0 b \ll \gamma mv^2/2$. For nonrelativistic motion, this says that the kinetic energy of q_2 must be much larger than the electrical potential energy of the particles at closest approach. It is reasonable



Figure 104

that the condition takes this form, because these are the only two energy scales in the problem.

5.25. Decreasing velocity

Force equals the rate of change of momentum. Therefore, since there is no force in the x direction in the lab frame, p_x must be constant. Now, $p_x = \gamma m v_x$, so if v_x were to remain constant, then the increase in γ (due to the increase in v_y due to the E_y field) would cause p_x to increase, in contradiction with the fact that p_x is constant. Therefore, v_x must actually decrease. Remember that the γ factor in $p_x = \gamma m v_x$ involves the square of the *entire* velocity, not just the v_x component. So p_x is indeed affected by the y motion.

We can also demonstrate this result by looking directly at the forces, without mentioning momentum. Let F be the lab frame, and let F' be the particle frame. The key point is that although the electric force on the particle points in the y direction in F, it does *not* point in the y' direction in F'. This can be seen as follows.

Except right at the start, the particle's velocity \mathbf{v} will be angled upward in F, as shown in Fig. 105. The E_{\parallel} and E_{\perp} components (with respect to the velocity \mathbf{v}) of the given vertical \mathbf{E} field in F are shown. If we transform these components to the particle frame F', the \perp component is larger by a factor γ (because it is smallest in the frame of the source, which is the lab frame; imagine two large capacitor plates). So we end up with the \mathbf{E}' vector shown. Since $\mathbf{F}' = q\mathbf{E}'$, the force on the particle in the particle's frame is therefore *tilted leftward*. Physically, the slightly leftward-pointing field in F'arises from the tilted pancakes that we encountered in Fig. 5.26. The particle therefore has an acceleration in the negative x' direction in its frame, so it picks up a negative x' velocity relative to the inertial frame it was just in. The x speed in the lab frame therefore decreases (via the velocity addition formula).

5.26. Charges in a wire

The β value associated with $\gamma = 1.2$ is $\beta = \sqrt{1 - 1/\gamma^2} = 0.553$. So the test charge is moving at speed v = (0.553)c with respect to the lab frame. We also know that the electrons are moving at speed $v_0 = \beta_0 c = (0.8)c$ with respect to the lab frame. β'_0 , which gives the speed of the electrons with respect to the test charge, is therefore given by the velocity addition (or subtraction) formula,

$$\beta_0' = \frac{\beta_0 - \beta}{1 - \beta_0 \beta} = \frac{0.8 - 0.553}{1 - (0.8)(0.553)} = 0.443.$$
(402)

From Eq. (5.24) we have

$$\lambda' = \gamma \beta \beta_0 \lambda_0 = (1.2)(0.553)(0.8)\lambda_0 = (0.531)\lambda_0.$$
(403)

5.27. Equal velocities

If $\beta = \beta_0$, then $\gamma = \gamma_0$. So the positive charge density $\gamma \lambda_0$ in the test-charge frame becomes $\gamma_0 \lambda_0$. And the negative charge density $-\gamma \lambda_0 (1 - \beta \beta_0)$ in Eq. (5.24) (the second term in that equation) becomes $-\gamma_0 \lambda_0 (1 - \beta_0^2) = -\lambda_0 / \gamma_0$.

These results make sense, because in the test charge's frame the positive charges are moving backward at speed v_0 , so their separation is contracted by γ_0 (because they were at rest in the lab frame). And the negative charges are at rest in the test charge's frame, so their separation is uncontracted by a factor γ_0 (because they were moving at speed v_0 in the lab frame).



Figure 105

5.28. Stationary rod and moving charge

- (a) The force is always largest in the rest frame of the particle. It is smaller in any other frame by the γ factor associated with the speed v of the particle. So the force in the new frame (the charge's frame) is larger; it is $\gamma q \lambda / 2\pi r \epsilon_0$. This force is repulsive, assuming q and λ have the same sign.
- (b) In the new frame the charge q isn't moving, so even though the magnetic *field* is nonzero, the magnetic *force* is zero, due to the v in $F_B = qvB$.

We therefore need only worry about the electric force. In the new frame the linear charge density on the rod is increased to $\gamma\lambda$, due to length contraction. So the electric field is $\gamma\lambda/2\pi r\epsilon_0$. This field produces a repulsive electric force of $F_E = \gamma q \lambda/2\pi r\epsilon_0$, which agrees with the result in part (a).

5.29. Protons moving in opposite directions

In the rest frame of one of the protons, the other proton moves with a speed given by $\beta_2 = 2\beta/(1+\beta^2)$, from the velocity addition formula. In this frame, Eq. (5.15) gives the electric field of the moving proton as $\gamma_2 e/4\pi\epsilon_0 r^2$. That is, the field is larger by a factor γ_2 in the transverse direction, compared with the naive Coulomb result. The γ_2 factor equals

$$\gamma_2 = \frac{1}{\sqrt{1 - \beta_2^2}} = \frac{1}{\sqrt{1 - \left(\frac{2\beta}{1 + \beta^2}\right)^2}} = \frac{1 + \beta^2}{\sqrt{(1 + \beta^2)^2 - 4\beta^2}} = \frac{1 + \beta^2}{1 - \beta^2}.$$
 (404)

The force on each proton in its rest frame is therefore

$$F_{\rm rest} = \frac{1+\beta^2}{1-\beta^2} \cdot \frac{e^2}{4\pi\epsilon_0 r^2} = \gamma^2 (1+\beta^2) \frac{e^2}{4\pi\epsilon_0 r^2} \,, \tag{405}$$

where $\gamma \equiv 1/\sqrt{1-\beta^2}$.

The relative speed of the proton rest frame and the lab frame is βc , so the force in the lab frame is smaller than F_{rest} by a factor $1/\gamma = \sqrt{1-\beta^2}$. (Remember, the force is always largest in the rest frame of the particle on which it acts.) The repulsive force on each of the protons in the lab frame is therefore

$$F_{\rm lab} = \frac{F_{\rm rest}}{\gamma} = \gamma (1 + \beta^2) \frac{e^2}{4\pi\epsilon_0 r^2} \,. \tag{406}$$

This is the correct total force in the lab frame. As stated in the problem, it is not equal to $\gamma e^2/4\pi\epsilon_0 r^2$. The task now is to explain the force as the sum of the electric and magnetic forces in the lab frame.

As mentioned in the problem, Eq. (5.15) tells us that the repulsive *electric* force in the lab frame is $F_{\text{lab},E} = \gamma e^2/4\pi\epsilon_0 r^2$. This must not be the whole force, because it doesn't equal the correct total force in Eq. (406). Apparently there must be an additional repulsive force in the lab frame that equals the difference between these two repulsive forces. This difference is

$$F_{\text{lab}} - F_{\text{lab},E} = \gamma (1+\beta^2) \frac{e^2}{4\pi\epsilon_0 r^2} - \gamma \frac{e^2}{4\pi\epsilon_0 r^2} = \gamma \beta^2 \frac{e^2}{4\pi\epsilon_0 r^2}.$$
 (407)

This is exactly the force produced by a magnetic field with strength $B = (\beta/c)\gamma e/4\pi\epsilon_0 r^2$. (This is consistent with the Lorentz transformations in Chapter 6.) This field does indeed produce the desired force, because the proton is moving with speed βc through this B field, which yields a magnetic force with magnitude

$$F_{\text{lab},B} = qvB = e(\beta c)B = \frac{\gamma \beta^2 e^2}{4\pi \epsilon_0 r^2}.$$
 (408)

This is the quantity that appears on the right-hand side of Eq. (407), so that equation is correctly the statement that

$$F_{\text{lab}} - F_{\text{lab},E} = F_{\text{lab},B} \implies F_{\text{lab}} = F_{\text{lab},E} + F_{\text{lab},B}.$$
(409)

The direction of the magnetic force will be correct (repulsive) provided that the \mathbf{B} field at the location of each proton, due to the other proton, points out of the page.

Note that the electric and magnetic forces in the lab frame have exactly the same magnitudes that they had in the second example in Section 5.9, because the speeds are the same in the two setups. But the difference in direction of the top charge's motion in the two setups causes its magnetic field to point in the opposite direction. The $q\mathbf{v} \times \mathbf{B}$ forces on both charges switch sign, so in the present setup the magnetic force is added to the electric force instead of subtracted from it. The total force in the lab frame is therefore different in the two setups.

5.30. Transformations of λ and I

The speed of the charges in the new frame is given by $\beta'_k = (\beta_k + \beta)/(1 + \beta_k\beta)$. The γ factor associated with this speed is $\gamma'_k = \gamma_k \gamma(1 + \beta_k\beta)$ (see below). The density in the rest frame of the charges is λ_k/γ_k , so the density in frame F' is

$$\lambda'_{k} = \frac{\lambda_{k}}{\gamma_{k}}\gamma'_{k} = \frac{\lambda_{k}}{\gamma_{k}}\gamma_{k}\gamma(1+\beta_{k}\beta) = \gamma\left(\lambda_{k} + \frac{\beta}{c}(\lambda_{k}\beta_{k}c)\right) = \gamma\left(\lambda_{k} + \frac{\beta I_{k}}{c}\right), \quad (410)$$

as desired. The current in F' is

$$I'_{k} = \lambda'_{k}\beta'_{k}c = \lambda_{k}\gamma(1+\beta_{k}\beta)\left(\frac{\beta_{k}+\beta}{1+\beta_{k}\beta}\right)c$$
$$= \lambda_{k}\gamma(\beta_{k}+\beta)c = \gamma(\lambda_{k}\beta_{k}c+\beta c\lambda_{k}) = \gamma(I_{k}+\beta c\lambda_{k}),$$
(411)

as desired. Since these two transformations are linear in I_k and λ_k , they hold for any corresponding linear combinations of the I_k and λ_k (for example, $2I_1 - 7I_5 + 3I_8$ and $2\lambda_1 - 7\lambda_5 + 3\lambda_8$). In particular, they hold for the sums of the I_k and the λ_k . But these sums are just the total current I and the total density λ . So the Lorentz transformations hold for the total I and λ , as we wanted to show.

Here is the algebra that produces γ'_k :

$$\gamma_k' = \frac{1}{\sqrt{1 - \left(\frac{\beta_k + \beta}{1 + \beta_k \beta}\right)^2}} = \frac{1 + \beta_k \beta}{\sqrt{(1 + 2\beta_k \beta + \beta_k^2 \beta^2) - (\beta_k^2 + 2\beta_k \beta + \beta^2)}}$$
$$= \frac{1 + \beta_k \beta}{\sqrt{(1 - \beta_k^2)(1 - \beta^2)}} = \gamma_k \gamma (1 + \beta_k \beta). \tag{412}$$

5.31. Moving perpendicular to a wire

In the notation of Fig. 5.25, the velocity components of the electrons in the charge's frame are $v_x = v_0/\gamma$ and $v_y = v$. So $\beta_e^2 c^2 = v_0^2/\gamma^2 + v^2$. And the angle α in Fig. 5.26 is given by $\tan \alpha = v/(v_0/\gamma) = \gamma v/v_0$. The angle θ' in Eq. (5.15) is measured

with respect to the direction of motion of the electrons. So for the left electron in Fig. 5.26 we have $\theta' = \phi - \alpha$. This holds for the right electron too, provided that we consistently measure ϕ with respect to the positive x axis (unlike how it is defined for the right electron in Fig. 5.26). The angle α is fixed by the parameters in the setup; the variable we will integrate over is ϕ . The range $0 \le \phi \le \pi$ corresponds to the range $-\infty \le x \le \infty$ on the wire.

If the charge q is a distance ℓ from the wire in the lab frame, then it is $\ell' = \ell/\gamma$ from the wire in its own frame, due to length contraction. (Imagine a transverse stick of length ℓ attached to the wire as it moves toward the charge in the charge's frame, and consider the moment when the end of the stick coincides with the charge.) In the charge's frame, the distance r' from the charge q to an electron is $r' = \ell' / \sin \phi = \ell / \gamma \sin \phi$. The position of an electron along the wire is given by $x = -\ell' / \tan \phi = -\ell / \gamma \tan \phi$. Taking the differential of this gives $dx = \ell d\phi / \gamma \sin^2 \phi$.

We now have a handle on all the necessary quantities, so we can use Eq. (5.15) to find the force on the charge q (as measured in its own frame) due to a small interval of the wire with length dx. The (negative) electron charge contained in this length is $(-\lambda_0)dx = -\lambda_0 \ell d\phi/\gamma \sin^2 \phi$. (There is no length contraction along the wire in the charge's frame.) We are concerned with the x component of the force, which brings in a factor of $\cos \phi$, so from Eq. (5.15) we obtain

$$dF_x = \frac{q(-\lambda_0 \ell \, d\phi/\gamma \sin^2 \phi)}{4\pi\epsilon_0 (\ell/\gamma \sin \phi)^2} \frac{1 - \beta_e^2}{(1 - \beta_e^2 \sin^2(\phi - \alpha))^{3/2}} \cos \phi$$

= $-\frac{\gamma q \lambda_0 (1 - \beta_e^2)}{4\pi\epsilon_0 \ell} \frac{\cos \phi \, d\phi}{(1 - \beta_e^2 \sin^2(\phi - \alpha))^{3/2}}.$ (413)

Note that the γ here is associated with the speed v of the charge q in the lab frame, whereas the β_e is the total speed of the electrons in the charge q frame, which is given by $\beta_e^2 c^2 = v_0^2 / \gamma^2 + v^2$.

To find the total force on the charge q in its own frame, we need to integrate Eq. (413) from $\phi = 0$ to $\phi = \pi$. The integral is given in Appendix K, so the total horizontal force from all of the electrons in the wire is

$$F = -\frac{\gamma q \lambda_0 (1 - \beta_e^2)}{4\pi \epsilon_0 \ell} \frac{(2 - \beta_e^2) \sin \phi + \beta_e^2 \sin(2\alpha - \phi)}{2(1 - \beta_e^2) \sqrt{1 - \beta_e^2 \sin^2(\alpha - \phi)}} \bigg|_0^{\mu}.$$
 (414)

In the numerator, the $\sin \phi$ term is zero at both limits, and $\sin(2\alpha - \pi) = -\sin 2\alpha$. In the denominator, $\sin(\alpha - \pi) = -\sin \alpha$, but we're squaring this, so the minus sign doesn't matter. The denominators are therefore equal at both limits. Hence we obtain twice the value at the upper limit:

$$F = \frac{\gamma q \lambda_0}{4\pi\epsilon_0 \ell} \frac{\beta_e^2 \sin 2\alpha}{\sqrt{1 - \beta_e^2 \sin^2 \alpha}} = \frac{\gamma q \lambda_0}{2\pi\epsilon_0 \ell} \frac{\beta_e^2 \sin \alpha \cos \alpha}{\sqrt{1 - \beta_e^2 \sin^2 \alpha}}.$$
 (415)

From $\tan \alpha = \gamma v / v_0$ we have $\sin \alpha = \gamma v / \sqrt{\gamma^2 v^2 + v_0^2}$ and $\cos \alpha = v_0 / \sqrt{\gamma^2 v^2 + v_0^2}$.

Since β_e^2 can be written as $(\gamma^2 v^2 + v_0^2)/\gamma^2 c^2$, we therefore have

$$F = \frac{\gamma q \lambda_0}{2\pi\epsilon_0 \ell} \frac{\frac{\gamma^2 v^2 + v_0^2}{\gamma^2 c^2} \frac{\gamma v \cdot v_0}{\gamma^2 v^2 + v_0^2}}{\sqrt{1 - \frac{\gamma^2 v^2 + v_0^2}{\gamma^2 c^2} \frac{\gamma^2 v^2}{\gamma^2 v^2 + v_0^2}}} = \frac{q \lambda_0 v v_0}{2\pi\epsilon_0 \ell c^2} \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{\gamma q \lambda_0 v v_0}{2\pi\epsilon_0 \ell c^2}.$$
(416)

This is the total force on the charge q in its own frame. The force is positive, so it points rightward, consistent with the field lines in Fig. 5.26. Now, back in the lab frame, $\lambda_0 v_0$ equals the current I in the wire (which is directed to the left, since the electrons are moving to the right). Transforming the above force to the lab frame involves dividing by the γ factor associated with the speed v, which is the γ factor that appears in Eq. (416). So we end up with a rightward directed force with magnitude $qvI/2\pi\epsilon_0\ell c^2$, as desired. As mentioned in the text, if the magnetic field **B** points out of the page, then $\mathbf{v} \times \mathbf{B}$ points to the right. So the force can be interpreted as the $q\mathbf{v} \times \mathbf{B}$ force from a magnetic field with strength $I/2\pi\epsilon_0\ell c^2$.

Chapter 6

The magnetic field

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

6.29. Motion in a B field

FIRST SOLUTION: The magnitude of the magnetic force is F = qvB, so the magnitude of the change in p during a short time dt is dp = F dt = qvB dt. The momentum itself is $p = \gamma mv$. Fig. 106(a) shows the **r** and **p** vectors at two nearby times. In Fig. 106(b) the angle θ is the same in the two triangles, because each **p** is perpendicular to the corresponding **r**. So from similar triangles we have

$$\frac{|d\mathbf{r}|}{|\mathbf{r}|} = \frac{|d\mathbf{p}|}{|\mathbf{p}|} \implies \frac{v\,dt}{R} = \frac{qvB\,dt}{p} \implies R = \frac{p}{qB} = \frac{\gamma mv}{qB} \,. \tag{417}$$

The time to complete one revolution is

$$t = \frac{2\pi R}{v} = \frac{2\pi}{v} \frac{\gamma m v}{qB} = \frac{2\pi \gamma m}{qB}.$$
(418)

If **B** is uniform, then Eq. (417) actually *proves* that the particle travels in a circle, because it gives the radius of curvature at any point as $R = \gamma m v/qB$. Since v is constant (because the magnetic force is always perpendicular to the velocity), we see that R is constant, which means that the path is a circle.

SECOND SOLUTION: We can also calculate R in a more mathematical way. The Lorentz-force law $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ combined with Newton's third law $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\frac{d(\gamma m \mathbf{v})}{dt} = q \mathbf{v} \times \mathbf{B} \implies \frac{d \mathbf{v}}{dt} = \frac{q}{\gamma m} \mathbf{v} \times \mathbf{B}.$$
(419)

Note that we are in fact allowed to take the γ outside the derivative because we know that the speed v is constant.

Assume that **B** is uniform. Let the motion be in the *x*-*y* plane, with the magnetic field pointing in the *z* direction. Then $\mathbf{v} = (v_x, v_y, 0)$ and $\mathbf{B} = (0, 0, B)$. So $\mathbf{v} \times \mathbf{B} = B(v_y, -v_x, 0)$. The *x* and *y* components of Eq. (419) can then be written as

$$\frac{dv_x}{dt} = \frac{qB}{\gamma m} v_y \qquad \text{and} \qquad \frac{dv_y}{dt} = -\frac{qB}{\gamma m} v_x. \tag{420}$$





 $d\mathbf{r}$



(b)

Taking the derivative of the first of these equations, and then substituting in the value of dv_y/dt from the second, gives

$$\frac{d^2 v_x}{dt^2} = -\left(\frac{qB}{\gamma m}\right)^2 v_x. \tag{421}$$

This is a simple-harmonic-oscillator type equation, for which the general solution takes the form, $^{-R}$

$$v_x(t) = A\cos(\omega t + \phi), \quad \text{where } \omega = \frac{qB}{\gamma m}.$$
 (422)

The first of the equations in Eq. (420) then quickly gives $v_y(t) = -A\sin(\omega t + \phi)$. A and ϕ are arbitrary constants, determined by the initial conditions. However, if the momentum $p = \gamma m v$ is given, then v_x and v_y must each have an amplitude of $p/\gamma m$. Hence $A = p/\gamma m$.

The period is $2\pi/\omega = 2\pi\gamma m/qB$, in agreement with the result in part (a). Integrating $v_x(t)$ and $v_y(t)$ to find x and y gives (up to arbitrary additive constants, which only affect the position of the center of the circle)

$$(x(t), y(t)) = \frac{A}{\omega} \Big(\sin(\omega t + \phi), \cos(\omega t + \phi) \Big).$$
(423)

This describes a circle with radius $R = A/\omega = (p/\gamma m)/(qB/\gamma m) = p/qB$, in agreement with the result in part (a).

6.30. Proton in space

The speed is essentially c, so from Exercise 6.29 the radius of curvature is

$$R = \frac{p}{qB} = \frac{\gamma mv}{eB} = \frac{10^7 (1.67 \cdot 10^{-27} \,\mathrm{kg})(3 \cdot 10^8 \,\mathrm{m/s})}{(1.6 \cdot 10^{-19} \,\mathrm{C})(3 \cdot 10^{-10} \,\mathrm{T})} = 1.0 \cdot 10^{17} \,\mathrm{m}.$$
 (424)

(It was fine to set the factor of v in the numerator equal to c, but we of course can't set v = c inside the γ factor!) The time to complete one revolution is

$$t = \frac{2\pi R}{v} = \frac{2\pi (1.0 \cdot 10^{17} \,\mathrm{m})}{3 \cdot 10^8 \,\mathrm{m/s}} = 2.1 \cdot 10^9 \,\mathrm{s} \approx 70 \,\mathrm{years.}$$
(425)

6.31. Field from three wires

At point P_1 at the center of the square, the magnetic fields due to wires A and C in Fig. 107 cancel. The field due to B is $\mu_0(2I)/2\pi(d/\sqrt{2}) = \sqrt{2} \mu_0 I/\pi d$, directed diagonally down to the left, as shown.

At point P_2 at the lower right-hand corner, the field due to B is half of the field at P_1 , so it is $\mu_0 I/\sqrt{2} \pi d$, directed diagonally down to the left, as shown. The field due to Ais $\mu_0 I/2\pi d$, directed upward. The field due to C is $\mu_0 I/2\pi d$, directed rightward. The sum of these two fields is $\mu_0 I/\sqrt{2} \pi d$, directed diagonally up to the right. The vector sum of all three fields is therefore zero at P_2 .

6.32. Oersted's experiment

The compass needle initially points in the direction of the earth's magnetic field, which has strength 0.2 gauss (in the horizontal direction). In Oersted's experiment, the wire runs parallel to the initial orientation of the needle. If the needle ends up rotated by 45° , the magnetic field from the wire must be 0.2 gauss in the perpendicular direction.



Figure 107

In other words, we have a current-carrying wire that produces a magnetic field of $2 \cdot 10^{-5}$ T at a distance of about 2 cm. Therefore,

$$B = \frac{\mu_0 I}{2\pi r} \implies I = \frac{2\pi r B}{\mu_0} = \frac{2\pi (0.02 \,\mathrm{m})(2 \cdot 10^{-5} \,\mathrm{T})}{4\pi \cdot 10^{-7} \frac{\mathrm{kg} \,\mathrm{m}}{\mathrm{C}^2}} = 2 \,\mathrm{A}.$$
 (426)

6.33. Force between wires

The magnetic field due to one of the wires in Fig. 5.1(b), at the location of the other, is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\left(4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^2}\right)(20 \text{ A})}{2\pi (0.05 \text{ m})} = 8 \cdot 10^{-5} \text{ T.}$$
(427)

The force per unit length on each wire is then $IB = (20 \text{ A})(8 \cdot 10^{-5} \text{ T}) = 1.6 \cdot 10^{-3} \text{ N/m}$, and it is repulsive.

6.34. Torque on a loop

From Eq. (6.17) the force on a small piece of the loop is $d\mathbf{F} = I \, d\mathbf{l} \times \mathbf{B}$. The $\hat{\mathbf{z}}$ component of \mathbf{B} produces a force in the plane of the loop, which contributes nothing to the torque around the x axis (or the y axis) and can therefore be ignored. (As an exercise, you can also show that it produces no torque around the z axis.) So for the present purposes we have $\mathbf{B} = \hat{\mathbf{y}}B_y$. Now, $d\mathbf{l} \times \hat{\mathbf{y}}B_y = \hat{\mathbf{z}} \, dl \, B_y \sin \theta$, where θ is the angle that $d\mathbf{l}$ makes with the y axis. But $dl \sin \theta$ is simply the dx span of the little interval $d\mathbf{l}$. Hence $dF_z = IB_y \, dx$. The torque about the x axis is then $dN_x = y \, dF_z = IB_y \, dx$. We haven't been keeping track of the signs, but this result for dN_x does indeed have the correct sign according to the right-hand-rule convention.

We must integrate dN_x around the entire loop to find the total torque. But $\int_{\text{loop}} y \, dx$ equals the area a of the loop. This is true because in Fig. 108 the product $y \, dx$ for segment A equals the area of the thin rectangle up to A, whereas the product $y \, dx$ for segment B equals the *negative* (since dx is negative) of the area of the thin rectangle up to B. The sum of these signed areas is the area of the rectangle from B to A. Adding up all such rectangles gives the area a of the whole loop. This works even if B is below the x axis; y is now negative, so the areas add, which again yields the entire area of the thin rectangle.

We therefore have $N_x = IaB_y$. Physically, this N_x component arises because the relevant force on the loop points in the $+\hat{z}$ direction for the top part (the part with larger values of y), and in the $-\hat{z}$ direction for the bottom part. The loop will therefore tend to spin around an axis pointing in the x direction. This physical reasoning makes it fairly clear that the loop won't tend to spin around an axis pointing in the y direction (assuming **B** has no x component). Mathematically, dN_y for our loop equals $x dF_z = IB_y x dx$. So N_y involves the integral $\int x dx$ around the loop, which you can quickly show is zero; the upper and lower parts of the loop now exactly cancel instead of only partially canceling.

The total torque therefore points only in the x direction, and it equals $\mathbf{N} = \hat{\mathbf{x}} I a B_y$. If we define the magnetic moment as $\mathbf{m} = I \mathbf{a} = I a(-\hat{\mathbf{z}})$, then we can write N in the general form of $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ because

$$\mathbf{m} \times \mathbf{B} = (-Ia\hat{\mathbf{z}}) \times (\hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) = -IaB_y\hat{\mathbf{z}} \times \hat{\mathbf{y}} + 0 = \hat{\mathbf{x}}IaB_y,$$
(428)

as desired. The minus sign in the relation $\mathbf{a} = -a\hat{\mathbf{z}}$ comes from the given orientation of the current, along with the right-hand convention. Since $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ is a vector relation, its validity can't depend on our specific choice of coordinate system. So if it



Figure 108

it true for a particular choice of axes (as we just demonstrated), then it must be true for any choice.

Let's now look at the net force on the loop. If ${\bf B}$ is uniform over the loop, then the net force is

$$\int_{\text{loop}} d\mathbf{F} = \int I \, d\mathbf{l} \times \mathbf{B} = I\left(\int d\mathbf{l}\right) \times \mathbf{B}.$$
(429)

But $\int d\mathbf{l} = 0$ because we have a closed loop. The net force is therefore zero. Note that this result holds even if the loop isn't planar.

6.35. Determining c

If we follow the wires around in Fig. 6.42, we see that the voltage across both capacitors is equal to $\mathcal{E}_0 \cos 2\pi ft$. Let's first determine the magnetic force between the rings. Neglecting the inductance (the subject of Chapter 7) and resistance of the two rings and leads, the charge on capacitor C_2 at any time t takes the form, $Q_2 = \mathcal{E}_0(\cos 2\pi ft)C_2$. The current through C_2 is then $I = dQ_2/dt = -2\pi f\mathcal{E}_0 C_2 \sin 2\pi ft$, with positive defined as flowing into the left terminal of C_2 , or equivalently out of the right terminal and into the rings. The two rings are in series, so this current flows through each.

If $h \ll b$ we are justified in computing the magnetic force between the rings as if they were parallel straight wires, with force per unit length $\mu_0 I^2/2\pi h$. The length is $2\pi b$, so the magnetic force pulling the upper ring down (note that the currents are in the same direction, so the force is attractive) is

$$F_{\rm m} = \frac{\mu_0 I^2}{2\pi h} (2\pi b) = \frac{\mu_0 (2\pi f \mathcal{E}_0 C_2)^2 b}{h} \sin^2 2\pi f t.$$
(430)

The time average of $\sin^2 2\pi ft$ is 1/2, so the average attractive magnetic force between the rings is

$$\overline{F}_{\rm m} = \frac{2\mu_0 \pi^2 f^2 \mathcal{E}_0^2 C_2^2 b}{h} \,. \tag{431}$$

Now let's determine the electric force between the capacitor plates. As mentioned above, the potential difference between the plates is $\mathcal{E}_0 \cos 2\pi ft$. The field between the plates is therefore $E = (\mathcal{E}_0 \cos 2\pi ft)/s$. The downward electric force on the upper plate can be obtained in various ways. From Section 3.7 the force is the energy density times the area, which gives $F_e = (\epsilon_0 E^2/2)(\pi a^2)$. Alternatively, the force is the charge times the average of the fields on either side of the plate, namely E/2 (see Section 1.14), which gives $(\sigma \pi a^2)(E/2)$. (Equivalently, the force is the charge times the field from the other plate.) These two expressions agree because $\sigma = E\epsilon_0$. Using the above expression for E, and noting that the time average of $\cos^2 2\pi ft$ is 1/2, we find the average attractive electric force between the rings to be

$$F_{\rm e} = \frac{\epsilon_0 \pi a^2}{2} \left(\frac{\mathcal{E}_0 \cos 2\pi f t}{s} \right)^2 \implies \overline{F}_{\rm e} = \frac{\epsilon_0 \pi a^2 \mathcal{E}_0^2}{4s^2} \,. \tag{432}$$

We can eliminate s in favor of the capacitance C_1 , which is given by Eq. (3.15) as $C_1 = \epsilon_0 \pi a^2/s$. So $1/s^2 = C_1^2/\epsilon_0^2 \pi^2 a^4$, and the electric force becomes

$$\overline{F}_{\rm e} = \frac{\mathcal{E}_0^2 C_1^2}{4\pi\epsilon_0 a^2} \,. \tag{433}$$

When the forces are balanced (which might be brought about by varying C_2), we have

$$\overline{F}_{e} = \overline{F}_{m} \implies \frac{\mathcal{E}_{0}^{2}C_{1}^{2}}{4\pi\epsilon_{0}a^{2}} = \frac{2\mu_{0}\pi^{2}f^{2}\mathcal{E}_{0}^{2}C_{2}^{2}b}{h}$$

$$\implies \frac{1}{\mu_{0}\epsilon_{0}} = \frac{8\pi^{3}a^{2}bf^{2}C_{2}^{2}}{hC_{1}^{2}}$$

$$\implies \frac{1}{\sqrt{\mu_{0}\epsilon_{0}}} = (2\pi)^{3/2}a\left(\frac{b}{h}\right)^{1/2}\left(\frac{C_{2}}{C_{1}}\right)f, \qquad (434)$$

as desired. Note that a constant voltage wouldn't be useful here, because the capacitors would quickly reach their maximum charge, which would mean the current would be zero. The alternating voltage allows there to be (except at discrete moments during each cycle) both a nonzero charge on the capacitor C_1 and a nonzero current in the rings.

Let's see what some reasonable numbers give. If a = 0.1 m, b/h = 10, and $C_2/C_1 = 10^6$, we find that the righthand side equals $1/\sqrt{\mu_0\epsilon_0} = c = 3 \cdot 10^8 \text{ m/s}$ when $f = 60.2 \text{ s}^{-1}$. So given all the other parameters, we can determine c by sweeping though frequency values until we find the f (which is 60.2 s^{-1} in the present case) that makes things balance. If the current rings consist of N turns each, this magnifies the magnetic force by a factor N^2 (the magnetic field is N times as large, and there are N times as many loops that it acts on), which decreases the necessary value of f (or C_2/C_1 , etc.) by a factor N.

6.36. Field at different radii

The radius is 2 cm, so 1/4 of the cross-sectional area, and hence current (so 2000 A), is enclosed within r = 1 cm. The current enclosed in both the r = 2 cm and r = 3 cm cases is 8000 A. So we have

$$B_{1} = \frac{\mu_{0}I_{1}}{2\pi r_{1}} = \frac{\left(4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^{2}}\right)(2000 \text{ A})}{2\pi (0.01 \text{ m})} = 0.04 \text{ T},$$

$$B_{2} = \frac{\mu_{0}I_{2}}{2\pi r_{2}} = \frac{\left(4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^{2}}\right)(8000 \text{ A})}{2\pi (0.02 \text{ m})} = 0.08 \text{ T},$$

$$B_{3} = \frac{\mu_{0}I_{3}}{2\pi r_{3}} = \frac{\left(4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^{2}}\right)(8000 \text{ A})}{2\pi (0.03 \text{ m})} = 0.0533 \text{ T}.$$
(435)

These fields are 400, 800, and 533 gauss, respectively.

6.37. Off-center hole

The given setup is equivalent to the superposition of a complete solid rod with current flowing into the page plus a smaller rod (where the hole is) with current flowing out of the page. If the two current densities are equal and opposite, then there will be zero current in the hole, in agreement with the given setup. Given the ratio of the areas of the two circular cross sections, currents of 1200 A into the page and 300 A out of the page will yield the given 900 A into the page. The large rod produces zero field on its axis, so the desired field is due entirely to the smaller rod with 300 A coming out of the page. The magnitude of the field is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\left(4\pi \cdot 10^{-7} \,\frac{\text{kg m}}{\text{C}^2}\right)(300 \,\text{A})}{2\pi (0.02 \,\text{m})} = 0.003 \,\text{T},\tag{436}$$

or 30 gauss, and it points to the left. A more remarkable fact (see Exercise 6.38) is that the field is 30 gauss pointing to the left not only at P but everywhere inside the cylindrical hole.

6.38. Uniform field in off-center hole

Inside a solid cylinder (with no cavity), the current enclosed within a radius r is $I(r^2/R^2)$, because area is proportional to length squared. So Ampere's law gives $B(2\pi r) = \mu_0 I r^2/R^2 \implies B = \mu_0 I r/2\pi R^2$. Assuming that the z axis (and hence current) points out of the page, the magnetic field points in the tangential direction, $\hat{\theta}$, which can be written as $\hat{\mathbf{z}} \times \hat{\mathbf{r}}$. If we combine the magnitude r with the unit vector $\hat{\mathbf{r}}$ to make the full vector \mathbf{r} , the complete vector form of the field can be written as $\mathbf{B} = (\mu_0 I/2\pi R^2)\hat{\mathbf{z}} \times \mathbf{r}$. And since $I/\pi R^2$ equals the current density J, we can write this as $\mathbf{B} = (\mu_0 J/2)\hat{\mathbf{z}} \times \mathbf{r}$.

The given setup can be considered to be the superposition of the given solid rod with radius R and another smaller rod with current flowing in the opposite direction with the same current density J. The current densities will then cancel in the region of the smaller rod, creating the desired cavity.

Consider a point inside the cavity, at position \mathbf{r}_1 with respect to the center of the given rod, and at position \mathbf{r}_2 with respect to the center of the cavity, as shown in Fig. 109. Using the above expression for **B**, the field at this point is the sum of the fields from the two rods we are superposing. Since the currents flow in opposite directions, the net field is

$$\frac{\mu_0 J}{2}\hat{\mathbf{z}} \times \mathbf{r}_1 - \frac{\mu_0 J}{2}\hat{\mathbf{z}} \times \mathbf{r}_2 = \frac{\mu_0 J}{2}\hat{\mathbf{z}} \times (\mathbf{r}_1 - \mathbf{r}_2) = \frac{\mu_0 J}{2}\hat{\mathbf{z}} \times \mathbf{a}.$$
 (437)

All of the quantities in this expression are constants, so the field is uniform, as desired. It is perpendicular to \mathbf{a} and proportional to $|\mathbf{a}|$. If $\mathbf{a} = 0$, the field is zero, as expected, because the final object is a thick annular ring. The field doesn't depend on the radius of either rod, as long as one rod is contained inside the other.

6.39. Constant magnitude of B

If I_r is the current inside radius r, then Ampere's law gives

$$B \cdot 2\pi r = \mu_0 I_r \implies B = \frac{\mu_0 I_r}{2\pi r} .$$
(438)

If we want B to be independent of r, then we need I_r to be proportional to r. I_r is found by integrating the current density J(r):

$$I_r = \int J \, da = \int_0^r J(r') \cdot (2\pi r' \, dr'). \tag{439}$$

It is easiest to guess and check the form of J(r'). If J(r') is proportional to 1/r', then it takes the form of $J(r') = \alpha/r'$, so

$$I_r = \int_0^r (\alpha/r') (2\pi r' \, dr') = 2\pi \alpha r, \qquad (440)$$

as desired. The field is then

$$B = \frac{\mu_0 I_r}{2\pi r} = \frac{\mu_0 (2\pi\alpha r)}{2\pi r} = \mu_0 \alpha.$$
(441)



Figure 109

The above "1/r" result for the current density is the same result that holds for the charge density in the case of the electric field due to a charged cylinder or sphere. In both of these cases the electric field is independent of r if the density ρ is proportional to 1/r.

Note that even though the current density diverges at r = 0, the actual current does not. There is a finite amount of current in any cross section with radius r, and it is given (by construction) by $I_r = 2\pi\alpha r$. Any ring (at any radius) with thickness drcontains the same amount of current, $dI = 2\pi\alpha dr$.

We can also solve this exercise by using the differential form of Ampere's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Since **B** points tangentially and has a uniform value, it can be written as $\mathbf{B} = B_0 \hat{\boldsymbol{\theta}}$. Equation F.2 in Appendix F then gives

$$\nabla \times \mathbf{B} = \frac{1}{r} \frac{\partial (rB_0)}{\partial r} \hat{\mathbf{z}} = \frac{B_0}{r} \hat{\mathbf{z}}.$$
(442)

Setting this equal to $\mu_0 J \hat{\mathbf{z}}$ gives $J = B_0/(\mu_0 r)$, consistent with the 1/r dependence we found above. The factor of B_0/μ_0 here equals the α from above.

6.40. The pinch effect

If the conduction electrons are forced closer to the axis, there will be uncompensated negative charge near the axis. This will generate an inward radial electric field E that pushes outward on the electrons, preventing further constriction when the outward electric force balances the inward magnetic force, that is, when $eE = evB \implies E = vB$. The magnetic field at radius r is $B_r = \mu_0 I_r / 2\pi r$, where I_r is the current contained within radius r. Assuming no redistribution of the charge, I_r is given by $I_r = \pi r^2 J$, where J = nev is the current density (n is the number of electrons per unit volume, and v is the drift velocity). The B field is therefore $B_r = \mu_0 (\pi r^2 nev) / 2\pi r = \mu_0 rnev / 2$. Suppose that the cloud of electrons at radius r is squeezed inward by a small distance Δr . The cylinder of radius r will now contain, per unit length, an excess of negative charge in the amount of $\Delta \lambda = (ne)(2\pi r \Delta r)$; this is the volume charge density times the cross-sectional area. This causes an inward electric field equal to $E_r = \Delta \lambda / 2\pi\epsilon_0 r = ne \Delta r / \epsilon_0$. The condition for equilibrium in then (using $\mu_0 \epsilon_0 = 1/c^2$)

$$E_r = vB_r \implies \frac{ne\,\Delta r}{\epsilon_0} = v\frac{\mu_0 rnev}{2} \implies \frac{\Delta r}{r} = \frac{\mu_0\epsilon_0 v^2}{2} = \frac{v^2}{2c^2}.$$
 (443)

In solid conductors we always have $v/c \ll 1$. In metal conduction, v/c is seldom much greater than 10^{-10} , so $(\Delta r)/r \approx 10^{-20}$ is too small to detect. In highly ionized gases, however, the "pinch effect," as it is called, can be not only detectable but important.

If the effect were large enough to measure, a Hall probe in the spirit of Fig. 6.33 could be used, with one lead connected to the axis (by drilling a thin tube in the rod), and the other lead connected to the surface of the rod. If $v \approx 10^{-3} \,\mathrm{m/s}$ and $B \approx 1 \,\mathrm{T}$, the resulting $E \approx 10^{-3} \,\mathrm{V/m}$ would be large enough to generate a measurable voltage difference.

6.41. Integral of A, flux of B

Using Stokes' theorem, along with $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\int_{C} \mathbf{A} \cdot d\mathbf{s} = \int_{S} \nabla \times \mathbf{A} \cdot d\mathbf{a} = \int_{S} \mathbf{B} \cdot d\mathbf{a} = \Phi, \qquad (444)$$

as desired. This relation is similar to Ampere's law because the differential form of that law, $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$, takes the same form as the above $\mathbf{B} = \nabla \times \mathbf{A}$ relation.

6.42. Finding the vector potential

Since $\mathbf{B} = \nabla \times \mathbf{A}$, we want

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0, \qquad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0, \qquad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0.$$
(445)

From inspection, a few choices for **A** that satisfy these equations are $\mathbf{A} = (0, B_0 x, 0)$, or $(-B_0 y, 0, 0)$, or $(-B_0 y/2, B_0 x/2, 0)$. In general, any vector of the form (-ay, bx, 0) works if $a + b = B_0$. And even more generally, adding on any vector with zero curl also works.

6.43. Vector potential inside a wire

Since area is proportional to r^2 , the current contained within a radius r is $I_r = Ir^2/r_0^2$. The magnitude of the magnetic field at radius r is then

$$B(r) = \frac{\mu_0 I_r}{2\pi r} = \frac{\mu_0 I r}{2\pi r_0^2},$$
(446)

and it points in the positive $\hat{\theta}$ direction. The $\hat{\theta}$ vector equals (-y/r, x/r, 0) because this vector has length 1 and has zero dot product with the radial vector (x, y, 0). So the Cartesian components of **B** are

$$B_x = -\frac{y}{r}B = -\frac{\mu_0 I y}{2\pi r_0^2}, \quad \text{and} \quad B_y = \frac{x}{r}B = \frac{\mu_0 I x}{2\pi r_0^2}.$$
 (447)

The magnetic field associated with the potential $\mathbf{A} = A_0 \hat{\mathbf{z}} (x^2 + y^2)$ is

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{x}} \frac{\partial A_z}{\partial y} - \hat{\mathbf{y}} \frac{\partial A_z}{\partial x} = 2A_0 y \hat{\mathbf{x}} - 2A_0 x \hat{\mathbf{y}}.$$
 (448)

This agrees with the **B** in Eq. (447) if $A_0 = -\mu_0 I/4\pi r_0^2$.

Alternatively, in cylindrical coordinates we have $\mathbf{A} = A_0 \hat{\mathbf{z}} r^2$. From Eq. (F.2) in Appendix F the associated magnetic field is $\mathbf{B} = \nabla \times \mathbf{A} = -(\partial A_z/\partial r)\hat{\boldsymbol{\theta}} = -2A_0 r\hat{\boldsymbol{\theta}}$. Comparing this with the *B* in Eq. (446), which points in the positive $\hat{\boldsymbol{\theta}}$ direction, we find $A_0 = -\mu_0 I/4\pi r_0^2$, as above.

Since A_0 is negative, **A** points in the direction opposite to the current (which points in the positive $\hat{\mathbf{z}}$ direction). You might be wondering how this can be, in view of the fact that Eq. (6.44) seems to say that **A** points in the same direction as **J**. The answer is that we can add an arbitrary constant to the **A** in Eq. (6.44), and it will still yield the same value of $\mathbf{B} = \nabla \times \mathbf{A}$. Adding on a sufficiently large vector pointing in the negative $\hat{\mathbf{z}}$ direction will make **A** point opposite to **J**.

6.44. Line integral along the axis

The magnetic field on the axis is $B_z = \mu_0 I b^2 / 2(b^2 + z^2)^{3/2}$, so the given line integral is (using the integral table in Appendix K)

$$\int_{-\infty}^{\infty} B_z \, dz = \frac{\mu_0 I b^2}{2} \int_{-\infty}^{\infty} \frac{dz}{(b^2 + z^2)^{3/2}} = \frac{\mu_0 I b^2}{2} \frac{z}{b^2 (b^2 + z^2)^{1/2}} \Big|_{-\infty}^{\infty} = \frac{\mu_0 I b^2}{2} \frac{2}{b^2} = \mu_0 I,$$
(449)

as desired. If you want, you can derive this integral with a trig substitution, $z = b \tan \theta$.

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To see why the integral along the axis should indeed be equal to $\mu_0 I$, consider the closed path shown in Fig. 110, which involves a semicircle touching the points $z = \pm r$. Assume that $r \gg b$. Along the z axis, B_z behaves like $1/z^3$ for $z \gg b$. And $|\mathbf{B}|$ also behaves like $1/r^3$ along the (large) semicircle. Accepting that this is true (see below), then since the length of the semicircle is proportional to r, the line integral along the semicircle is at least as small (in order of magnitude) as $r/r^3 = 1/r^2$, which goes to zero as $r \to \infty$. We can therefore ignore the return semicircular path. So the line integral along the whole loop (which encloses a current I) equals the line integral along the z axis, in the $r \to \infty$ limit.

Let's now argue why $|\mathbf{B}|$ behaves like $1/r^3$ for large r. Consider the point at the "side" of the semicircle in Fig. 110. In order of magnitude, the field at this point, due to the ring, is the same as the field due to a square with side b. But the field due to the square has contributions from two opposite sides (the sides perpendicular to the $\hat{\mathbf{r}}$ vector) that nearly cancel, because the current moves in opposite directions along these sides. The Biot-Savart law says that each side gives a contribution of order $1/r^2$. Taking the difference of these contributions is essentially the same as taking a derivative, and the derivative of $1/r^2$ is proportional to $1/r^3$, as desired. Additionally, the two sides parallel to the $\hat{\mathbf{r}}$ vector also happen to produce a contribution of order $1/r^3$; see Problem 6.14. At points in between the axis and the "side" point on the semicircle, there will be various angles that come into play. But these simply bring in factors of order 1 that morph the $1/z^3$ result on the axis to the $1/r^3$ result at the side point, so they don't change the general $1/r^3$ result.

6.45. Field from an infinite wire

Consider a small piece of the wire at angle θ , subtending an angle $d\theta$, as shown in Fig.111. If r is the distance from a given point P to the small piece, then Fig.112 shows that the length of the piece is $dl = r d\theta / \cos \theta$. But r equals $b / \cos \theta$, so $dl = b d\theta / \cos^2 \theta$. (This can also be obtained by taking the differential of $l = b \tan \theta$.) In the Biot-Savart law, the cross product between dl and $\hat{\mathbf{r}}$ brings in a factor of $\sin \phi$, which is the same as $\cos \theta$. If the current points rightward, then we have (with $\hat{\mathbf{z}}$ pointing out of the page)

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{(b \, d\theta/\cos^2) \cos\theta}{(b/\cos\theta)^2} \hat{\mathbf{z}}$$
$$= \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi b} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi b} \sin\theta \Big|_{-\pi/2}^{\pi/2} = \hat{\mathbf{z}} \frac{\mu_0 I}{2\pi b}, \qquad (450)$$

which agrees with the standard result obtained more much quickly via Ampere's law.

6.46. Field from a wire frame

- (a) In Fig. 113 the contributions to the field at P from the diagonally opposite edges AB and EF cancel, as do those from CD and GH. The pair BC and FG produces a field that points in the y direction at P, as does the pair HA and DE. So the total magnetic field at P points in the positive y direction.
- (b) Imagine superposing on the given setup the current in the two horizontal squares shown in Fig. 114. The currents along six of the edges cancel, and we end up with the desired square loop of current. But the field at P due to the two squares in Fig. 114 is zero, due to a symmetry argument. (Imagine rotating the setup by 180° around either the x or y axis. The setup is unchanged, so the magnetic field must point along both the x and y axes. The zero vector is the only vector with







Figure 111



Figure 112





Figure 114

this property.) Or you can just note that diagonally opposite edges combine to give zero field at P, as we saw in part (a). Therefore, since we added on zero field at P, the field at P in the original setup must be the same as the field at P in the case of the single square loop.

6.47. Field at the center of an orbit

The time for one revolution is $t = 2\pi r/v$, so the average current is $I = e/t = ev/2\pi r$. From Eq. (6.54) the field at the center of the orbit is therefore

$$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 ev}{4\pi r^2} = \frac{\left(4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^2}\right) (1.6 \cdot 10^{-19} \,\text{C}) (0.01 \cdot 3 \cdot 10^8 \,\text{m/s})}{4\pi (10^{-10} \,\text{m})^2} = 4.8 \,\text{T.}$$
(451)

6.48. Fields from two rings

The Biot-Savart law is $d\mathbf{B} = (\mu_0/4\pi)I \, d\mathbf{l} \times \hat{\mathbf{r}}/r^2$. Consider corresponding pieces of the two rings that subtend the same angle $d\theta$. The dl for the larger piece is twice the dl for the smaller piece. And the I for the larger ring is also twice the I for the smaller ring, because I is proportional to the speed of the ring, which in turn is proportional the radius, because the ω 's are the same. These two powers of 2 in the numerator cancel the two rings are the same. This reasoning works for any ratio of ring sizes, of course. In terms of the various parameters, you can show that the field at the center is $B = \mu_0 \lambda \omega/2$, which is independent of r, as we just showed.

6.49. Field at the center of a disk

Consider a ring with radius r and thickness dr. The effective linear charge density along the ring is $d\lambda = \sigma dr$. The speed of all points on the ring is $v = \omega r$, so the current in the ring is $dI = (d\lambda)v = (\sigma dr)(\omega r)$. From the Biot-Savart law, a small piece of the ring with length dl produces a $d\mathbf{B}$ field at the center that points perpendicular to the ring and has magnitude $(\mu_0/4\pi)I dl/r^2$. Integrating over the whole ring turns the dl into $2\pi r$, so the field at the center due to the ring is $(\mu_0/4\pi)(\sigma\omega r dr)(2\pi r)/r^2 =$ $\mu_0\sigma\omega dr/2$. Integrating over r (that is, integrating over all the rings in the disk) turns the dr into an R, so the field at the center equals $\mu_0\sigma\omega R/2$. It points perpendicular to the disk, with the direction determined by the righthand rule.

6.50. Hairpin field

Each of the two straight segments contributes half the field of an infinite wire. (The contributions do indeed add and not cancel.) The semicircle contributes half the field of an entire ring at the center, which is given by Eq. (6.54). The desired field therefore points out of the page and has magnitude

$$B = 2 \cdot \frac{1}{2} \frac{\mu_0 I}{2\pi r} + \frac{1}{2} \frac{\mu_0 I}{2r} = \left(\frac{1}{2\pi} + \frac{1}{4}\right) \frac{\mu_0 I}{r} = (0.409) \frac{\mu_0 I}{r} \,. \tag{452}$$

6.51. Current in the earth

If the current is essentially all located on the equator of the core, then we can use the expression for the *B* field due to a ring. From Eq. (6.53) the field along the axis of the ring is $B_z = \mu_0 I b^2 / 2 (b^2 + z^2)^{3/2}$. In the situation at hand, b = R/2 and z = R, where *R* is the radius of the earth. So we have

$$B_z = \frac{\mu_0 I(R/2)^2}{2((R/2)^2 + R^2)^{3/2}} = \frac{\mu_0 I}{5\sqrt{5}R}.$$
(453)

If this field equals 0.5 gauss = $5\cdot 10^{-5}$ T, then I must be (taking the earth's radius to be $R=6\cdot 10^6\,{\rm m})$

$$I = \frac{5\sqrt{5}RB_z}{\mu_0} = \frac{5\sqrt{5}(6\cdot10^6\,\mathrm{m})(5\cdot10^{-5}\,\mathrm{T})}{4\pi\cdot10^{-7}\,\frac{\mathrm{kg\,m}}{\mathrm{C}^2}} = 2.7\cdot10^9\,\mathrm{A},\tag{454}$$

which is huge. For comparison, the peak current in a bolt of lightning is on the order of 10^5 A.

6.52. Right-angled wire

Consider the contribution from the positive x-axis part of the wire. In the notation of Fig. 115, the Biot-Savart law gives

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I \hat{\mathbf{z}}}{4\pi} \int_0^\infty \frac{\sin \theta \, dx'}{r^2} = \frac{\mu_0 I \hat{\mathbf{z}}}{4\pi} \int_0^\infty \frac{y \, dx'}{r^3} \,. \tag{455}$$

If y is positive (or negative), then this **B** points in the positive (or negative) $\hat{\mathbf{z}}$ direction, that is, out of (or into) the page. Writing r in terms of the Cartesian coordinates gives (with $x'' \equiv x' - x$, and using the integral table in Appendix K)

$$B_{z} = \frac{\mu_{0}Iy}{4\pi} \int_{0}^{\infty} \frac{dx'}{[y^{2} + (x' - x)^{2}]^{3/2}} = \frac{\mu_{0}Iy}{4\pi} \int_{-x}^{\infty} \frac{dx''}{[y^{2} + x''^{2}]^{3/2}}$$
$$= \frac{\mu_{0}Iy}{4\pi} \frac{x''}{y^{2}[y^{2} + x''^{2}]^{1/2}} \Big|_{-x}^{\infty} = \frac{\mu_{0}I}{4\pi} \left(\frac{1}{y} + \frac{x}{y\sqrt{x^{2} + y^{2}}}\right).$$
(456)

If you write out the contribution from the positive y axis, you will find that it takes the same form, except with x and y reversed. We therefore end up with the desired result.

We can check a limit: If $x \gg y$, we should end up with the field at a distance y from an infinite wire lying along the x axis. And indeed, if $x \gg y$, two of the terms in the result in Eq. (6.98) are negligible, and two of the terms are essentially equal to $(\mu_0 I/4\pi)(1/y)$. So we end up with $\mu_0 I/2\pi y$, as expected.

6.53. Superposing right angles

An infinite straight wire carrying current in the positive direction along the x axis is the superposition of the two right-angled wires shown in Fig. 116. So we need to find the fields at (x, y) due to these two wires. The field from the right wire is just the field derived in Exercise 6.52. The field from the left wire is the same as the field at the point shown in the setup in Fig. 117 (we have simply rotated the setup in Fig. 116 by 90°). But this field can be obtained by taking the result from Exercise 6.52 and setting the x value equal to y, and the y value equal to -x, as shown. So the field due to the left right-angled wire is

$$B_z = \frac{\mu_0 I}{4\pi} \left(\frac{1}{y} + \frac{1}{-x} + \frac{y}{-x\sqrt{y^2 + x^2}} + \frac{-x}{y\sqrt{y^2 + x^2}} \right).$$
(457)

When we add this to the field from the right wire given in Eq. (6.98), the last three terms in Eq. (457) cancel. We therefore end up with $B_z = (\mu_0 I/4\pi)(2/y) = \mu_0 I/2\pi y$, as desired, because y is the distance from the wire (the x axis).



Figure 115



Figure 116



Figure 117

6.54. Force between a wire and a loop

Consider a little segment in the right-hand side of the square. The current points into the page, and the magnetic field due to the infinite straight wire has magnitude $B_1 = \mu_0 I_1/2\pi R$ and points down to the left, as shown in Fig. 118. From the right-hand rule, the force $q\mathbf{v} \times \mathbf{B}$ on the charges in the current points up to the left, as shown (toward the infinite wire; parallel currents attract). In the left-hand side of the square, the current points out of the page, and the magnetic field due to the infinite wire points up to the left, as shown. The force $q\mathbf{v} \times \mathbf{B}$ on the charges in the current now points down to the left (away from the infinite wire; antiparallel currents repel). The vertical components of the preceding two forces cancel, but the leftward components add. So the net force is leftward, as desired. You can quickly show that the net force on each of the other two sides of the square is zero.



Figure 118

In short, it is the vertical component of **B** that matters, because this component changes sign from the right half to the left half of the square. And the direction of the square's current into and out of the page also changes sign. So these two negative signs cancel in $q\mathbf{v} \times \mathbf{B}$, yielding a net leftward force. In contrast, the horizontal component of **B** does *not* change sign, so the negation of the current causes a negation of the vertical force. The net vertical force is therefore zero.

Quantitatively, the general form of the force on a wire is $F = IB\ell$. The "B" we are concerned with here is the vertical component, which is $B \sin \beta$. The force comes from two sides, so the total horizontal force is

$$F = 2I_2(B_1 \sin \beta)\ell = 2I_2\left(\frac{\mu_0 I_1}{2\pi R} \cdot \frac{\ell/2}{R}\right)\ell = \frac{\mu_0 I_1 I_2 \ell^2}{2\pi R^2},$$
(458)

where we have used $\sin \beta = (\ell/2)/R$.

The above reasoning shows where the two factors of R in the denominator come from. One comes from the distance to the wire, and the other comes from the fact that the **B** field becomes more horizontal (which means that the vertical component decreases) as R gets large.

We weren't concerned with torques in this exercise, but from looking at the vertical forces on the left and right sides (which come from the horizontal component of \mathbf{B}), it is clear that there is a torque on the square. It will rotate counterclockwise when viewed

from the side. This is consistent with conservation of angular momentum, because the straight wire will gain angular momentum (relative to, say, an origin chosen to be the center of the square) as it moves to the right. This angular momentum will have a clockwise sense, consistent with the fact that the total angular momentum of the system remains constant.

6.55. Helmholtz coils

Let the symmetry axis of the setup be the z axis, and let the centers of the rings be located at $z = \pm b/2$. If the currents in the rings are equal and point in the same direction, then from Eq. (6.53) the field along the axis at position z is given by

$$B_z(z) \propto \frac{1}{[a^2 + (z+b/2)^2]^{3/2}} + \frac{1}{[a^2 + (z-b/2)^2]^{3/2}}.$$
(459)

If we expand this function in a Taylor series around z = 0, the first derivative and all other odd derivatives are zero at z = 0, because $B_z(z)$ is an even function of z, due to the symmetry of the setup. So the function will be most uniform near z = 0 if the second derivative is zero there. The deviations will then be of order z^4 . That is, the Taylor series will look like $B_z(z) = B_z(0) + Cz^4 + \cdots$. Differentiating the first term in Eq. (459) twice and evaluating the result at z = 0 yields

$$3\frac{5(z+b/2)^2 - [a^2 + (z+b/2)^2]}{[a^2 + (z+b/2)^2]^{7/2}}\Big|_{z=0} = \frac{3(b^2 - a^2)}{[a^2 + b^2/4]^{7/2}}.$$
(460)

The second derivative of the second term in Eq. (459) simply involves replacing b/2 with -b/2, so we end up with the same result, because there are no odd powers of b in Eq. (460). We therefore see that the second derivative is zero at z = 0 if a = b. You can show that if a = b, the field halfway from z = 0 to the plane of each ring (that is, at $z = \pm b/4$) is only 0.4% smaller than the field at z = 0. And at $z = \pm b/8$ the field is only 0.03% smaller. Two coils arranged with a = b are called Helmholtz coils.

A continuity argument shows why there must exist a point where the second derivative of $B_z(z)$ equals zero. If the rings are far apart (for example, if b = 4a), then the plot of B_z consists of two bumps, as shown in Fig. 119. But if the rings are close together (for example, if b = a/4), then they act effectively like one ring with twice the current, so there is just one bump. The second derivative at z = 0 is positive in the former case, and negative in the latter, so somewhere in between it must be zero.



Figure 119

6.56. Field at the tip of a cone

Consider a circular strip with width dx, a slant-distance x from the tip. The velocity of any point in this strip is $v = \omega(x \sin \theta)$. The amount of charge in the strip that passes a given point during time dt is $dq = \sigma(dx)(v dt) = \sigma(dx)(\omega x \sin \theta) dt$. The current in the strip is therefore $I = dq/dt = \sigma \omega x \sin \theta dx$. (Equivalently, you can use the general result $I = \lambda v$, where $\lambda = \sigma dx$ is the effective linear charge density of the ring.)

From the Biot-Savart law, a small piece of the strip with length ds at the location shown in Fig. 120 produces a $d\mathbf{B}$ field at the tip that points up to the right, with magnitude $(\mu_0/4\pi)I ds/x^2$. When we integrate over the whole strip, the horizontal components of the $d\mathbf{B}$'s cancel, and we are left with only a vertical component. This brings in a factor of $\sin \theta$.

For a given strip, the ds in the Biot-Savart law integrates up to the length of the strip, which is $s = 2\pi(x \sin \theta)$. The contribution to the (vertical) field from a given strip at slant-distance x, with width dx, is therefore

$$\hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{Is}{x^2}\sin\theta = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{(\sigma\omega x\sin\theta\,dx)(2\pi x\sin\theta)}{x^2}\sin\theta = \hat{\mathbf{z}}\frac{1}{2}\mu_0\sigma\omega\sin^3\theta\,dx.$$
 (461)

Integrating from x = 0 to x = L simply gives a factor of L, so the field at the tip is

$$\mathbf{B} = \hat{\mathbf{z}} \frac{1}{2} \mu_0 \sigma \omega L \sin^3 \theta. \tag{462}$$

If $\theta = 0$ we correctly obtain zero field. If $\theta = \pi/2$ we obtain $\mathbf{B} = \hat{\mathbf{z}}\mu_0 \sigma \omega L/2$. In this case we just have a flat disk with radius L, and this is indeed the field at the center; see Exercise 6.49.

To check that the units of **B** are correct, we can compare it with the *B* due to a wire, which is $\mu_0 I/2\pi r$. And indeed, $\sigma \omega L$ correctly has the same units as I/r.

Note that the result in Eq. (461) is independent of x, so all rings with the same thickness dx give the same contribution to the field. The reason for this is that the larger a ring is, the larger the current I and length s are, and these effects cancel the effect of the x^2 in the denominator of the Biot-Savart law. Note also where the three factors of $\sin \theta$ come from. For given values of the other parameters, a larger θ means a larger velocity (and hence current), a larger circumference s, and a larger vertical component of each of the $d\mathbf{B}$'s.

6.57. A rotating cylinder

Eq. (6.57) gives the magnetic field inside an infinite solenoid as $B = \mu_0 n I$, where n is the number of turns per unit length. The surface current density (per unit length) is $\mathcal{J} = n I$, so we can write the field as $B = \mu_0 \mathcal{J}$.

What is the current density in our rotating cylinder? The amount of charge that passes a given segment of length ℓ on the cylinder in a time dt is $dq = \sigma \ell(v dt)$. The current per unit length (that is, the surface current density) is therefore $\mathcal{J} = (1/\ell)(dq/dt) = \sigma v$. In terms of the angular frequency, \mathcal{J} equals $\sigma \omega R$.

To find the field inside the rotating cylinder, we simply need to replace the current density $\mathcal{J} = nI$ in the original solenoid formula with the present current density $\mathcal{J} = \sigma \omega R$. This yields a field of $B = \mu_0 \sigma \omega R$.




6.58. Rotating cylinders

We must first find the charge per unit length (in the direction of the axis), λ , on the inner cylinder. The electric field between the cylinders is $E(r) = \lambda/2\pi\epsilon_0 r$. The magnitude of the potential difference is therefore $\phi = \int E dr = (\lambda/2\pi\epsilon_0) \ln(r_2/r_1)$, where r_1 and r_2 are the radii of the inner and outer cylinders, respectively. So

$$\lambda = \frac{2\pi\epsilon_0\phi}{\ln(r_2/r_1)} = \frac{2\pi \left(8.85 \cdot 10^{-12} \frac{\mathrm{s}^2 \mathrm{C}^2}{\mathrm{kg} \mathrm{m}^3}\right) (1.5 \cdot 10^4 \,\mathrm{V})}{\ln(4/3)} = 2.9 \cdot 10^{-6} \,\mathrm{C/m}. \tag{463}$$

If the inner cylinder rotates at 30 rev/sec, then it is a solenoidal surface with a current density equal to $\mathcal{J} = (30 \,\mathrm{s}^{-1})(2.9 \cdot 10^{-6} \,\mathrm{C/m}) = 8.7 \cdot 10^{-5} \,\mathrm{C/(s\,m)}$, because in 1 meter of the cylinder, each revolution carries a charge of $2.9 \cdot 10^{-6} \,\mathrm{C}$. Within the inner cylinder the field is therefore $B = \mu_0 \mathcal{J} = 1.09 \cdot 10^{-10} \,\mathrm{T}$. (The continuum limit of the $B = \mu_0 \mathcal{I}$) If the cylinder rotates counterclockwise as we look along the axis, and if the inner cylinder is the positive one, the field points out of the page (toward us). Outside the inner cylinder (that is, for $r > r_1$), the field is zero.

If both cylinders rotate at 30 rev/sec in the counterclockwise direction, the outer cylinder produces a field of equal strength but opposite direction in its interior (because it has the opposite charge per unit length, assuming the cylinders started neutral before the potential difference was created). The fields therefore cancel inside the inner cylinder. So B = 0 for $r < r_1$. And B = 0 for $r > r_2$, of course. But between the cylinders (that is, for $r_1 < r < r_2$), the field is $B = 1.09 \cdot 10^{-10}$ T, pointing into the page.

6.59. Scaled-down solenoid

(a) The resistance of the winding in the small solenoid is 10 times that of the large solenoid. This is true because the resistance is given by $R = \rho L/A$, and the small wire is 1/10 as long, with 1/100 the cross-sectional area. So if we apply the same voltage of 120 V to the small solenoid, we get 1/10 the current. This is just what is needed to produce a magnetic field equal to that in the large solenoid, because the field is proportional to nI, and the small coil has 10 times as many turns per unit length, each with 1/10 the current. Equivalently, the surface current density \mathcal{J} is the same in the small coil (it has 10 times as many wires per unit length, with 1/10 the current in each), and \mathcal{J} is all that matters for a solenoid, because the field inside is $B = \mu_0 \mathcal{J}$.

Symbolically, we have $B \propto nI = n(V/R) = nV/(\rho L/A) = (V/\rho)(nA/L)$. But the quantity nA/L is dimensionless (the units are m⁻¹m²/m), so it can't depend on the length scale of the solenoid. Therefore, if V and ρ are the same in both setups, then B is also the same.

(b) The power is P = IV, so it is smaller by a factor of 10 in the smaller solenoid, because V is the same and I is smaller by a factor of 10. But the smaller solenoid has only 1/100 the surface area (because area is proportional to length squared), so it will be harder to keep it cool; any cooling mechanism operates by interacting with the surface of the wire.

6.60. Zero field outside a solenoid

Let the solenoid extend in the z direction. Consider one of the small patches. The current in this patch flows in some direction in the xy plane. That is, the dl vector in the Biot-Savart law lies in the xy plane. (We can slice the patch into many thin strips represented by dl's.) The dl vector gets crossed with the $\hat{\mathbf{r}}$ vector directed to the point

P. Now, $\hat{\mathbf{r}}$ has a *z* component, but the $\hat{\mathbf{r}}$ vector associated with the corresponding patch defined by the thin cone on the other side of *P* has the opposite *z* component. These components therefore yield canceling contributions to the total magnetic field. So we need only worry about the component of $\hat{\mathbf{r}}$ that lies in the *xy* plane. Let's call this vector $\hat{\mathbf{r}}_{xy}$.

We need to compute the cross product of $d\mathbf{l}$ and $\hat{\mathbf{r}}_{xy}$, both of which lie in the xy plane. (The resulting cross product will therefore point in the z direction, so we have just proved that the B field from the solenoid must be longitudinal.) In general, $d\mathbf{l}$ has a component parallel to $\hat{\mathbf{r}}_{xy}$ and a component perpendicular to $\hat{\mathbf{r}}_{xy}$. The parallel component yields zero in the cross product $d\mathbf{l} \times \hat{\mathbf{r}}_{xy}$, so we need only worry about the component perpendicular to $\hat{\mathbf{r}}_{xy}$. In other words, if we project the area of the patch onto the (vertical) plane orthogonal to $\hat{\mathbf{r}}_{xy}$, then the cross product $d\mathbf{l} \times \hat{\mathbf{r}}_{xy}$ remains the same. We can do the same with the other patch in the same cone.

We therefore need to compare the Biot-Savart contributions from the two "projected" patches of area defined by a given cone. If the projected patches are distances r_1 and r_2 from the point P, then their areas are proportional to r_1^2 and r_2^2 , because areas are proportional to length squared, and because the patches cut the line from P at the same angle (perpendicular, by construction).

Now, if we imagine a small rectangular patch (any patch can be built up from rectangles), the I dl product in the Biot-Savart law is proportional to the area, because I is proportional to the height dh of the rectangle (since $I = \mathcal{J} dh$), and because dh dl is the area of the rectangle. The numerators in the Biot-Savart law for the corresponding patches are therefore proportional to r_1^2 and r_2^2 . These factors exactly cancel the r^2 in the denominator of the Biot-Savart law. So the magnitudes of the contributions from the two patches equal. And since the currents flow in opposite directions in the projected patches, the contributions therefore cancel. The entire solenoid can be considered to be built up from small patches subtended by cones, so the external field is zero, as desired.

If the solenoid isn't convex, then a given cone may define 4, 6, 8, etc., patches. But there will still be equal numbers of patches having currents in each direction (which can be traced to the fact that P has the property of being outside the solenoid), so the sum of the contributions will still be zero.

6.61. Rectangular torus

Consider two loops of current that are located symmetrically with respect to a plane through the axis of the torus; a top view is shown in Fig. 121. At any point on this plane (inside or outside the torus) the vector sum of the field due to Loop 1 and the field due to Loop 2 is perpendicular to the plane (or is zero). You can check this by looking at the Biot-Savart contributions from corresponding little pieces of the two loops; the components parallel to the plane cancel. In general, the \mathbf{B}_1 and \mathbf{B}_2 vectors shown also have components perpendicular to the page, but you can show that these components are equal and opposite.

This result is actually true for two similar loops of *any* (planar) shape carrying equal currents in the same orientation; the cross section of the torus doesn't have to be rectangular. The same Biot-Savart reasoning involving corresponding little pieces holds.

The entire coil can be decomposed into pairs of loops located symmetrically with respect to a given plane. Hence the total magnetic field at any point must be perpendicular to the plane containing that point and the axis (or be zero). In other words, the field points in the circumferential direction.



Figure 121

To find the magnitude of the field, we can use Ampere's law. By symmetry, the magnetic field must have the same magnitude B everywhere on a circle of radius r around the axis. The line integral of **B** around this circle equals μ_0 times the current enclosed. Since **B** points in the tangential direction, the line integral equals $2\pi rB$. If the circle doesn't lie inside the torus, the current enclosed is zero. This is true because either the disk defined by the circle doesn't intersect the torus, in which case the current enclosed is clearly zero; or the disk does intersect the torus, in which case a current of NI passes through the disk in one direction, but another NI also passes through in the other direction. Therefore, **B** = 0 everywhere outside the torus.

On the other hand, if the circle lies inside the torus, the current enclosed is NI, because the disk defined by the circle intersects only the inner boundary of the torus. Therefore, $2\pi rB = \mu_0 NI \implies B = \mu_0 NI/2\pi r$ inside the torus. This expression for B holds for a torus of any (uniform) cross section. Note that B depends only on r, and not on the "height" inside the torus.

In the limit where $b-a \ll a$, the curvature of the torus is negligible, so we essentially have an infinite straight solenoid with rectangular cross section. The field should therefore equal $\mu_0 nI$ (see the solution to Problem 6.19), where n is the number of turns per unit length. And indeed, in the above result, $n = N/2\pi r$ is the number of turns per unit length (where r is essentially equal to both a and b), so we do obtain $B = \mu_0 nI$.

6.62. Creating a uniform field

Since 10 milligauss is about 2% of the earth's field, we need a compensating field of approximately 0.55 gauss that is uniform to about 2% over the region of interest. Let's try a solenoid 1 meter long and 50 cm in diameter; see Fig. 122. To see whether it meets the requirement, compare the field at the center C with the field on the axis 15 cm from the center, at D. From Eq. (6.56) we have

$$\frac{\text{field at } C}{\text{field at } D} = \frac{2\cos\theta_0}{\cos\theta_1 + \cos\theta_2} \,. \tag{464}$$

The various angles are given by

$$\theta_0 = \tan^{-1}(25/50) \implies \cos \theta_0 = .8944, \theta_1 = \tan^{-1}(25/35) \implies \cos \theta_1 = .8137, \theta_2 = \tan^{-1}(25/65) \implies \cos \theta_2 = .9333.$$
 (465)

The ratio of the fields is therefore 1.024. So the deviation is about 2.4%. This is a little too large for comfort, especially as we have no easy way to estimate the deviation at off-axis points such as E. Let's lengthen the solenoid to 1.2 meters. The denominators in the above arctans are now 60, 45, and 75, respectively, and you can quickly show that the ratio of the fields is now 1.013.

We expect the departure from uniformity in the radial direction to be of the same magnitude, roughly, as the variation in the axial direction. But it has the opposite sign; the field strength at E is *larger* than that at C. This makes sense, because in the limit where the solenoid is very squat, so that it basically looks like a ring, we know that the field increases (without bound, in fact) as we move away from the center toward the circumference. An exact calculation of the field strength very close to the solenoid at F, which involves an elliptic integral, shows it to be 1.4% greater than the field strength at C, in the case of the 1.2 meter solenoid.



Figure 122

The number of ampere turns, NI, required to make the field of the solenoid at C equal to the earth's field, $5.5 \cdot 10^{-5}$ T, is found from Eq. (6.56). The number of turns per unit length is n = N/(1.2 m), so we have

$$\frac{\iota_0(N/(1.2 \text{ m}))I}{2} (2\cos\theta_0) = 5.5 \cdot 10^{-5} \text{ T}$$

$$\implies NI = \frac{(1.2 \text{ m})(5.5 \cdot 10^{-5} \text{ T})}{(4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^2})(0.923)}$$

$$= 57 \text{ ampere-turns.}$$
(466)

6.63. Solenoids and superposition

(a) Imagine adding a similar solenoid on the left, as shown in Fig. 123. This exactly doubles the field strength at P_2 . But now the field strengths at P_2 and P_1 are approximately equal, because both points lie well inside a fairly long solenoid, the field at P_2 being slightly stronger. Therefore, the original field at P_2 must have been slightly more than half the field at P_1 .

This "more than half" result is consistent with the extreme case where the solenoid is very short, basically just a ring. In this case the fields at P_2 and P_1 are essentially equal, both taking on the value of the field at the center of a ring, namely $\mu_0 I/2r$. So the field at P_2 is certainly more than half of the field at P_1 .

A less elegant way of solving this exercise is to use Eq. (6.56). The field at the center is proportional to $2 \cos \alpha_1$, and the field at the end is proportional to $\cos \alpha_2$ (plus $\cos 90^\circ$, which is zero), where these angles are defined in Fig. 124. For small angles, both of these cosines are essentially equal to 1, hence the ratio of 1/2 in the fields. But $\cos \alpha_2 > \cos \alpha_1$, hence the "more than half."

(b) Let's assume (in search of a contradiction) that there exists a field line that crosses the line GH with a vertical component, as shown in Fig. 125(a). Imagine flipping the solenoid upside down to obtain the situation in Fig. 125(b), and then reversing the direction of the current (so that it now has the same direction as in Fig. 125(a)) to obtain the situation in Fig. 125(c). Note that the field at the given point on the line GH has a downward component in both figures (a) and (c) (or upward in both, if we had initially drawn it upward).



Figure 125

Now join the two semi-infinite solenoids in figures (a) and (c) end to end, thereby creating an infinite solenoid. By superposition, the fields simply add, so we end up with a downward component at the given point along GH. But this is a contradiction, because we know that the field of an infinite solenoid is zero outside the solenoid. We conclude that the field due to the semi-infinite solenoid

$$\bullet P_2 \bullet P_1$$





Figure 124

- (c) The argument used in part (a), applied to the semi-infinite solenoid, shows that the axial component of the field, at any point on the end face is exactly $B_0/2$, where B_0 is the (uniform) field throughout the inside the corresponding infinite solenoid. This is true because adding another semi-infinite solenoid simply doubles the axial field at any point on the end face (see the reasoning in part (b)), and cancels the radial field, resulting in a purely axial field. As far as the flux goes, when calculating the flux through the end face, only the axial field component is involved. Therefore, the flux must be exactly half the interior flux.
- (d) From the reasoning in part (c), a given flux tube that starts with area A far back in the solenoid must flare out as it approaches the end face, so that its cross section there (where the axial field is half as large as the field far back in the solenoid) has area 2A and thus the same amount of flux. (There can be no net flux into or out of the tube, since div $\mathbf{B} = 0$.) In the special case of an axial tube with circular cross section everywhere, this tells us that $\pi r_1^2 = 2\pi r_0^2 \implies r_1 = \sqrt{2} r_0$. Of course, this holds only if $r_0 < R/\sqrt{2}$, where R is the radius of the solenoid. Otherwise, the field line exits the solenoid before it reaches the end.

REMARK: The arguments used in parts (b) and (c) lead to more general statements about the field of a semi-infinite solenoid. Consider two points P and P' symmetrically located with respect to the end plane, as shown in Fig. 126. The fields **B** and **B'** are related as follows: The radial components of **B** and **B'** are equal. The sum of the axial components of **B** and **B'** is equal to B_0 if P lies inside the solenoid, or to zero if P lies outside the solenoid (that is, above the top "edge" of the solenoid in the figure). The conclusions of parts (b) and (c) follow in the special case where P and P' coincide.

6.64. Equal magnitudes

This setup could be created by taking the setup in Fig. 6.22(a) and superposing a magnetic field pointing out of the page, with magnitude equal to the magnitude of the fields in Fig. 6.22(a) (which is $\mu_0 \mathcal{J}/2$). The field on the left then points out of the page and down at a 45° angle, and the field on the right points out of the page and up at a 45° angle. So they are perpendicular, as desired.

From Eq. (6.63), the force per unit area on the sheet is $(B_1^2 - B_2^2)/2\mu_0$. But the magnitudes B_1 and B_2 are equal, so the force is zero. This is no surprise, after all, because the *B* field we superposed pointed out of the page, which was exactly the direction of the current in the sheet in Fig. 6.22(a). So the force on the moving charges is zero. In general, if we superpose a *B* field that points in the same direction as the current, the force will be zero. And consistent with this, B_1 and B_2 will be equal, although in only one special case will the two fields be perpendicular. In general, they will simply make equal angles with the direction of the current.

Similarly, if we superpose a B field perpendicular to the sheet, the magnitudes of B_1 and B_2 will be equal. So the force on the sheet will be zero, consistent with the fact that the $q\mathbf{v} \times \mathbf{B}$ force on the charges in the current sheet lies in the plane of the sheet.

The only way for the magnitudes of B_1 and B_2 to be unequal is for the superposed B field to have a component along the direction of the original fields. There will then be a nonzero force on the sheet. Consistent with this, the $q\mathbf{v} \times \mathbf{B}$ force on the charges in the current sheet now has a component perpendicular to the sheet.



Figure 126

6.65. Proton beam

(a) The total energy of each proton is 3 GeV, so the γ factor is 3 (because the total energy of a particle is γmc^2). Hence $\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{8/9} = 0.943$. The current is 10^{-3} C/s, so $I = \lambda v$ gives the linear charge density as

$$\lambda = \frac{I}{v} = \frac{10^{-3} \,\mathrm{C/s}}{0.943 \cdot 3 \cdot 10^8 \,\mathrm{m/s}} = 3.53 \cdot 10^{-12} \,\mathrm{C/m}.$$
 (467)

The electric field 1 cm from the axis of the beam is

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{3.53 \cdot 10^{-12} \,\mathrm{C/m}}{2\pi \left(8.85 \cdot 10^{-12} \,\frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg \,m^3}}\right)(0.01 \,\mathrm{m})} = 6.35 \,\mathrm{V/m}.$$
 (468)

(b) The magnetic field is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\left(4\pi \cdot 10^{-7} \,\frac{\text{kg m}}{\text{C}^2}\right) (10^{-3} \,\text{C/s})}{2\pi (0.01 \,\text{m})} = 2 \cdot 10^{-8} \,\text{T.}$$
(469)

(c) In F' there is no current because the protons are at rest, so B = 0. The spacing between the protons is "uncontracted," so the density is smaller; it is $\lambda' = \lambda/\gamma$. The electric field is therefore $E' = E/\gamma = 2.12$ V/m.

6.66. Fields in a new frame

In frame F, the electric field components are $E_x = 100 \cos 30^\circ \text{ V/m} = 86.6 \text{ V/m}$, $E_y = 100 \sin 30^\circ \text{ V/m} = 50 \text{ V/m}$, and $E_z = 0$. And also $\mathbf{B} = 0$. The transformations to frame F' are given by Eq. (6.76). The " \parallel " direction is along the y axis, and the " \perp " direction is in the x-z plane. \mathbf{v} is the velocity of F' with respect to F, so $\mathbf{v} = (0.6c)\hat{\mathbf{y}}$. Since $\mathbf{B} = 0$ the transformations reduce to:

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \qquad \mathbf{E}'_{\perp} = \gamma \mathbf{E}_{\perp}, \qquad \mathbf{B}'_{\parallel} = 0, \qquad \mathbf{B}'_{\perp} = -(\gamma/c^2)\mathbf{v} \times \mathbf{E}_{\perp}.$$
(470)

These yield (with $\gamma = 5/4$)

$$\mathbf{E}'_{\parallel} = \hat{\mathbf{y}} E_y \implies E'_y = E_y = 50 \,\mathrm{V/m}.$$

$$\mathbf{E}'_{\perp} = \gamma \hat{\mathbf{x}} E_x \implies E'_x = \gamma E_x = 108.3 \,\mathrm{V/m},$$
and
$$E'_z = 0.$$

$$\mathbf{B}'_{\parallel} = 0 \implies B'_y = 0.$$

$$\mathbf{B}'_{\perp} = -(\gamma/c^2) \mathbf{v} \times \mathbf{E}_{\perp} \implies B'_z = -(5/4)(1/c^2)(\hat{\mathbf{y}} 3c/5) \times (\hat{\mathbf{x}} E_x)$$

$$= (3/4)(E_x/c)\hat{\mathbf{z}} = 2.17 \cdot 10^{-7} \,\mathrm{T},$$
and
$$B'_z = 0.$$
(471)

The magnitude of \mathbf{E}' is $\sqrt{108.3^2 + 50^2} = 119.3 \text{ V/m}$. The angle that \mathbf{E}' makes with the $\hat{\mathbf{x}}'$ axis is $\tan^{-1}(50/108.3) = 24.8^{\circ}$. And \mathbf{B}' points directly along the $\hat{\mathbf{z}}'$ axis with magnitude $2.17 \cdot 10^{-7} \text{ T}$.

6.67. Fields from two ions

(a) FIRST SOLUTION: The electric field can be found via Eq. (5.15) in Chapter 5. If $\theta = 90^{\circ}$, we get an extra factor of γ compared with the static case, so the electric field at $(3\ell, 0, 0)$ is (with $\gamma = 5/4$ and $\ell = 1$ m)

$$E = \hat{\mathbf{x}} \frac{1}{4\pi\epsilon_0} \frac{\gamma e}{\ell^2} + \hat{\mathbf{x}} \frac{1}{4\pi\epsilon_0} \frac{\gamma e}{(3\ell)^2} = \hat{\mathbf{x}} \frac{1}{4\pi\epsilon_0} \frac{25}{18} \frac{e}{\ell^2} = 2 \cdot 10^{-9} \text{ V/m.}$$
(472)

SECOND SOLUTION: Let F' be the lab frame, and consider the frame F traveling upward with the left ion. Consider the field at (3, 0, 0) due to just the left ion. In frame F there is no **B** field from the left ion, so the transformations in Eq. (6.76) yield $\mathbf{E}'_{\perp} = \gamma \mathbf{E}_{\perp}$. That is, the electric field due to the left ion is larger in F' (the lab frame) by a factor γ . The same reasoning holds for the field due to the right ion (because the direction of the relative velocity of the frames doesn't matter in γ), so the solution proceeds as above.

(b) Consider the frames F and F' defined above. Eq. (6.76) gives $\mathbf{B}'_{\perp} = -(\gamma/c^2)\mathbf{v} \times \mathbf{E}_{\perp}$, where \mathbf{v} is the velocity of F' (the lab frame) with respect to F (the ion frame). For the left ion traveling upward, we have $\mathbf{v} = -v\hat{\mathbf{y}}$, where v = 3c/5. So the magnetic field in F' (the lab frame) at $(3\ell, 0, 0)$ due to the left ion is (using $\mu_0 = 1/\epsilon_0 c^2$)

$$\mathbf{B}'_{\perp,\text{left}} = -\frac{\gamma}{c^2} (-v\hat{\mathbf{y}}) \times \left(\frac{\hat{\mathbf{x}}e}{4\pi\epsilon_0(3\ell)^2}\right) = -\hat{\mathbf{z}}\frac{\gamma\mu_0 ev}{4\pi(3\ell)^2} \,. \tag{473}$$

Similarly, the magnetic field due to the right ion (for which F' moves with velocity $\mathbf{v} = v\hat{\mathbf{y}}$ with respect to F) is $\hat{\mathbf{z}}\gamma\mu_0 ev/4\pi\ell^2$. The sum is

$$\mathbf{B}_{\perp,\text{total}}' = \hat{\mathbf{z}} \frac{\mu_0}{4\pi} (\gamma v) \left(\frac{e}{\ell^2} - \frac{e}{(3\ell)^2} \right) = \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{2c}{3} \frac{e}{\ell^2} = 3.2 \cdot 10^{-18} \text{ T.}$$
(474)

6.68. Force on electrons moving together

- (a) Since there is no transverse length contraction, the distance between the electrons in the frame in which they are at rest is still r. The force on each electron is repulsive, and the magnitude is simply $e^2/4\pi\epsilon_0 r^2$. To transform this force to the lab frame, we can use the fact that the transverse force on a particle is always greatest in the rest frame of the particle. It is smaller in any other frame by the factor γ . The repulsive force in the lab frame is therefore $(1/\gamma)(e^2/4\pi\epsilon_0 r^2)$.
- (b) In the lab frame F', consider the E' and B' fields at the location of the bottom electron arising from the top electron in Fig. 127. The electric field points upward toward the top electron, with magnitude $\gamma e/4\pi\epsilon_0 r^2$. This follows from Eq. (5.15) in Chapter 5, with $\theta = 90^{\circ}$. Alternatively, it follows from the transformations in Eq. (6.76); if F is the rest frame of the electrons, then $\mathbf{B} = 0$, so the \mathbf{E}' field in the lab frame is given by $\mathbf{E}'_{\perp} = \gamma \mathbf{E}_{\perp} = \gamma (e/4\pi\epsilon_0 r^2) \hat{\mathbf{y}}$.

The transformations in Eq. (6.76) also give the B' field. Since $\mathbf{B} = 0$ in the electrons' frame F, the \mathbf{B}' field in the lab frame F' is given by $\mathbf{B}'_{\perp} = -\gamma(\mathbf{v}/c^2) \times \mathbf{E}_{\perp}$. The \mathbf{v} here is the velocity of the lab frame F' with respect to the electrons' frame F. So \mathbf{v} points leftward, and the righthand rule gives the direction of \mathbf{B}'_{\perp} as pointing out of the page. The magnitude is $B' = (v/c^2)(\gamma e/4\pi\epsilon_0 r^2)$.

All of the fields and forces are shown in Fig. 127 (when calculating the forces, remember that electrons are negatively charged). The resulting force on the electron is the sum of the eE' repulsive electric force and the evB' attractive (as you can verify from the righthand rule) magnetic force. The net repulsive force is therefore

$$eE' - evB' = e\frac{\gamma e}{4\pi\epsilon_0 r^2} - ev\frac{v}{c^2}\frac{\gamma e}{4\pi\epsilon_0 r^2} = \left(1 - \frac{v^2}{c^2}\right)\frac{\gamma e^2}{4\pi\epsilon_0 r^2} = \frac{e^2}{\gamma 4\pi\epsilon_0 r^2}, \quad (475)$$

in agreement with the result in part (a). If you forgot to consider the magnetic field in the lab frame, then you would have a "paradox" where the γ appears in the denominator of the correct force in part (a), but in the numerator in the incomplete (that is, just electric) force in part (b).





Figure 127

(c) In the limit $v \to c$, we have $\gamma \to \infty$, so the force in the lab frame goes to zero.

6.69. Relating the forces

LAB FRAME: In the lab frame (the frame of the top stick and the charge), there is no magnetic force on the charge q, because it is at rest. Label the top stick as "1" and the bottom stick as "2." The electric field due to the top stick is $\lambda/2\pi\epsilon_0 r$, so the electric force is $F_{E,1}^{\text{lab}} = \lambda q/2\pi\epsilon_0 r$ downward (assuming both λ and q are positive). Due to length contraction, the bottom stick has charge density $\gamma\lambda$ in the lab frame, so the electric force is $F_{E,2}^{\text{lab}} = \gamma\lambda q/2\pi\epsilon_0 r$ upward. The total force in the lab frame is therefore

$$F_{\text{tot}}^{\text{lab}} = F_{E,1}^{\text{lab}} + F_{E,2}^{\text{lab}} = -\frac{\lambda q}{2\pi\epsilon_0 r} + \frac{\gamma\lambda q}{2\pi\epsilon_0 r} = (\gamma - 1)\frac{\lambda q}{2\pi\epsilon_0 r} \,. \tag{476}$$

This is positive, so the total force is upward.

BOTTOM-STICK FRAME: In this frame, the situation is shown in Fig. 128. The charge q and the top stick are now moving. The density of the top stick is $\gamma\lambda$ due to length contraction. The top stick produces a current $(\gamma\lambda)v$, so the magnetic field is $\mu_0(\gamma\lambda v)/2\pi r$, and it points into the page at the location of the charge q. The magnetic force qvB therefore points upward with magnitude $qv \cdot \mu_0(\gamma\lambda v)/2\pi r$. Using $\mu_0 = 1/\epsilon_0 c^2$, this force in the bottom-stick frame (bsf) can be written as $F_{B,1}^{\text{bsf}} = (v^2/c^2)\gamma\lambda q/2\pi\epsilon_0 r$.

The electric forces are quickly found to be $F_{E,1}^{\text{bsf}} = \gamma \lambda q / 2\pi \epsilon_0 r$ downward and $F_{E,2}^{\text{bsf}} = \lambda q / 2\pi \epsilon_0 r$ upward. The total force in the bottom-stick frame is therefore

$$F_{\text{tot}}^{\text{bsf}} = F_{B,1}^{\text{bsf}} + F_{E,1}^{\text{bsf}} + F_{E,2}^{\text{bsf}}$$

$$= \frac{v^2}{c^2} \frac{\gamma \lambda q}{2\pi \epsilon_0 r} + \left(-\frac{\gamma \lambda q}{2\pi \epsilon_0 r}\right) + \frac{\lambda q}{2\pi \epsilon_0 r}$$

$$= \left(\gamma \left(\frac{v^2}{c^2} - 1\right) + 1\right) \frac{\lambda q}{2\pi \epsilon_0 r}$$

$$= \left(-\frac{1}{\gamma} + 1\right) \frac{\lambda q}{2\pi \epsilon_0 r}.$$
(477)

This is positive, so the total force is upward. It is $1/\gamma$ times the total force in the lab frame. This is correct, because the force on a particle is largest in the rest frame of the particle (the lab frame here); it is smaller in any other frame by the factor $1/\gamma$.

6.70. Drifting motion

If there is a frame in which the electric field is zero, then we know from Exercise 6.29 that the ion moves in a circle in that frame. Let F be the lab frame, and consider the frame F' that moves in the positive y direction with speed $v = (0.1 \text{ m})/(1 \mu \text{s}) = 10^5 \text{ m/s} = c/3000$. Since F' moves with the average velocity of the ion, it is the only frame in which the ion could possibly be moving in a circle (because in any other frame the ion would drift away). F sees F' moving with velocity $v\hat{\mathbf{y}}$, so if we demand that the electric field be zero in F', then Eq. (6.76) tells us how \mathbf{E}_{\perp} and \mathbf{B}_{\perp} in the lab frame F must relate to each other:

$$\begin{aligned}
\mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}) \\
\implies 0 &= \gamma\left(\mathbf{E}_{\perp} + (v\hat{\mathbf{y}}) \times (0.6 \,\mathrm{T})\hat{\mathbf{z}}\right) \\
\implies \mathbf{E}_{\perp} &= -(10^5 \,\mathrm{m/s})(0.6 \,\mathrm{T})(\hat{\mathbf{y}} \times \hat{\mathbf{z}}) \\
&= -(6 \cdot 10^4 \,\mathrm{V/m})\hat{\mathbf{x}}.
\end{aligned}$$
(478)



Figure 128

Note that \mathbf{E}_{\perp} points in the (negative) x direction, whereas the drift of the motion is in the y direction. See Problem 6.26 for more details.

6.71. Rowland's experiment

With the disk at 10 kV, the electric field strength in the space above and below the disk in Fig. 129 is

$$E = \frac{V}{d} = \frac{10^4 \,\mathrm{V}}{0.006 \,\mathrm{m}} = 1.67 \cdot 10^6 \,\mathrm{V/m}.$$
 (479)

The charge density on each surface (top and bottom) of the disk is therefore

$$\sigma = \epsilon_0 E = \left(8.85 \cdot 10^{-12} \, \frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg} \,\mathrm{m}^3}\right) (1.67 \cdot 10^6 \,\mathrm{V/m}) = 1.5 \cdot 10^{-5} \,\mathrm{C/m^2}. \tag{480}$$



Figure 129

The left end of the charged region on the rotating disk is determined by the left end of the electrode on the glass plate, which is at radius 4.45 cm. The right end of the charged region is determined by the right end of the disk, which is at radius 10.55 cm. The mean radius of the charged part of the disk is therefore $\bar{r} = (4.45 \text{ cm} + 10.55 \text{ cm})/2 = 7.5 \text{ cm}$. So the average velocity of the charges is

$$\overline{v} = 2\pi \overline{r}\nu = 2\pi (0.075 \,\mathrm{m})(61 \,\mathrm{s}^{-1}) = 28.7 \,\mathrm{m/s}.$$
 (481)

(Actually, this is the velocity at the average radius, and not the average velocity of all points in the disk. But we're just doing things roughly, so this distinction isn't important. For that matter, we could just pick a round number like $\bar{r} \approx 0.1 \text{ m}$.) The effective surface current density is then

$$\mathcal{J} = \sigma \overline{v} = (1.5 \cdot 10^{-5} \,\mathrm{C/m^2})(28.7 \,\mathrm{m/s}) = 4.3 \cdot 10^{-4} \,\mathrm{C/(s \,m)}. \tag{482}$$

The combined effect of the two current sheets, each with surface current density \mathcal{J} (in the same direction), is to produce a horizontal field $B = \mu_0 \mathcal{J}$ both above and below the disk. So

$$B = (4\pi \cdot 10^{-7} \,\mathrm{kg}\,\mathrm{m/C^2}) (4.3 \cdot 10^{-4} \,\mathrm{C/(s}\,\mathrm{m})) = 5.4 \cdot 10^{-10} \,\mathrm{T}, \tag{483}$$

or $5.4\cdot 10^{-6}$ gauss. In terms of the various quantities, a few equivalent symbolic expressions for the magnetic field are

$$B = \frac{2\pi \overline{r}\nu\mu_0\epsilon_0 V}{d} = \frac{2\pi \overline{r}\nu V}{c^2 d} = \frac{\overline{v}V}{c^2 d}.$$
(484)

We have used the speed at the mean radius to estimate the field strength B immediately above the disk in that region. For a more accurate calculation of the field strength to be expected at the location of the magnetometer needle, one could divide the disk into circular rings and integrate over the whole distribution. That is what Rowland did.

6.72. Transverse Hall field

In Gaussian units the relation between current density \mathbf{J} and charge carrier velocity \mathbf{v} is still $\mathbf{J} = nq\mathbf{v}$, so $\mathbf{v} = \mathbf{J}/nq$. Here \mathbf{J} is measured in esu/(cm² s), n in cm⁻³, q in esu, and \mathbf{v} in cm/s. The force on a charge carrier in Gaussian units is $q(\mathbf{E}_{t} + (\mathbf{v}/c) \times \mathbf{B})$ which is zero if

$$\mathbf{E}_{\mathrm{t}} = -\frac{\mathbf{v} \times \mathbf{B}}{c} = -\frac{\mathbf{J} \times \mathbf{B}}{nqc} \,. \tag{485}$$

6.73. Hall voltage

Our strategy will be to find the current density, then the drift velocity, then the transverse field, then the transverse (Hall) voltage. The current density is

$$J = \frac{I}{A} = \frac{V/R}{A} = \frac{V/(\rho L/A)}{A} = \frac{V}{\rho L} = \frac{1 \,\mathrm{V}}{(0.016 \,\mathrm{ohm} \cdot \mathrm{m})(0.005 \,\mathrm{m})} = 1.25 \cdot 10^4 \,\mathrm{A/m^2}.$$
(486)

The drift velocity is then

$$v = \frac{J}{ne} = \frac{1.25 \cdot 10^4 \,\mathrm{C/s}\,\mathrm{m}^2}{(2 \cdot 10^{21}\,\mathrm{m}^{-3})(1.6 \cdot 10^{-19}\,\mathrm{C})} = 39\,\mathrm{m/s}.$$
(487)

The induced electric field is $E_t = vB = (39 \text{ m/s})(0.1 \text{ T}) = 3.9 \text{ V/m}$. The Hall voltage across the ribbon of width 0.002 m is therefore $(3.9 \text{ V/m})(0.002 \text{ m}) = 7.8 \cdot 10^{-3} \text{ V}$, or 7.8 millivolts. Symbolically, the Hall voltage equals $VBw/\rho Lne$, where w is the width.

Chapter 7

Electromagnetic induction

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

7.20. Induced voltage from the tides

Assume that the speed of the tidal current is 1 m/s. The force per unit charge in the water is $vB = (1 \text{ m/s})(5 \cdot 10^{-5} \text{ T}) = 5 \cdot 10^{-5} \text{ V/m}$. This is the effective Efield, so the induced voltage across the length of the wire is (using 960 ft \approx 300m) $E\ell = (5 \cdot 10^{-5} \text{ V/m})(300 \text{ m}) = 0.015 \text{ V}$, or 15 millivolts.

7.21. Maximum emf

The maximum emf equals the maximum value of $d\Phi/dt$. The flux is given by $\Phi = NAB\cos(\omega t + \phi)$, where N is the number of turns and $A = \pi r^2$ is the area. (We'll assume that the coil is oriented optimally, with its axis of rotation lying perpendicular to the field.) The maximum value of $d\Phi/dt = -\omega NAB\sin(\omega t + \phi)$ is then ωNAB , so

$$\mathcal{E}_{\max} = \omega NAB = (2\pi \cdot 30 \,\mathrm{s}^{-1})(4000) \left(\pi (0.12 \,\mathrm{m})^2\right) (5 \cdot 10^{-5} \,\mathrm{T}) = 1.7 \,\mathrm{V}.$$
(488)

7.22. Oscillating E and B

Consider a circle of radius r centered on the axis. The flux through this circle is $\pi r^2 B$. Since B takes the form of $B_0 \cos(\omega t + \phi)$, the amplitude of $dB/dt = -\omega B_0 \sin(\omega t + \phi)$ is ωB_0 . So the amplitude of the emf around the circle is $\mathcal{E}_{\max} = (d\Phi/dt)_{\max} = \pi r^2(\omega B_0)$. The electric field along the circle is related to \mathcal{E} by $2\pi r E = \mathcal{E}$. Therefore, the amplitude of E is

$$E_{\rm max} = \frac{\mathcal{E}_{\rm max}}{2\pi r} = \frac{\omega r B_0}{2} = \frac{(2\pi \cdot 2.5 \cdot 10^6 \,\mathrm{s}^{-1})(0.03 \,\mathrm{m})(4 \cdot 10^{-4} \,\mathrm{T})}{2} = 94 \,\mathrm{V/m.} \quad (489)$$

7.23. Vibrating wire

The amplitude of the vibration is $x_0 = 3 \cdot 10^{-4}$ m, the frequency is $\nu = 2000$ Hz, the length of wire within the gap is $\ell = 0.018$ m, and the magnetic field is B = 0.5 T.

The position of the wire takes the general form of $x_0 \cos(\omega t + \phi)$, so taking the derivative tells us that the maximum speed is $v_{\text{max}} = \omega x_0 = 2\pi\nu x_0$. Imagine connecting the ends of the wire with another wire to form a complete loop, which is in fact what you would be doing if you measured the voltage between the ends by connecting them to a voltmeter. Then the movement of the wire implies that an area is being swept out;

the area enclosed by the loop is changing. The maximum rate of swept area equals the maximum speed times ℓ . So the maximum induced voltage is

$$\mathcal{E}_{\max} = \left(\frac{d\Phi}{dt}\right)_{\max} = B\ell v_{\max} = 2\pi B\ell \nu x_0$$

= $2\pi (0.5 \,\mathrm{T})(0.018 \,\mathrm{m})(2000 \,\mathrm{s}^{-1})(3 \cdot 10^{-4} \,\mathrm{m}) = 0.034 \,\mathrm{V}.$ (490)

This result depends linearly on all four of the given quantities, which makes intuitive sense.

We can also solve this exercise by looking at the qvB magnetic force on the charges in the wire. Multiplying this force by the distance ℓ along the wire over which it acts, and dividing by q to obtain the work per charge, gives a voltage difference of $vB\ell$, as above. This is maximum when v is maximum, since B and ℓ are constants.

7.24. Pulling a frame

We could solve this exercise piecemeal, but let's instead derive a single expression for the force, which will take care of all the questions. Let ℓ be the total perimeter of the rectangle, and let b be the length of the side that sweeps through the field. The current in the frame is $I = \mathcal{E}/R$, where $\mathcal{E} = d\Phi/dt = Bbv$, and where $R = \rho\ell/A = \rho\ell/\pi r^2$. So $I = Bbv/(\rho\ell/\pi r^2) = Bbv\pi r^2/\rho\ell$. The force on the trailing side of the frame is F = IBb, and you can show with Lenz's law and the right-hand rule that this force is directed to the left; that is, it is a drag force. The force required to balance the magnetic drag force therefore equals

$$F = \frac{B^2 b^2 v \pi r^2}{\rho \ell} \,. \tag{491}$$

(You can check that this does indeed have units of force.) Since we are ignoring the inertia of the frame, the applied force must be exactly equal to the magnetic force, in magnitude. (If m = 0, then $\sum \mathbf{F} = m\mathbf{a}$ implies that $\sum \mathbf{F} = 0$.) For any particular F that you pick, Eq. (491) can be solved for the velocity v that the frame will have. We can now answer the various questions.

Eq. (491) implies that twice the force means twice the velocity. So a force of 2 N will pull the frame out in half the time, or 0.5 sec.

Keeping everything else the same, doubling ρ means halving F (there is half as much current). So a brass frame will be pulled out in 1 sec by a force of 0.5 N.

Doubling the radius increases F by a factor $2^2 = 4$ (there is four times as much current). So a 1 cm aluminum frame will be pulled out in 1 sec by a force of 4 N. (We effectively have four of the original frames stacked on top of each other, each of which requires 1 N.)

7.25. Sliding loop

In Fig. 130 the y axis points into the page. We've arbitrarily chosen the current in the wire to flow in the negative y direction (out of the page), but the sign doesn't matter since all we care about is the magnitude of the emf. At the leading edge of the square loop, the magnitude of B is $\mu_0 I/2\pi r$, where $r = \sqrt{h^2 + (b/2)^2}$. Only the z component matters in the flux, and this brings in a factor of (b/2)/r. So

$$B_z = \frac{\mu_0 I}{2\pi r} \frac{b/2}{r} = \frac{\mu_0 I b}{4\pi (h^2 + b^2/4)}.$$
(492)





At the trailing edge, B_z has the opposite sign. If the loop moves a small distance v dt, there is additional positive flux through a thin rectangle with area b(v dt) at the leading edge, and also less negative flux through a similar rectangle at the trailing edge. Both of these effects cause the upward flux to increase. Therefore,

$$\mathcal{E} = \frac{d\Phi}{dt} = 2\frac{b(v\,dt)B_z}{dt} = 2bvB_z = \frac{\mu_0 I b^2 v}{2\pi (h^2 + b^2/4)}\,.$$
(493)

The flux is increasing upward. So for our choice of direction of the current in the wire, the induced emf is clockwise when viewed from above, because that creates a downward field inside the loop which opposes the change in flux. For h = 0 (or in general for $h \ll b$) \mathcal{E} reduces to $2\mu_0 Iv/\pi$. This is independent of b because the field at the leading and trailing edges decreases with b, while the length of the thin rectangles at these edges increases with b.

You can show that our result for \mathcal{E} has the correct units, either by working them out explicitly, or by noting that \mathcal{E} has the units of B (which are the same as $\mu_0 I/2\pi r$) times length squared divided by time, which correctly gives flux per time.

7.26. Sliding bar

(a) Let v be the instantaneous velocity of the bar. The area of the circuit increases at a rate b(v dt)/dt = bv, so the induced emf is $\mathcal{E} = d\Phi/dt = Bbv$. The current is therefore $I = \mathcal{E}/R = Bbv/R$. The general expression for the force on piece of wire (the bar in our setup) is F = IBb, which yields B^2b^2v/R here. So F = magives (including the minus sign because the force opposes the motion, as you can check with Lenz's law and the right-hand rule)

$$-F = m\frac{dv}{dt} \implies -\frac{B^2b^2v}{R} = m\frac{dv}{dt} \implies -\int_0^t \frac{B^2b^2}{mR} dt' = \int_{v_0}^v \frac{dv'}{v'}$$
$$\implies -\frac{B^2b^2t}{mR} = \ln\left(\frac{v}{v_0}\right) \implies v = v_0e^{-t/T}, \quad \text{where} \quad T \equiv \frac{mR}{B^2b^2}.$$
(494)

(You can check that T has units of time.) We see that the velocity decreases exponentially, so technically the rod never stops moving (in an ideal world). This exponential decay of v is a familiar result for forces that are proportional to (the negative of) v.

(b) The total distance traveled in the limit $t \to \infty$ is

$$x = \int_0^\infty v \, dt = \int_0^\infty v_0 e^{-t/T} \, dt = -v_0 T e^{-t/T} \Big|_0^\infty = v_0 T = \frac{v_0 m R}{B^2 b^2} \,. \tag{495}$$

So the rod travels a finite distance in an infinite time.

(c) The initial kinetic energy of the rod is $mv_0^2/2$. This must eventually show up as heat in the resistor, so let's check this. The instantaneous power dissipated in the resistor is $P = I^2 R$, where I is given above as $I = Bbv/R = (Bbv_0/R)e^{-t/T}$. The total energy loss in the resistor is therefore

$$\int_{0}^{\infty} I^{2}R \, dt = \frac{B^{2}b^{2}v_{0}^{2}}{R} \int_{0}^{\infty} e^{-2t/T} \, dt = -\frac{B^{2}b^{2}v_{0}^{2}}{R} \frac{T}{2} e^{-2t/T} \Big|_{0}^{\infty}$$
$$= \frac{B^{2}b^{2}v_{0}^{2}}{R} \frac{T}{2} = \frac{B^{2}b^{2}v_{0}^{2}}{R} \frac{mR}{2B^{2}b^{2}} = \frac{1}{2}mv_{0}^{2}.$$
(496)

7.27. Ring in a solenoid

(a) The magnetic field inside the solenoid is $B(t) = \mu_0 n I(t) = \mu_0 n I_0 \cos \omega t$. Faraday's law applied to the given ring yields

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\pi r^2 \frac{dB}{dt} = \pi r^2 \mu_0 n I_0 \omega \sin \omega t.$$
(497)

With the given positive direction of I, the right-hand rule gives the positive direction of B as upward, and then also gives the positive direction of \mathcal{E} as counterclockwise when viewed from above (as for I). The current in the loop is $I_{\text{loop}}(t) = \mathcal{E}/R = (\pi r^2 \mu_0 n I_0 \omega/R) \sin \omega t$.

(b) The force on a little piece of the ring is $F(t) = I_{loop}(t) d\mathbf{l} \times \mathbf{B}$. With positive I counterclockwise and positive B upward, this force is radial and equals

$$F(t) = \frac{\pi r^2 \mu_0 n I_0 \omega}{R} \sin \omega t \cdot dl \cdot \mu_0 n I_0 \cos \omega t = \frac{\pi r^2 \mu_0^2 n^2 I_0^2 \omega \, dl}{R} \sin \omega t \cos \omega t. \tag{498}$$

The force is radially outward if this quantity is positive, inward if it is negative. Since $\sin \omega t \cos \omega t = (1/2) \sin(2\omega t)$ we see that the force is maximum outward when $\omega t = \pi/4$ (plus multiplies of π), and maximum inward when $\omega t = 3\pi/4$ (plus multiplies of π).

(c) Since the force lies in the horizontal plane, it serves only to stretch/shrink the ring (negligibly, if the ring is rigid).

7.28. A loop with two surfaces

Surface (a) has two sides, whereas surface (b) has only one side; it is a Mobius strip. The two-sided surface is the one we must use to calculate the flux through the loop. The Mobius strip can't be used, because the direction of the area vector $d\mathbf{a}$ isn't well defined; if you travel around the strip and return to your starting point, $d\mathbf{a}$ will point in the opposite direction.

If surface (a) is stretched vertically at the twist and then viewed from the side, it looks like the surface shown in Fig. 131(a). So it can be considered a two-turn coil. The three-turn coil in shown in Fig. 131(b), and so on for N turns. If you trace along the wire as it spirals upward, it takes the same shape as the windings of a solenoid. All of the "pancakes" have the same orientation, so the total flux through all of them is simply N times the flux through one. The pancakes are truly all part of a single surface. We therefore obtain the result that a coil of N turns has an emf that is N times that of a single loop.

7.29. Induced emf in a loop

The magnetic field due to an infinite current-carrying wire is $\mu_0 I/2\pi r$, so the fields at the near and far sides of the rectangle are

$$B_1 = \frac{\mu_0(100 \text{ A})}{2\pi(0.15 \text{ m})} = 1.33 \cdot 10^{-4} \text{ T}, \quad \text{and} \quad B_2 = \frac{\mu_0(100 \text{ A})}{2\pi(0.25 \text{ m})} = 0.8 \cdot 10^{-4} \text{ T}.$$
 (499)

The induced emf is therefore

$$\mathcal{E} = \frac{d\Phi}{dt} = wv(B_1 - B_2) = (0.08 \,\mathrm{m})(5 \,\mathrm{m/s})(0.53 \cdot 10^{-4} \,\mathrm{T}) = 2.1 \cdot 10^{-5} \,\mathrm{V}.$$
(500)

The flux is downward and decreasing, so \mathcal{E} will be in the direction to drive a current that would make more flux downward. The current is therefore clockwise when viewed from above.





Figure 131

Let's now estimate roughly how large the resistance must be to make the effect of the current in the loop negligible. The current in the loop at any instant is $I' = \mathcal{E}/R$. This causes a field B' and a flux Φ' through the loop. Because \mathcal{E} is changing with time as the loop moves away from the wire, Φ' is changing too, resulting in an extra emf \mathcal{E}' , which we have so far ignored. The question is, how large must R be so that \mathcal{E}' is negligible compared with \mathcal{E} ?

As a very rough estimate, we have $B' \approx \mu_0 I'/2\pi d$, where d is a typical dimension of the loop, say, d = 5 cm. The flux¹ is then $\Phi' \approx B'A = (\mu_0 I'/2\pi d)w\ell$, where ℓ is the length of the loop (we could set $\ell \approx w \approx d$ here since we're being rough, but we'll keep them separate). The (very rough) time characteristic of the change in Φ' is $\tau = h/v$, where h is the mean distance from the loop to the wire (20 cm), and v is the speed of the loop. So in order of magnitude, we have (using $I' = \mathcal{E}/R$)

$$\mathcal{E}' = \frac{d\Phi'}{dt'} \approx \frac{\Phi'}{\tau} = \frac{\mu_0 I' w \ell}{2\pi d} \cdot \frac{v}{h} = \frac{\mu_0 \mathcal{E} w \ell v}{2\pi R h d}.$$
(501)

Our goal is to have $\mathcal{E}' \ll \mathcal{E}$, which is equivalent to

$$\frac{\mu_0 w \ell v}{2\pi R h d} \ll 1 \implies R \gg \frac{\mu_0 w \ell v}{2\pi h d}$$

$$\implies R \gg \frac{\left(4\pi \cdot 10^{-7} \frac{\text{kg m}}{C^2}\right) (0.08 \text{ m}) (0.1 \text{ m}) (5 \text{ m/s})}{2\pi (0.2 \text{ m}) (0.05 \text{ m})} = 8 \cdot 10^{-7} \,\Omega. \tag{502}$$

This is roughly equal to $10^{-6} \Omega$, so the condition that the current in the loop is negligible (more precisely, the condition that $\mathcal{E}' \ll \mathcal{E}$) can be written as $R \gg 10^{-6} \Omega$. This is a rather small resistance, so this bound is easily satisfied by a typical copper wire.

We can alternatively write the condition in Eq. (502) as

$$\frac{\mu_0 w\ell}{2\pi d} \cdot \frac{1}{R} \ll \frac{h}{v} \,. \tag{503}$$

But from the definition of the self-inductance L, the above expression for Φ' yields $L = \mu_0 w \ell / 2\pi d$. So the condition can be written as $L/R \ll \tau$. In other words, the inductive time constant of the loop itself, L/R, should be short compared with the time scale of the change of the externally induced emf.

Note that in the case where all of the above lengths (w, ℓ, h, d) are of the same order of magnitude (which is the case here), the condition reduces to the simple expression (ignoring the 2π): $R \gg \mu_0 v$. We therefore see that the smallness of the above lower bound on R (in SI units) comes from the smallness of μ_0 (in SI units).

7.30. Work and dissipated energy

The induced emf is $\mathcal{E} = wv(B_1 - B_2)$, so the current is $I = \mathcal{E}/R = wv(B_1 - B_2)/R$. From Lenz's law it is counterclockwise when viewed from above. From the righthand rule, the forward-directed force that must be applied to the loop to balance the retarding magnetic force and keep the loop moving at constant speed is $F = IB_1w - IB_2w =$ $Iw(B_1 - B_2)$. Using $(B_1 - B_2) = IR/wv$, the rate at which work is done is therefore

$$Fv = Iw(B_1 - B_2)v = Iw\left(\frac{IR}{wv}\right)v = I^2R,$$
(504)

¹Strictly speaking, we should expect the flux of B' to involve a factor like $\ln(r_{\text{loop}}/r_{\text{wire}})$; see the comments at the end of Section 7.8. But unless the wire is extremely thin, this logarithm won't be a very large number.

which is the power dissipated in the resistance, as we wanted to show.

In Fig. 7.14 the energy that is dissipated in the stationary loop has to be supplied by whatever agency is moving the coil. A force is indeed required to move the coil because of the magnetic field arising from the induced current in the loop. To see how this works out in a simple case, let the coil have the same rectangular shape as the loop, but with N turns. And let the coil have current I_0 . Then the difference in the B fields (due to the loop) at the leading and trailing edges of the coil is smaller than the difference in the B fields (due to the coil) at the leading and trailing edges of the loop by a factor of I/I_0 (because these are the currents that produce the B fields) and also by a factor of N. So the $Fv = Iw(B_1 - B_2)v$ relation in Eq. (504) for the rate at which work is done in moving the N-loop coil becomes

$$Fv = N \cdot I_0 w \left((B_1 - B_2) \cdot \frac{I}{I_0} \cdot \frac{1}{N} \right) v = I w (B_1 - B_2) v,$$
(505)

which agrees with the expression in Eq. (504), and hence equals I^2R .

7.31. Sinusoidal emf

If the field is uniform, the emf will be sinusoidal, regardless of the shape of the planar loop (assuming a constant rate of rotation). To see why, imagine slicing the loop into strips oriented perpendicular to the axis of rotation. Each strip is essentially a thin rectangle, so the flux through each strip varies sinusoidally. The fluxes through all the different strips have the same frequency and phase, so the total flux also varies sinusoidally.

Actually, the emf will be sinusoidal even if the loop isn't planar. This is true for the following reason. Consider a nonplanar surface bounded by the loop, and divide it into many tiny planar patches. The flux through each planar patch varies sinusoidally, so it takes the form of $\Phi_i \sin(\omega t + \phi_i)$. You can quickly verify, by using the trig sum formula for sine, that the sum of two such fluxes (with arbitrary Φ_i and ϕ_i values) again takes the same form. Therefore, by mathematical induction, the sum of an arbitrary number of such fluxes takes this form. The point is that the frequency of all the individual sinusoidal fluxes is the same, which means that there is no way for any other frequency to creep into the total flux. Alternatively, imagine looking at the loop along the line of the *B* field. If you close one eye, so that you don't have any depth perception, then for all you know the loop could be planar.

However, if the field isn't uniform, then the emf need not be sinusoidal. This is easily seen in an extreme case where B is zero except in a given region; see Fig. 132. The emf will be zero except when the loop is sweeping through that region, so the emf will look something like the curve in Fig. 133. The nonzero parts of the curve belong to a sine curve. The numbers shown correspond to the orientations in Fig. 132.

In the case where the field is created by a ring (instead of a solenoid which creates a uniform field), the field will weaker in the middle and stronger near the ring. The former fact will lessen the peak in the standard sinusoidal curve, and the latter fact will increase \mathcal{E} near its zeros. The emf will therefore look more like a "square wave."

7.32. Emfs and voltmeters

The three cases are:

(a) The electric field caused by the changing flux is directed clockwise around the solenoid, so the line integral $\int_a^b \mathbf{E} \cdot d\mathbf{s}$ along path 1 equals $+\mathcal{E}_0$. The voltage







Figure 133

difference $V_b - V_a \equiv -\int_a^b \mathbf{E} \cdot d\mathbf{s}$ therefore equals $-\mathcal{E}_0$. Path 2 encloses zero changing flux, so $V_a - V_b$ equals zero along path 2. We are assuming that *a* and *b* are infinitesimally close to each other.

- (b) We now have a simple electrostatic setup, so the voltage differences are path independent. Point *a* at the higher potential, so we have $V_b V_a = -\mathcal{E}_0$, and $V_a V_b = \mathcal{E}_0$ (along any paths).
- (c) The answers here are the same as in part (b), except with $-\mathcal{E}_0$ replaced with $-\mathcal{E}_0/N$, and \mathcal{E}_0 replaced with \mathcal{E}_0/N . So in the $N \to \infty$ limit, the potential differences are zero.

The results are compiled in this table:

	Path 1	Path 2
	$V_b - V_a$	$V_a - V_b$
Case (a)	$-\mathcal{E}_0$	0
Case (b)	$-\mathcal{E}_0$	\mathcal{E}_0
Case (c)	0	0

Cases (a) and (b) are equivalent for path 1, but not for path 2. Cases (a) and (c) are equivalent for path 2, but not for path 1. Cases (b) and (c) are different for both paths. The total voltage drop around a closed path is zero in cases (b) and (c) (consistent with the fact that these are electrostatic setups), but nonzero in case (a).

7.33. Getting a ring to spin

The flux through the ring at any given time is $\Phi = \pi a^2 B$. From Faraday's law, the induced emf is $\mathcal{E} = d\Phi/dt = \pi a^2(dB/dt)$ (ignoring the sign). But by definition, the emf is also $\mathcal{E} = \int \mathbf{E} \cdot d\mathbf{s} = 2\pi a E$. The tangential field is therefore $E = \mathcal{E}/2\pi a = (a/2)(dB/dt)$. A little element of charge dq feels a tangential force E dq, and hence a torque Ea dq. So the total torque is $\tau = Eaq = (qa^2/2)(dB/dt)$. The final angular momentum acquired by the ring is therefore

$$L = \int_0^\infty \tau \, dt = \int_0^\infty \frac{qa^2}{2} \frac{dB}{dt} \, dt = \frac{qa^2}{2} \int_{B_0}^0 dB = -\frac{qa^2B_0}{2} \,. \tag{506}$$

We haven't been keeping track of signs, so the minus sign here doesn't mean much. The correct statement to make is that Lenz's law implies that the ring will spin in the direction that has a positive right-hand-rule relation to the direction of the initial magnetic flux. So you can quickly show that if q is positive, the direction of **L** is the same as the direction of \mathbf{B}_0 . The angular momentum can be written as $L = I\omega = (ma^2)\omega$, so we have

$$\omega = \frac{L}{ma^2} = \frac{qB_0}{2m}, \qquad (507)$$

as desired. We see that L depends only on the net change in B, and not on the rate at which this change comes about. Intuitively, if B changes more slowly, then \mathcal{E} (and hence the torque) is smaller, so the angular momentum increases at a lesser rate. But the process takes longer, so the increase goes on for a longer time. These two competing effects exactly cancel.

Note that ω is independent of a. This is due to the fact that E is proportional to a, which means that the linear acceleration in the tangential direction, and hence the tangential speed v at a given time, is proportional to a. The angular velocity, $\omega = v/a$,

is therefore independent of a. (The $E \propto a$ fact quickly follows from Faraday's law, although it isn't terribly intuitive. In the end, it comes down to whether or not you find $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ to be intuitive.)

7.34. Faraday's experiment

We have enough information to calculate the resistance of each coil, if we know the resistivity of copper. From Table 4.1 let's use the rough value of $2 \cdot 10^{-8}$ ohm-m at room temperature. The length of each wire is $\ell = (203 \text{ ft})(12 \text{ in/ft})(0.0254 \text{ m/in}) = 62 \text{ m}$. And the radius is $r = ((1/40) \text{ in})(0.0254 \text{ m/in}) = 6.4 \cdot 10^{-4} \text{ m}$, which gives a cross-sectional area of $A = \pi r^2 = 1.3 \cdot 10^{-6} \text{ m}^2$. The resistance of each wire is then

$$R = \frac{\rho \ell}{A} = \frac{(2 \cdot 10^{-8} \text{ ohm-m})(62 \text{ m})}{1.3 \cdot 10^{-6} \text{ m}^2} \approx 1 \text{ ohm.}$$
(508)

We don't know the coil size or the number of turns. Suppose the coil is cylindrical, with length h and radius a. Then if N is the number of turns in each wire, the length of each wire (which we know is 62 m) is $\ell = 2\pi a N$.

We have another clue: the two coils are wound closely together separated only by twine. The winding must look something like what is shown in Fig. 134. If we assume that the twine is about as thick as the wire, then the turns in one coil are spaced at intervals of 4/20 inch, or about 0.005 m. The number of turns in each coil is then N = h/(0.005 m) = 200h/(1 m).

It seems likely that Faraday's "block of wood" would have been roughly "squarish" in proportions, so let's assume h = 2a. Then the three relations, $\ell = 2\pi aN$, N = 200h/(1 m), and h = 2a, quickly give $\ell(1 \text{ m}) = 200\pi h^2$. From $\ell = 62 \text{ m}$ we find $h \approx 0.3 \text{ m}$. And then a = 0.15 m, and $N \approx 60 \text{ turns}$. So the coil is about a foot long and a foot in diameter – very reasonable.

An approximate formula for the inductance L of a coil with N turns is easily derived. Using the long-solenoid field of $B = \mu_0 nI = \mu_0 (N/h)I$, we have $\Phi = N\pi a^2 B = \pi a^2 \mu_0 N^2 I/h$. The inductance is obtained by erasing the I, so we have

$$L = \frac{\mu_0 \pi a^2 N^2}{h} = \frac{\left(4\pi \cdot 10^{-7} \, \frac{\text{kg m}}{\text{C}^2}\right) \pi (0.15 \,\text{m})^2 (60)^2}{0.3 \,\text{m}} \approx 1 \cdot 10^{-3} \,\text{H.}$$
(509)

Because end effects were neglected, this somewhat overestimates the inductance (see Exercise 7.40). But it would be silly to worry about this error, given all the other approximations we've made. This L is also the mutual inductance of the two coils, because they link the same flux. The reconstructed circuit is shown in Fig. 135.

We'll assume that the 100-plate battery has an emf of roughly 100 volts. Nothing would be gained by using so large a battery if its internal resistance were much greater than 1 ohm, but it probably wasn't much less. So let's assume $R_{\rm b} \approx 1 \,\Omega$. Then with the switch closed, the steady current through coil #1 is 50 amps. When the switch is opened (causing the current in coil #1 to drop to zero), the current must rise instantaneously to 50 amps in coil #2, because the flux through the coil cannot decrease discontinuously.² Thereafter, the current in circuit #2 decays with the time constant $L/(R + R_{\rm g})$. Assuming $R_{\rm g} \ll R$ (but maybe it wasn't!) we have $L/R = (10^{-3} \,\mathrm{H})/(1 \,\Omega) = 10^{-3}$ seconds. The current pulse through the galvanometer therefore looks something like the curve in Fig. 136.











Figure 136

²If coil #2 weren't present, then the current in coil #1 would *not* stop abruptly; charge would build up on either side of the switch, and a spark might jump across. But the existence of coil #2 allows the current in coil #1 to stop.

7.35. M for two rings

From Eq. (6.53), the magnetic field along the axis of a ring of radius a, a distance b from the center, is $B = \mu_0 I a^2 / 2(a^2 + b^2)^{3/2}$. For $b \gg a$ this can be approximated as $B = \mu_0 I a^2 / 2b^3$. In this limit we can also neglect the variation of B over the interior of the other ring. The flux through the other ring is therefore $\Phi = \pi a^2 B = \mu_0 \pi I a^4 / 2b^3$. The mutual inductance is then $\Phi/I = \mu_0 \pi a^4 / 2b^3$.

7.36. Connecting two circuits

(a) In Fig. 7.40(a), if I_2 is increasing we have an increasing upward flux through the top circuit. This will induce an \mathcal{E}_1 in a direction to drive current that makes a downward flux; this direction is opposite to the positive direction assigned to \mathcal{E}_1 . Hence the sign in the first equation must be a "-." A similar argument shows that the sign in the second equation is also a "-."

Had we assigned the opposite positive directions for I_2 and \mathcal{E}_2 , the sign in front of M in both equations would be a "+."

(b) In Fig. 7.40(b), the current I is the same in both coils, so $I = I_1 = I_2$. And the emfs add, so the total emf is $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$. If we add the two given equations, we obtain

$$\mathcal{E} = -L_1 \frac{dI}{dt} - M \frac{dI}{dt} - L_2 \frac{dI}{dt} - M \frac{dI}{dt} = -(L_1 + L_2 + 2M) \frac{dI}{dt}.$$
 (510)

The circuit is therefore equivalent to a single coil with $L' = L_1 + L_2 + 2M$. In Fig. 7.40(c), the current is now $I = I_1 = -I_2$, and the emf is $\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2$. If we subtract the two given equations, we obtain

$$\mathcal{E} = -L_1 \frac{dI}{dt} - M \frac{d(-I)}{dt} + L_2 \frac{d(-I)}{dt} + M \frac{dI}{dt} = -(L_1 + L_2 - 2M) \frac{dI}{dt}.$$
 (511)

The circuit is therefore equivalent to a single coil with $L'' = L_1 + L_2 - 2M$. Since M is positive, we see that L' is larger than L''.

(c) A circuit with L < 0 would violate Lenz's law; it would be unstable. That is, an increasing current would cause an emf that would increase the current even more. This would violate conservation of energy. Therefore, we must have $L'' \ge 0$, which implies that $M \le (L_1 + L_2)/2$ for any pair of circuits. Equality is achieved if we have two identical coils wound right on top of each other. (An even stronger inequality, $M^2 \le L_1L_2$, can be derived by considering coils connected in parallel. This is indeed stronger, due to the arithmetic-geometric-mean inequality.)

Another (less enlightening) way of obtaining the $M \leq (L_1 + L_2)/2$ result is the following. The energy stored in a system of two inductors is $U = L_1 I_1^2/2 + L_2 I_2^2/2 + M I_1 I_2$ (see Problem 7.10). As you can check, this energy can be written as

$$U = \left(\frac{L_1 + L_2}{2} - M\right) I_1^2 + M(I_1^2 + I_1I_2) - \frac{L_2}{2}(I_1^2 - I_2^2).$$
(512)

This holds for any I_1 and I_2 . In particular, if $I_2 = -I_1$, then only the first of the three terms is nonzero. Since the energy must be positive, we must therefore have $M \leq (L_1 + L_2)/2$.

7.37. Flux through two rings

Figure 137 shows a side view of the field due to the inner ring. (The dots are the intersections of the rings with the plane of the paper.) The key point here is that the

flux through the outer ring comes not only from the field lines pointing upward in the interior of the inner ring, but also from the field lines pointing downward in the region between the rings. The latter flux partially cancels the former flux. The larger the outer ring is, the larger this canceling effect is, and so the smaller the net flux is. The field lines within the dotted curves yield a net flux of zero through the outer ring, so it is only the lines in the central region that contribute to the net flux. The larger the outer ring is, the smaller this central region is.



Figure 137

7.38. Using the mutual inductance for two rings

With current I_1 in the outer ring, Eq. (6.54) tells us that the field at the location of the (much smaller) inner ring is $B_1 = \mu_0 I_1/2R_1$. The flux through the inner ring is therefore $\Phi_{21} = \pi R_2^2 B_1 = \mu_0 \pi R_2^2 I_1/2R_1$. If we increase R_1 by ΔR_1 , then this flux decreases by an amount

$$\Delta \Phi_{21} = \frac{\partial \Phi_{21}}{\partial R_1} \,\Delta R_1 = -\frac{\mu_0 \pi R_2^2 I_1}{2R_1^2} \,\Delta R_1.$$
(513)

Now consider a current I_2 in the inner ring. Let B_2 be the desired field due to the inner ring at the location of the outer ring at radius R_1 . If we expand the outer ring by ΔR_1 , the flux Φ_{12} through this ring *decreases* by the amount of flux in the annular region between radii R_1 and $R_1 + \Delta R_1$ (Exercise 7.37 explains why it is a *decrease*). The area of this region is $2\pi R_1 \Delta R_1$, so the decrease in flux is $\Delta \Phi_{12} = -B_2 2\pi R_1 \Delta R_1$. Now, if our theorem $\Phi_{21}/I_1 = \Phi_{12}/I_2$ always holds, in particular if it holds for any value of R_1 , then it must also hold if the Φ 's are replaced with $\Delta \Phi$'s. That is, we must also have

$$\frac{\Delta\Phi_{21}}{I_1} = \frac{\Delta\Phi_{12}}{I_2} \implies -\frac{\mu_0\pi R_2^2}{2R_1^2} \,\Delta R_1 = -\frac{B_2 2\pi R_1 \,\Delta R_1}{I_2} \implies B_2 = \frac{\mu_0 R_2^2 I_2}{4R_1^3} \,. \tag{514}$$

We can pick any radius for the outer ring, so in more general notation we can write $B = \mu_0 R^2 I/4r^3$ at any point at radius r in the plane of a ring with radius R and current I, if $r \gg R$. Note that this result can be written as $B = \mu_0 (\pi R^2) I/4\pi r^3 = (\mu_0/4\pi)(IA/r^3)$, where A is the area of the ring. We will have more to say about this form of B in Chapter 11.

7.39. Small L

163

Figure 138

Figure 139

One way to wind resistance wire into a "non-inductive" coil is shown in Fig. 138. Of course, the inductance is not exactly zero. The residual inductance is approximately that of the long, narrow "hair-pin" configuration shown in Fig. 139. Technically, if the wire is infinitely thin, then the self-inductance is actually infinite, due to the issue discussed at the end of Section 7.8. But real wires have thickness, so the self-inductance of the hair pin will indeed be small.

Note that the configuration in Fig. 138 effectively consists of two solenoids with currents in opposite directions. So there is essentially no B field inside the cylinder. However, this is actually irrelevant, because the area relevant to the flux is *not* the cross-sectional areas of all the circular loops. Rather, the area spanned by the wire in Fig. 138 is the hair-pin area in Fig. 139, which wraps around the surface of the cylinder. This area has nothing to do with the inside of the cylinder.

7.40. L for a cylindrical solenoid

The field inside a long solenoid is $B = \mu_0 nI = \mu_0 (N/\ell)I$. The flux is $\Phi = N\pi r^2 B = \mu_0 \pi r^2 N^2 I/\ell$. The self-inductance is obtained by erasing the "I," so we have

$$L = \frac{\mu_0 \pi r^2 N^2}{\ell} = \frac{\left(4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^2}\right) \pi (0.05 \,\text{m})^2 (1200)^2}{2 \,\text{m}} = 7.1 \cdot 10^{-3} \,\text{H.}$$
(515)

We have neglected the fact that the field inside the solenoid is not constant. It decreases near each end, to the point where the flux through the last turn is only about half the flux through a turn in the middle (see Exercise 6.63). This means that we have *over*estimated the inductance; the true value is smaller than the above approximate result. We might expect the error to be on the order of the diameter divided by the length, because the diameter should determine the length scale of the region near the end where the field differs appreciably from the idealized $B = \mu_0 nI$ value. This is 5% in the present example. The actual error is only about 2% in this case (which is consistent with 5%, as far as order of magnitude goes), as you can discover by referring to tables that give exact values for the inductance of cylindrical coils.

7.41. Opening a switch

After the switch has been closed a while, the currents are steady and the inductor is irrelevant. So 10 V is the initial voltage across each of the branches of the circuit. The initial currents across the 150Ω and 50Ω resistors are therefore 0.067 A and 0.2 A, respectively. They are both directed downward. Initially both A and B are at 10 V with respect to ground.

Right after the switch is opened, we have the circuit shown in Fig. 140. The current through the inductor cannot change abruptly (otherwise there would be an infinite $d\Phi/dt$ and hence infinite \mathcal{E} , which would cause the current to not change abruptly after all). Therefore, the current through the circuit is 0.2 A in the clockwise direction. The current *does* change abruptly in the 150 Ω resistor; it goes from 0.067 A downward to 0.2 A upward. The potential at B with respect to ground is still $V_B = (0.2 \text{ A})(50 \Omega) = 10 \text{ V}$, but the potential of A is now $V_A = -(0.2 \text{ A})(150 \Omega) = -30 \text{ V}$.









The circuit in Fig. 140 is a simple RL circuit, so as time goes on, the current equals $I(t) = I_0 e^{-(R/L)t}$, where $I_0 = 0.2$ A and where the time constant L/R equals $(0.1 \text{ H})/(200 \Omega) = 5 \cdot 10^{-4} \text{ s} = 0.5$ millisec. The potentials at A and B are proportional to I, so they decrease like $e^{-(R/L)t}$. After 0.5 millisec they have decreased by a factor 1/e = 0.37, and after 1 millisec by $1/e^2 = 0.14$. After 5 millisec the factor is $1/e^{10} = 4.5 \cdot 10^{-5}$, which is negligible. The plots are shown in Fig. 141. We have only plotted up to t = 2 millisec, because the curves are essentially zero after that. Note the discontinuity in V_A .

7.42. *RL* circuit

From Eq. (7.69) the current is $I(t) = I_0 (1 - e^{-(R/L)t})$, where $I_0 = \mathcal{E}_0/R$. In the problem at hand,

$$I_0 = \frac{\mathcal{E}_0}{R} = \frac{12 \,\mathrm{V}}{0.01 \,\Omega} = 1200 \,\mathrm{A} \qquad \text{and} \qquad \frac{R}{L} = \frac{0.01 \,\Omega}{0.5 \cdot 10^{-3} \,\mathrm{H}} = 20 \,\mathrm{s}^{-1}. \tag{516}$$

So the time scale is L/R = 0.05 s. The current reaches a value of $(0.9)I_0$ when

$$e^{-(R/L)t} = 0.1 \implies (20 \,\mathrm{s}^{-1})t = \ln 10 \implies t = 0.115 \,\mathrm{s}.$$
 (517)

At this time, the current is I = (0.9)(1200) = 1080 A, so the energy stored in the magnetic field is

$$\frac{1}{2}LI^2 = \frac{1}{2}(0.5 \cdot 10^{-3} \,\mathrm{H})(1080 \,\mathrm{A})^2 = 292 \,\mathrm{J}.$$
(518)

The instantaneous power delivered by the battery is $\mathcal{E}_0 I$, but since I is changing we must perform an integral to find the energy delivered by the battery between t = 0 and t = 0.115 s:

$$\int_{0}^{t=0.115 \,\mathrm{s}} \mathcal{E}_{0}I(t') \, dt' = \mathcal{E}_{0}I_{0} \int_{0}^{t} \left(1 - e^{-(R/L)t'}\right) dt'$$

$$= \mathcal{E}_{0}I_{0} \left(t' + \frac{L}{R}e^{-(R/L)t'}\right) \Big|_{0}^{t}$$

$$= \mathcal{E}_{0}I_{0} \left(t + \frac{L}{R}e^{-(R/L)t} - \frac{L}{R}\right)$$

$$= (12 \,\mathrm{V})(1200 \,\mathrm{A}) \left(0.115 \,\mathrm{s} + (0.05 \,\mathrm{s})(0.1) - (0.05 \,\mathrm{s})\right)$$

$$= 1008 \,\mathrm{J}. \tag{519}$$

From conservation of energy, apparently 1008 J - 292 J = 716 J has been dissipated in the resistor. The task of Problem 7.15 is to show that the energy delivered by the battery does indeed equal the energy stored in the magnetic field plus the energy dissipated in the resistor, at any general time t.

7.43. Energy in an *RL* circuit

The energy delivered by the battery is $\mathcal{E}_0 Q = \mathcal{E}_0 \int_0^t I \, dt$, and the power dissipated in the resistor is $I^2 R$. So our task is to show that

$$\mathcal{E}_0 \int_0^t I \, dt = \frac{1}{2} L I^2 + \int_0^t I^2 R \, dt.$$
 (520)

Using the expression for I given in Eq. (7.69), this becomes

$$\mathcal{E}_{0} \int_{0}^{t} \frac{\mathcal{E}_{0}}{R} \left(1 - e^{-(R/L)t} \right) dt = \frac{1}{2} L \frac{\mathcal{E}_{0}^{2}}{R^{2}} \left(1 - 2e^{-(R/L)t} + e^{-2(R/L)t} \right) + \int_{0}^{t} \frac{\mathcal{E}_{0}^{2}}{R^{2}} \left(1 - 2e^{-(R/L)t} + e^{-2(R/L)t} \right) R dt \iff t + \frac{L}{R} \left(e^{-(R/L)t} - 1 \right) = \frac{L}{2R} \left(1 - 2e^{-(R/L)t} + e^{-2(R/L)t} \right) + \left(t + \frac{2L}{R} \left(e^{-(R/L)t} - 1 \right) - \frac{L}{2R} \left(e^{-2(R/L)t} - 1 \right) \right),$$
(521)

which is indeed true, as you can check.

7.44. Magnetic energy in the galaxy

The energy density is

$$\frac{B^2}{2\mu_0} = \frac{(3 \cdot 10^{-10} \,\mathrm{T})^2}{2(4\pi \cdot 10^{-7} \,\frac{\mathrm{kg}\,\mathrm{m}}{\mathrm{C}^2})} = 3.6 \cdot 10^{-14} \,\mathrm{J/m^3}.$$
(522)

The volume of the galaxy is $\pi r^2 h = \pi (5 \cdot 10^{20} \text{ m})^2 (10^{19} \text{ m}) \approx 8 \cdot 10^{60} \text{ m}^3$, so the total energy contained in the magnetic field is $(3.6 \cdot 10^{-14} \text{ J/m}^3)(8 \cdot 10^{60} \text{ m}^3) \approx 3 \cdot 10^{47} \text{ J}$. The radiation power of starlight in the galaxy is 10^{37} J/s , so the magnetic energy is worth $(3 \cdot 10^{47} \text{ J})(10^{37} \text{ J/s}) = 3 \cdot 10^{10} \text{ s} \approx 1000 \text{ years}$.

7.45. Magnetic energy near a neutron star

The energy density is

$$\frac{B^2}{2\mu_0} = \frac{(10^{10}\,\mathrm{T})^2}{2\left(4\pi \cdot 10^{-7}\,\frac{\mathrm{kg}\,\mathrm{m}}{\mathrm{C}^2}\right)} = 4 \cdot 10^{25}\,\mathrm{J/m^3}.$$
(523)

Using $E = mc^2$, the amount of energy contained in 1 kg is $9 \cdot 10^{16}$ J. So the above energy density is equivalent to a mass density of

$$\rho = \frac{4 \cdot 10^{25} \,\mathrm{J/m^3}}{9 \cdot 10^{16} \,\mathrm{J/kg}} = 4.4 \cdot 10^8 \,\mathrm{kg/m^3} = 4.4 \cdot 10^5 \,\mathrm{g/cm^3}.$$
 (524)

This is very large. By comparison, the mass density of water is 1 g/cm^3 .

7.46. Decay time for current in the earth

From Eq. (6.54) the *B* field at the center of a ring is $\mu_0 I/2r$, which gives $B = \mu_0 I/2(a/2) = \mu_0 I/a$ here. The stored energy is then

$$U = \frac{B^2}{2\mu_0} \left(\text{volume} \right) = \frac{1}{2\mu_0} \left(\frac{\mu_0 I}{a} \right)^2 (\pi a^2 \cdot a) = \frac{\mu_0 \pi a I^2}{2} \,. \tag{525}$$

Since $R = \pi/a\sigma$, the ohmic energy dissipation is $I^2 R = \pi I^2/a\sigma$. The decay time is therefore

$$\tau \approx \frac{U}{I^2 R} = \frac{\mu_0 \pi a I^2 / 2}{\pi I^2 / a \sigma} = \frac{\mu_0 a^2 \sigma}{2} \sim \mu_0 a^2 \sigma,$$
 (526)

up to numerical factors. With a = 3000 km and $\sigma = 10^6$ (ohm-m)⁻¹, we have

$$\tau \sim \left(4\pi \cdot 10^{-7} \,\frac{\mathrm{kg\,m}}{\mathrm{C}^2}\right) \left(3 \cdot 10^6 \,\mathrm{m}\right)^2 \left(10^6 \,\,(\mathrm{ohm-m})^{-1}\right) \approx 1 \cdot 10^{13} \,\mathrm{s},\tag{527}$$

which is about 3000 centuries.

7.47. A dynamo

The device on the bottom is the dynamo. To see why, suppose a current I is flowing in the coil, in the clockwise direction in the upper circular loop. Such a current produces a magnetic field **B** pointing downward through the disk. With **v** the velocity of any part of the rotating disk, $\mathbf{v} \times \mathbf{B}$ is a vector pointing radially outward. Positive charges in the disk will be pushed outward, negative charges pushed inward. Either effect causes current to flow in the direction postulated. If we assume the opposite direction for the coil current I, the field **B** and hence the cross product $\mathbf{v} \times \mathbf{B}$ will be reversed, and the force will again be in the direction to sustain or increase the current. This conclusion is independent of the sign of the mobile charges. You can quickly verify that in the device on the top, the effect of the $q\mathbf{v} \times \mathbf{B}$ force on the charges in the disk is to decrease whatever current happened to be flowing in the coil.

See if you can formulate an unambiguous rule to distinguish the potential dynamo on the bottom from the non-dynamo on the top, a rule that refers only to the relation of disk rotation to coil configuration. Would a mirror image of the figure on the top represent a dynamo?

A dynamo of this kind runs equally well with current in either direction. The current can also be zero. However, in any circuit not at absolute zero there are slight random motions of charge, or randomly fluctuating currents. Some fluctuation, tremendously amplified by the "positive feedback" of the dynamo action, becomes the steady dynamo current. It retains the direction of its initial excitation. (In a conventional dc generator there is some residual magnetic field in the iron poles, even at zero current, which suffices to determine the eventual polarity.)

The magnitude of the current in this purely ohmic dynamo is determined by the input mechanical power. The current will be such that the ohmic power loss in the coil and disk is precisely equal to the applied torque times the shaft's angular speed.

Chapter 8

Alternating-current circuits

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8.16. Voltages and energies

At t = 0 the voltage across the capacitor is $V_0 \cos(0) = V_0$. So the voltage across the inductor must be $-V_0$, because the net voltage change around the loop is zero. The charge on the (top plate of the) capacitor is $CV = CV_0 \cos \omega t$. The clockwise current is then $I(t) = -dQ/dt = \omega CV_0 \sin \omega t$. This is zero at t = 0, so none of the energy is stored in the $LI^2/2$ in the inductor. All of the energy is stored in the $CV^2/2$ in the capacitor. This energy equals $CV_0^2/2$.

When $\omega t = \pi/2$ the voltage across the capacitor is $V_0 \cos(\pi/2) = 0$. So the voltage across the inductor must also be zero. The current at $\omega t = \pi/2$ is $I = \omega CV_0 \sin(\pi/2) = \omega CV_0$. Since the voltage across the capacitor is zero, none of the energy is stored in the $CV^2/2$ in the capacitor. All of the energy (which we know equals $CV_0^2/2$) is stored in the $LI^2/2$ in the inductor. As a double check, this energy is $L(\omega CV_0)^2/2 = \omega^2 LC^2 V_0^2/2$. Using $\omega = 1/\sqrt{LC}$, this becomes $CV_0^2/2$, as expected. The results are summarized in this table:

	ΔV_C	ΔV_L	U_C	U_L
t = 0	V_0	$-V_0$	$CV_0^2/2$	0
$t = \pi/2\omega$	0	0	0	$CV_0^2/2$

Note: At t = 0 you can also work out the voltage across the inductor directly, to double check that it equals $-V_0$. Using above form of I(t), the voltage across the inductor is $-L(dI/dt) = -\omega^2 LCV_0 \cos \omega t$. With $\omega = 1/\sqrt{LC}$, this becomes $-V_0 \cos \omega t$, which equals $-V_0$ at t = 0. However, it's risky to trust this minus sign. The magnitude is certainly correct, but it's best to check the sign by thinking about things physically. At t = 0 the current is zero but is increasing in the clockwise direction. The voltage above the inductor must therefore be higher than the voltage below; this difference is what causes the current to increase.

8.17. Amplitude after Q cycles

After Q cycles, the angle ωt equals $2\pi Q$. But from Eq. (8.13) we know that $Q = \omega/2\alpha$. So after Q cycles, the angle ωt equals $2\pi(\omega/2\alpha) = \pi(\omega/\alpha)$. The time t is therefore given by $t = \pi/\alpha$. The exponential factor $e^{-\alpha t}$ that appears in I(t) and V(t) therefore equals $e^{-\pi}$ as desired.

8.18. Effect of damping on frequency

From Eq. (8.13) we have $Q = \omega L/R \implies R/L = \omega/Q$, so Eq. (8.9) becomes

$$\omega^2 = \omega_0^2 - \frac{\omega^2}{4Q^2} \implies \omega^2 = \frac{\omega_0^2}{1 + \frac{1}{4Q^2}}.$$
 (528)

For large Q we can make the approximation,

$$\omega = \omega_0 \left(1 + \frac{1}{4Q^2} \right)^{-1/2} \approx \omega_0 \left(1 - \frac{1}{2} \cdot \frac{1}{4Q^2} \right).$$
 (529)

The fractional shift in ω for Q = 1000 is

$$\frac{\omega_0 - \omega}{\omega_0} = \frac{1}{8Q^2} = \frac{1}{8 \cdot 10^6} = 1.25 \cdot 10^{-7},\tag{530}$$

or $1.25 \cdot 10^{-5}$ percent. The fractional shift in ω for Q = 5 (for which the above Taylor approximation is still quite good) is

$$\frac{\omega_0 - \omega}{\omega_0} = \frac{1}{8Q^2} = \frac{1}{8 \cdot 25} = 0.005,$$
(531)

or 0.5 percent, which is still rather small.

8.19. Decaying signal

(a) The impedance of the $10^5 \Omega$ resistor is much larger than the impedances of the other circuit elements we will find below, so we can neglect the current through the $10^5 \Omega$ resistor, at least during the initial period right after the switch is closed. During this period we therefore essentially have a series *RLC* circuit in the right loop.

There are two things we can estimate by looking at the given plot: the frequency of oscillation and the rate of decay of the signal. These two things will allow us to calculate C and R, respectively.

In cases where the voltage doesn't immediately become negligible after a few oscillations, the 1/LC term in Eq. (8.9) dominates (see Problem 8.5), so the frequency is essentially given by $\omega = 1/\sqrt{LC}$. In the given plot, four cycles are completed in 10^{-3} sec,¹ so $\omega = 2\pi \cdot 4/(10^{-3} \text{ s}) = 2.5 \cdot 10^4 \text{ s}^{-1}$. The capacitance is then

$$C = \frac{1}{\omega^2 L} = \frac{1}{(2.5 \cdot 10^4 \,\mathrm{s}^{-1})^2 (0.01 \,\mathrm{H})} = 1.6 \cdot 10^{-7} \,\mathrm{F}.$$
 (532)

(b) The decay constant is $\alpha = R/2L$, so the time for the voltage to decrease by a factor of 1/e is t = 2L/R. From the figure, this time is about $0.5 \cdot 10^{-3}$ s. Therefore,

$$t = \frac{2L}{R} \implies R = \frac{2L}{t} = \frac{2(0.01 \text{ H})}{0.5 \cdot 10^{-3} \text{ s}} = 40 \ \Omega.$$
 (533)

As promised, this is negligible compared with the $10^5 \Omega$ resistor. Additionally (although at this point we technically haven't gotten to impedances in this chapter), the impedances of the inductor and capacitor have magnitudes $\omega L = (2.5 \cdot 10^4 \, \text{s}^{-1})(0.01 \, \text{H}) = 250 \, \Omega$ and $1/\omega C = 1/(2.5 \cdot 10^4 \, \text{s}^{-1})(1.6 \cdot 10^{-7} \, \text{F}) = 250 \, \Omega$. (These are equal because we are using $\omega^2 = 1/LC$.) These are also negligible compared with $10^5 \, \Omega$. But it was fine to just assume that here.

¹In the first printing of the book, the millisec span was drawn a little too long.

(c) After a long time, we essentially have just two resistors of $10^5 \Omega$ and 40Ω in series with a 20 V dc battery. (No current passes through the capacitor in the eventual direct-current steady state. And there is no voltage drop across the inductor for constant current.) So the voltage across the oscilloscope is

$$V_{40\,\Omega} = \left(\frac{40}{10^5 + 40}\right)(20\,\mathrm{V}) = 0.008\,\mathrm{V},\tag{534}$$

or 8 millivolts. The 40 in the denominator is of course inconsequential.

8.20. Resonant cavity

As indicated in Fig. 8.33, let *a* be the inner radius, *b* the outer radius, *h* the height, and *s* the gap in the capacitor. From Eq. (7.62) with N = 1, the inductance of the single-turn toroid is $(\mu_0 h/2\pi) \ln(b/a)$. The capacitance of the gap is $\epsilon_0(\pi a^2)/s$. The resonant frequency is therefore

$$\omega = \frac{1}{\sqrt{LC}} = \sqrt{\frac{2s}{\mu_0 \epsilon_0 h a^2 \ln(b/a)}} = \frac{c}{a} \sqrt{\frac{2s}{h \ln(b/a)}}, \qquad (535)$$

where we have used $c^2 = 1/\mu_0\epsilon_0$. The fields are shown in Fig. 142. The electric field spans the *s* gap. From Exercise 6.61, we know that the magnetic field points tangentially around the toroid. The current flows "around" the toroid in the same manner as it did in Exercise 6.61, except that it doesn't complete a full loop; the charge simply piles up on the end of the inner cylinder and on the corresponding part of the top face of the torus (which completes the capacitor). The charge oscillates back and forth. At the instants when the electric field is maximum, the current is zero so there is no magnetic field. At the instants when the magnetic field is maximum, the charge on the capacitor is zero so there is no electric field.

8.21. Solving an *RLC* circuit

Let I_1 and I_2 be the loop currents in the left and right loops, respectively, with clockwise taken to be positive. Let V be the voltage across the capacitor, taken to be positive when the upper plate of the capacitor is positive. The statements involving the three equal voltage drops are

$$V = \frac{Q}{C}, \qquad V = R'(I_1 - I_2), \qquad V = L\frac{dI_2}{dt}.$$
 (536)

But $I_1 = -dQ/dt$, so if we take the second derivative of the first statement, and also the first derivative of the second statement, we obtain

$$\frac{d^2 V}{dt^2} = -\frac{1}{C} \frac{dI_1}{dt}, \qquad \frac{dV}{dt} = R' \left(\frac{dI_1}{dt} - \frac{dI_2}{dt} \right), \qquad V = L \frac{dI_2}{dt}.$$
(537)

We can now eliminate I_1 and I_2 in favor of V by plugging the results for dI_1/dt and dI_2/dt from the first and third equations into the second. The result is

$$\frac{dV}{dt} = R'\left(-C\frac{d^2V}{dt^2} - \frac{V}{L}\right) \implies \frac{d^2V}{dt^2} + \left(\frac{1}{R'C}\right)\frac{dV}{dt} + \left(\frac{1}{LC}\right)V = 0.$$
(538)

Using the same trial solution for V as in Eq. (8.4) (and Eq. (8.5)), substituting into Eq. (538), and demanding that the coefficients of $\sin \omega t$ and $\cos \omega t$ independently vanish, we obtain, in place of Eq. (8.7),

$$2\alpha\omega - \frac{\omega}{R'C} = 0 \quad \text{and} \quad \alpha^2 - \omega^2 - \frac{\alpha}{R'C} + \frac{1}{LC} = 0, \quad (539)$$



Figure 142

which give the conditions,

$$\alpha = \frac{1}{2R'C} \quad \text{and} \quad \omega^2 = \frac{1}{LC} - \frac{1}{4R'^2C^2}.$$
(540)

Alternatively, note that Eq. (538) is identical to Eq. (8.2) if 1/R'C = R/L, that is, if R = L/R'C. So we can quickly obtain the results in Eq. (540) by simply replacing the R in Eqs. (8.8) and (8.9) with L/R'C.

Now let's assume that L, C, and Q are the same in the series and parallel circuits. Since $Q = \omega/2\alpha$, we just need to equate the values of $\omega/2\alpha$ for the two circuits. Using Eqs. (8.8) and (8.9), along with Eq. (540), you can show that the values of $\omega/2\alpha$ are equal if $1/R'C = R/L \implies R' = L/RC$. In view of the preceding paragraph, this is no surprise, because the time dependence of V is exactly the same in the two circuits if R' = L/RC. And the energy (which appears in the definition of Q) is proportional to V^2 .

8.22. Overdamped oscillator

From Problem 8.4 the solutions for β are

$$\beta_{1,2} = \frac{1}{2} \left(\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \right) = \frac{R}{2L} \left(1 \pm \sqrt{1 - \frac{4L}{R^2C}} \right).$$
(541)

The roots are real if $R \ge 2\sqrt{L/C}$, in which case we have exponentially decaying motion instead of (decaying) oscillatory motion. Note that both roots are positive, as they must be, because a voltage that grows with time would violate conservation of energy. By linearity the most general solution for V is

$$V(t) = Ae^{-\beta_1 t} + Be^{-\beta_2 t}.$$
(542)

With $R = 600 \Omega$, $L = 10^{-4}$ H, and $C = 10^{-8}$ F, we have

$$\frac{R}{2L} = \frac{600\,\Omega}{2\cdot10^{-4}\,\mathrm{H}} = 3\cdot10^6\,\mathrm{s}^{-1} \qquad \text{and} \qquad \frac{4L}{R^2C} = \frac{4\cdot10^{-4}\,\mathrm{H}}{(600\,\Omega)^2(10^{-8}\,\mathrm{F})} = \frac{1}{9}\,.$$
(543)

Therefore,

$$\beta_1 = (3 \cdot 10^6 \,\mathrm{s}^{-1}) \left(1 + \sqrt{8/9} \right) = 5.83 \cdot 10^6 \,\mathrm{s}^{-1}, \beta_2 = (3 \cdot 10^6 \,\mathrm{s}^{-1}) \left(1 - \sqrt{8/9} \right) = 0.172 \cdot 10^6 \,\mathrm{s}^{-1}.$$
(544)

The constants A and B are determined by the initial conditions. In the setup in Fig. 8.4, the current is zero at t = 0 (because it can't increase abruptly, due to the inductor), so dQ/dt = 0. And since Q = CV, our initial condition is therefore dV/dt = 0 at t = 0. From Eq. (542), the value of dV/dt at t = 0 is $-\beta_1 A - \beta_2 B$. This equals zero if $B/A = -\beta_1/\beta_2 \approx -34$.

Since β_2 is much smaller than β_1 , the $Be^{-\beta_2 t}$ term goes to zero much more slowly than the $Be^{-\beta_1 t}$ term. After a microsecond or so, the $Be^{-\beta_2 t}$ term is essentially all that is left.

8.23. Energy in an *RLC* circuit

The total energy in the capacitor and inductor is $CV^2/2 + LI^2/2$. In the underdamped case, the voltage is given in Eq. (8.10) as $V(t) = Ae^{-\alpha t} \cos \omega t$ (we can ignore the $\sin \omega t$

term by adjusting the t = 0 origin). The current is I = -dQ/dt = -C(dV/dt), so we have $I(t) = ACe^{-\alpha t} (\omega \sin \omega t + \alpha \cos \omega t)$. The total energy in the capacitor and inductor is therefore

$$\frac{1}{2}CV^{2} + \frac{1}{2}LI^{2} = \frac{1}{2}CA^{2}e^{-2\alpha t} \Big(\cos^{2}\omega t + LC\big(\omega\sin\omega t + \alpha\cos\omega t\big)^{2}\Big).$$
 (545)

For the overdamped case, the voltage is given in Eq. (8.15) as $V(t) = Ae^{-\beta_1 t} + Be^{-\beta_2 t}$. The current is then $I(t) = -C(dV/dt) = AC\beta_1 e^{-\beta_1 t} + BC\beta_2 e^{-\beta_2 t}$. The total energy is therefore

$$\frac{1}{2}CV^2 + \frac{1}{2}LI^2 = \frac{1}{2}C\left(Ae^{-\beta_1 t} + Be^{-\beta_2 t}\right)^2 + \frac{1}{2}LC^2\left(A\beta_1 e^{-\beta_1 t} + B\beta_2 e^{-\beta_2 t}\right)^2.$$
 (546)

For the critically damped case, the voltage is given in Eq. (8.16) as $V(t) = (A + Bt)e^{-\beta t}$, where $\beta = 1/\sqrt{LC}$. This form of β follows from Eq. (12.389) in the solution to Problem 8.4 when $R = 2\sqrt{L/C}$. The current is then $I(t) = -C(dV/dt) = Ce^{-\beta t}(\beta(A + Bt) - B)$. The total energy is therefore

$$\frac{1}{2}CV^2 + \frac{1}{2}LI^2 = \frac{1}{2}Ce^{-2\beta t} \Big((A+Bt)^2 + LC\big(\beta(A+Bt) - B\big)^2 \Big).$$
(547)

How fast does the energy decay in these three cases? In the underdamped case, it decays like $e^{-2\alpha t}$. In the overdamped case, if β_2 is the smaller of the two β 's, then for large t the energy decays like $e^{-2\beta_2 t}$. In the critically damped case, it decays like $e^{-2\beta t}$. To show that the energy decays most quickly in the critically damped case, we must show that $\beta > \alpha$ (so that critical damping decays faster than underdamping), and also that $\beta > \beta_2$ (so that critical damping decays faster than overdamping).

It is indeed true that $\beta > \alpha$, because $\beta = 1/\sqrt{LC}$ and $\alpha = R/2L$, and the condition for underdamping is precisely $R/2L < 1/\sqrt{LC}$. Hence $\alpha < \beta$.

It is also true that $\beta > \beta_2$, because this statement is equivalent to (using the result for β_2 in Eq. (12.389))

$$\frac{1}{\sqrt{LC}} > \frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \iff \beta > \alpha - \sqrt{\alpha^2 - \beta^2} \iff \sqrt{\alpha^2 - \beta^2} > \alpha - \beta \Leftrightarrow \sqrt{\alpha - \beta}\sqrt{\alpha + \beta} > \sqrt{\alpha - \beta}\sqrt{\alpha - \beta} \Leftrightarrow \sqrt{\alpha + \beta} > \sqrt{\alpha - \beta}, \tag{548}$$

which is indeed true. Note that $\alpha > \beta$ because $R/2L > 1/\sqrt{LC}$ in the overdamped case, so these square roots are real.

For large t, the critically-damped energy in Eq. (547) is essentially equal to $Ce^{-2\beta t}B^2t^2$, where we have used $LC\beta^2 = 1$. Note that the capacitor and inductor have the same energy in the limit of large t.

Intuitively, it is believable that the energy decay is quickest for critical damping, because the decay is very slow at both extremes of very light damping and very heavy damping. So the maximum decay rate must occur somewhere in between, and critical damping is as good a guess as any. The decay is slow at the extremes because for fixed L and C, light damping means small R, so there is essentially no resistor in which the energy can dissipate; the energy just sloshes back and forth between the capacitor and inductor, with hardly any overall decrease. And large damping means large R, so there is essentially no current, making the I^2R term negligible; the energy slowly leaks out of the capacitor.

8.24. RC circuit with a voltage source

This exercise is a special case of the general RLC circuit we solved in Section 8.3. The loop equation here is

$$RI(t) + \frac{Q(t)}{C} = \mathcal{E}_0 \cos \omega t.$$
(549)

Let us replace $\cos \omega t$ with $e^{i\omega t}$, and then guess an exponential solution of the form $\tilde{I}(t) = \tilde{I}e^{i\omega t}$. If $\tilde{I}e^{i\omega t}$ satisfies the equation with an $e^{i\omega t}$ on the right side, then taking the real part of the entire equation tells us that $\operatorname{Re}(\tilde{I}e^{i\omega t})$ satisfies the equation with a $\cos \omega t$ on the right side.

If $\tilde{I}(t) = \tilde{I}e^{i\omega t}$, then $\tilde{Q}(t)$, which is the integral of $\tilde{I}(t)$, equals $\tilde{I}e^{i\omega t}/i\omega$. (There is no need for a constant of integration because we know that Q(t) oscillates around zero.) So we obtain

$$R\tilde{I}e^{i\omega t} + \frac{\tilde{I}e^{i\omega t}}{i\omega C} = \mathcal{E}_0 e^{i\omega t} \implies \tilde{I} = \frac{\mathcal{E}_0}{R+1/i\omega C}.$$
(550)

Getting the *i* out of the denominator, we can write \tilde{I} in polar form as

$$\tilde{I} = \frac{\mathcal{E}_0(R - 1/i\omega C)}{R^2 + 1/\omega^2 C^2} = \frac{\mathcal{E}_0}{R^2 + 1/\omega^2 C^2} \cdot (R + i/\omega C)$$
$$= \frac{\mathcal{E}_0}{R^2 + 1/\omega^2 C^2} \cdot \sqrt{R^2 + 1/\omega^2 C^2} e^{i\phi} = \frac{\mathcal{E}_0}{\sqrt{R^2 + 1/\omega^2 C^2}} e^{i\phi}, \quad (551)$$

where $\tan \phi = 1/R\omega C$. The actual current is then

$$I(t) = \operatorname{Re}(\tilde{I}e^{i\omega t}) = \operatorname{Re}\left(\frac{\mathcal{E}_0}{\sqrt{R^2 + 1/\omega^2 C^2}} e^{i\phi} e^{i\omega t}\right) = \frac{\mathcal{E}_0}{\sqrt{R^2 + 1/\omega^2 C^2}} \cos(\omega t + \phi).$$
(552)

For large ω , the amplitude of the current goes to \mathcal{E}_0/R , and the phase ϕ goes to zero. This makes sense, because the capacitor essentially isn't there (that is, it behaves like a short circuit) because the oscillations happen too quickly for any charge to build up on the capacitor. So we simply have a resistor in series with the voltage source.

For small ω , the amplitude of the current goes to zero, and the phase ϕ goes to $\pi/2$. In this case, the charge (which has a maximum value of $C\mathcal{E}_0$ on the capacitor) sloshes back and forth very slowly, so the current is very small. The resistor essentially isn't there (the voltage drop IR across it is very small). So we simply have a capacitor in series with the voltage source. And $\phi = \pi/2$ for such a circuit. (The current is ahead of the voltage, because the current reaches its maximum while charge is building up on the capacitor, and then a quarter cycle later the charge reaches its maximum. We are taking Q to be the charge on the top plate of the capacitor, as we did in Section 8.3.)

8.25. Light bulb

The normal current for a 60 watt, 120 volt light bulb is I = P/V = (60 W)/(120 V) = 0.5 A. The resistance of the filament is then $R = V/I = (120 \text{ V})/(0.5 \text{ A}) = 240 \Omega$. (This could also be obtained from $P = V^2/R \implies R = V^2/P$.) We want to have the same current, 0.5 A, when the bulb is connected in series with an impedance of $i\omega L$, across 240 volts. (We want the same current because the resistor has a fixed resistance, so the power is determined by the current flowing through it; and the power when operating normally is 60 watts.) The magnitude of the total impedance is $|Z| = \sqrt{R^2 + (\omega L)^2}$, so the current will equal 0.5 A if

$$0.5 = \frac{V}{|Z|} = \frac{V}{\sqrt{R^2 + (\omega L)^2}} = \frac{240 \,\mathrm{V}}{\sqrt{(240 \,\Omega)^2 + (\omega L)^2}} \,.$$
(553)

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(Ohm's law works with |Z|; see Eq. (8.77).) Solving for ωL gives $\omega L = 240\sqrt{3} \Omega = 416 \Omega$. And since $\omega = 2\pi\nu = 2\pi(60 \text{ s}^{-1}) = 377 \text{ s}^{-1}$, we have $L = (416 \Omega)/(377 \text{ s}^{-1}) = 1.10 \text{ H}$.

8.26. Label the curves

The main point here is that in a series RLC circuit, V_R , V_L , and V_C are 90° out of phase with each other, while the applied voltage doesn't (in general) have a nice phase relation with the other three voltages. So we quickly see that the third curve must be the applied \mathcal{E} .

Also, in a series RLC circuit, V_L and V_C are 180° out of phase. So they must correspond to the 2nd and 4th curves. We can determine which is which by using the fact that V_L is 90° ahead of V_R (which is in phase with the current), which is 90° ahead of V_C . The 4th curve reaches a maximum 90° before the 1st, which in turn is 90° ahead of the 2nd. So the 4th is L, the 1st is R, and the 2nd is C. The order of the curves is therefore R, C, \mathcal{E}, L .

Note that \mathcal{E} is slightly ahead of R (or equivalently I). So I is behind \mathcal{E} . This means that if $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$, then the angle ϕ in $I(t) = I_0 \cos(\omega t + \phi)$ is negative. So from Eq. (8.39), we see that ωL must be larger than $1/\omega C$. That is, the impedance of the inductor is larger than the impedance of the capacitor.

8.27. RLC parallel circuit

Admittances add in parallel, so

$$\frac{1}{Z} = Y = \frac{1}{R} - \frac{i}{\omega L} + i\omega C.$$
(554)

The given values are $R = 10^3 \Omega$, $C = 5 \cdot 10^{-10} \text{ F}$, and $L = 2 \cdot 10^{-3} \text{ H}$. For 10 kHz we have $\omega = 2\pi (10^4 \text{ s}^{-1}) = 6.28 \cdot 10^4 \text{ s}^{-1}$. Therefore,

$$\frac{1}{Z} = \frac{1}{1000} - \frac{i}{125.6} + (3.14 \cdot 10^{-5})i = 10^{-3} (1 - 7.93i) \,\Omega^{-1}.$$
 (555)

Note that the effect of the capacitor is very small. The impedance is

$$Z = \frac{1}{10^{-3}(1-7.93i)} = \frac{10^3(1+7.93i)}{1+7.93^2} = (15.7+124i)\,\Omega.$$
 (556)

The inductor dominates the admittance, so Z is roughly equal to $i\omega L$. The frequency is low enough so that the inductor lets current through easily compared with the resistor and capacitor.

For 10 MHz we have $\omega = 2\pi (10^7 \, \text{s}^{-1}) = 6.28 \cdot 10^7 \, \text{s}^{-1}$. Therefore,

$$\frac{1}{Z} = \frac{1}{1000} - \frac{i}{1.256 \cdot 10^5} + (3.14 \cdot 10^{-2})i = 10^{-3} (1+31.4i) \,\Omega^{-1}.$$
 (557)

The effect of the inductor is now negligible. The impedance is

$$Z = \frac{1}{10^{-3}(1+31.4i)} = \frac{10^3(1-31.4i)}{1+31.4^2} = (1.01-31.8i)\,\Omega.$$
 (558)

The capacitor now dominates the admittance, so Z is roughly equal to $1/i\omega C$. The frequency is high enough so that the capacitor lets current through easily compared with the resistor and inductor.

To find the frequency for which |Z| is largest, Eq. (554) gives

$$Z = \frac{1}{1/R + i(\omega C - 1/\omega L)} \implies |Z| = \frac{1}{\sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2}}.$$
 (559)

This is maximum when the denominator is smallest, that is, when

$$\omega C = \frac{1}{\omega L} \implies \omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2 \cdot 10^{-3} \,\mathrm{H})(5 \cdot 10^{-10} \,\mathrm{F})}} = 10^6 \,\mathrm{s}^{-1}, \qquad (560)$$

or equivalently $\nu = \omega/2\pi = 159$ kHz. The maximum impedance is simply $|Z|_{\text{max}} = R = 1000 \,\Omega$. At this frequency, the inductor and capacitor always have equal and opposite currents, so they cancel each other out and effectively don't exist.

8.28. Small impedance

We have a capacitor in series with the parallel combination of a resistor and an inductor. So the total impedance is

$$Z = -\frac{i}{\omega C} + \frac{R(i\omega L)}{R + i\omega L}.$$
(561)

Setting $\omega = 1/\sqrt{LC}$ and combining these two fractions yields

$$Z = \frac{L/C}{R + i\sqrt{L/C}} \,. \tag{562}$$

If we want this to be small, then we should make R be large. For $R \to \infty$, Z goes to zero. This is because our choice of ω causes the impedances of the inductor and capacitor to exactly cancel. So if $R = \infty$, the inductor and capacitor are in series, and their impedances add up to zero. The system is on resonance. Any non-infinite value of R will destroy the exact cancelation and produce a nonzero impedance. So we have the counterintuitive result that decreasing the resistance in the circuit increases the impedance.

In the limit where $R \approx 0$ we have $Z = -i\sqrt{L/C}$. In this case, the parallel combination of the R and L has zero impedance, so we are left with only the impedance of the capacitor, which is $1/i\omega C = -i\sqrt{LC}/C = -i\sqrt{L/C}$, as desired. This case has the largest possible |Z|.

8.29. Real impedance

We have an inductor in series with the parallel combination of a resistor and a capacitor. So the total impedance is

$$Z = i\omega L + \frac{1}{1/R + i\omega C} = i\omega L + \frac{R}{1 + i\omega CR} = i\omega L + \frac{R(1 - i\omega CR)}{1 + \omega^2 C^2 R^2}.$$
 (563)

Setting the imaginary part of this equal to zero gives

$$\omega L(1 + \omega^2 C^2 R^2) - \omega C R^2 = 0 \implies \omega^2 = \frac{1}{LC} - \frac{1}{R^2 C^2}.$$
 (564)

So the answer is "yes," provided that $R^2 > L/C$, so that ω^2 is a positive quantity. Note that $\omega = 0$ is also a solution that makes the imaginary part of Z be zero. In this case, the capacitor lets through no current (its impedance is infinite), and the inductor is effectively just a short-circuit (its impedance is zero). So we effectively have only the resistor.

8.30. Equal impedance?

Setting the impedances of the two circuits equal to each other gives

$$\frac{R(i\omega L)}{R+i\omega L} = R + \frac{1}{i\omega C} \implies \frac{iR^2\omega L + R\omega^2 L^2}{R^2 + \omega^2 L^2} = R - \frac{i}{\omega C}.$$
 (565)

Equating the real and imaginary parts on the left and right sides of this equation gives

$$\frac{R\omega^2 L^2}{R^2 + \omega^2 L^2} = R$$
 and $\frac{R^2 \omega L}{R^2 + \omega^2 L^2} = -\frac{1}{\omega C}$. (566)

The left equation is true if R = 0 or if $L = \infty$ (or if $\omega = \infty$, but it is understood that we want to find a condition that works for all ω). In the right equation, the negative sign implies that the only way the relation can be true is if both sides are zero. So we must have $C = \infty$ and either R = 0 or $L = \infty$. (Again, $\omega = \infty$ works too.) The condition for the two circuits to have equal impedances is therefore: $C = \infty$ and either R = 0 or $L = \infty$. In the case where $C = \infty$ and R = 0, the impedance of both circuits is zero; the capacitor and resistor are short circuits (the impedance of the inductor is nonzero, but that doesn't matter). In the case where $C = \infty$ and $L = \infty$, the impedance of both circuits is R; the capacitor is a short circuit and the impedance of the inductor is infinite.

Physically, the reason why there are only a couple special solutions, both of which involve some infinities, is that the magnitude of the impedance of the parallel combination is less than or equal to R, while for the series combination it is greater than or equal to R.

Note: if you solve this exercise by multiplying Eq. (565) through by $(R + i\omega L)(i\omega C)$ and simplifying, you must be careful. A spurious solution is introduced, and a valid solution is missed.

8.31. Zero voltage difference

Both branches have the same voltage difference V_0 , so the complex currents are given by

$$I_A = \frac{V_0}{R_1 + 1/i\omega C}$$
 and $I_B = \frac{V_0}{R_2 + i\omega L}$. (567)

The voltage at A is $V_A = V_0 - I_A(1/i\omega C)$, and at B is it $V_B = V_0 - I_B R_2$. Therefore,

$$V_B - V_A = -I_B R_2 + I_A(1/i\omega C) = -\frac{V_0}{R_2 + i\omega L} \cdot R_2 + \frac{V_0}{R_1 + 1/i\omega C} \cdot \frac{1}{i\omega C}$$
$$= -\frac{V_0 R_2}{R_2 + i\omega L} + \frac{V_0}{i\omega C R_1 + 1}.$$
(568)

This vanishes if

$$R_2(i\omega CR_1 + 1) = R_2 + i\omega L \implies i\omega CR_1R_2 = i\omega L \implies R_1R_2 = \frac{L}{C}, \quad (569)$$

as desired. Given a known capacitance C and known resistors R_1 and R_2 , we could determine an unknown L by adjusting C, or R_1 , or R_2 , until we obtain $V_B - V_A = 0$. But a real inductor generally has some resistance too, so we effectively have another resistor r in series with R_2 and L. This makes things more complicated.



8.32. Finding L

The setup is shown in Fig. 143. We can quickly determine the amplitude of the current (or the rms value, depending on what the voltmeter is calibrated to read; the final value of L won't depend on the choice). The frequency is $\omega = 2\pi (1000 \,\mathrm{s}^{-1}) = 6283 \,\mathrm{s}^{-1}$, so the $V_0 = I_0 |Z|$ statement for the capacitor alone yields

15.5 V =
$$\frac{I_0}{\omega C}$$
 \implies $I_0 = (15.5 \text{ V})(6283 \text{ s}^{-1})(10^{-6} \text{ F}) = 0.0974 \text{ A}.$ (570)

Since the elements are in series, this is the current through all of the components in the circuit. The $V_0 = I_0 |Z|$ statement for the whole circuit then tells us that (ignoring the units)

$$I_0 = \frac{V_0}{|Z|} \implies 0.0974 = \frac{10.1}{\sqrt{(35)^2 + (\omega L - 1/\omega C)^2}} \implies \omega L - 1/\omega C = \pm 97.6 \,\Omega.$$
(571)

Note that there are two roots. We therefore have

$$L = \frac{1}{\omega^2 C} \pm \frac{97.6}{\omega} \implies 0.0253 \pm 0.0155 \implies L = 0.041 \text{ H} \text{ or } 0.0098 \text{ H.}$$
(572)

So L could be 41 mH or 9.8 mH. The amplitude of the voltage across the inductor alone is $I_0\omega L$, which gives 25.1 V and 6.0 V for the two possibilities. If we then measure the voltage across the inductor and obtain 25.4 V, the second possibility is ruled out, and we have reasonably good agreement with the computed value of 25.1 V.

8.33. Equivalent boxes

=

The impedance of the top circuit is (ignoring the units)

$$Z = 1000 + \frac{1}{1/4000 + i\omega \cdot 10^{-6}} = \frac{1000(1 + 4 \cdot 10^{-3}i\omega) + 4000}{1 + 4 \cdot 10^{-3}i\omega}$$
$$= \frac{5000 + 4i\omega}{1 + 4 \cdot 10^{-3}i\omega} = \frac{(5000 + 4i\omega)(1 - 4 \cdot 10^{-3}i\omega)}{1 + 16 \cdot 10^{-6}\omega^2}$$
$$= \frac{5000 + 16 \cdot 10^{-3}\omega^2 - 16i\omega}{1 + 16 \cdot 10^{-6}\omega^2}.$$
(573)

The impedance of the bottom circuit is given by

$$\frac{1}{Z} = \frac{1}{5000} + \frac{1}{1250 + 10^6/(0.64)i\omega} = \frac{1}{5000} + \frac{0.64 \cdot 10^{-6}i\omega}{8 \cdot 10^{-4}i\omega + 1} \\
= \frac{(1 + 8 \cdot 10^{-4}i\omega) + 32 \cdot 10^{-4}i\omega}{5000(1 + 8 \cdot 10^{-4}i\omega)} \\
\Rightarrow Z = \frac{5000 + 4i\omega}{1 + 4 \cdot 10^{-3}i\omega},$$
(574)

in agreement with an intermediate step for the top circuit.

Now let's find the general rules for constructing the bottom circuit, given the top circuit. Consider the general circuits shown in Fig. 144. If these two circuits are to have the same impedance for all values of ω , then in particular they must have the same impedance in the $\omega \to 0$ and $\omega \to \infty$ limits. In the $\omega \to 0$ limit, no current goes through the capacitors, so we must have $R_1 + R_2 = R_3$ (which is indeed true for the given circuits). In the $\omega \to \infty$ limit, the capacitor has no impedance, so we



Figure 144

Figure 143

must have $R_1 = R_3 R_4/(R_3 + R_4)$. Using $R_3 = R_1 + R_2$ and solving for R_4 gives $R_4 = R_1(R_1 + R_2)/R_2$ (which is again true for the given circuits).

If the circuits are equivalent at every frequency, they must behave the same way for any pulse or transient. Consider a battery applied across the terminals which charges the capacitor in each circuit. When we remove the battery, the voltage between the terminals will decay with time constant R_2C_1 in the top circuit and $(R_3 + R_4)C_2$ in the bottom circuit. These times are equal if $C_2 = C_1R_2/(R_3 + R_4)$. Using the above values of R_3 and R_4 , this becomes $C_2 = C_1R_2^2/(R_1 + R_2)^2$. This is true for the given circuits.

Of course, demanding that the circuits are equivalent in the above three special cases doesn't actually prove that they are equivalent for all frequencies and scenarios, although it turns out that this is indeed the case. All we've shown is that *if* they are equivalent, *then* R_3 , R_4 , and C_2 must take on the above values. To be rigorous, we can set the two impedances equal:

$$R_1 + \frac{1}{\frac{1}{R_2} + i\omega C_1} = \frac{1}{\frac{1}{R_3} + \frac{1}{R_4 + \frac{1}{i\omega C_2}}}.$$
(575)

If you get everything on one side of the equation and simplify, you will find that there are three different powers of ω . The three coefficients must each be zero, if the equation is to be true for all ω . As you can verify, the term with the smallest power of ω yields the $\omega \to 0$ limit above, the term with the highest power yields the $\omega \to \infty$ limit, and the middle power (combined with the information from the other two) yields C_2 .

8.34. LC chain

The right inductor and Z_0 are in series, so they form an impedance of $Z_0 + i\omega L$, which is then in parallel with the capacitor, all of which is in series with the left inductor. The total impedance of the circuit between the input terminals is therefore

$$Z = i\omega L + \frac{(Z_0 + i\omega L)\frac{1}{i\omega C}}{(Z_0 + i\omega L) + \frac{1}{i\omega C}} = i\omega L + \frac{Z_0 + i\omega L}{iZ_0\omega C - \omega^2 LC + 1}.$$
(576)

Setting this equal to Z_0 gives

$$Z_0(iZ_0\omega C - \omega^2 LC + 1) = i\omega L(iZ_0\omega C - \omega^2 LC + 1) + \aleph_{\aleph} + i\omega L$$

$$\implies Z_0^2 C = L(-\omega^2 LC + 1) + L$$

$$\implies Z_0 = \sqrt{(2 - \omega^2 LC)(L/C)}.$$
(577)

 Z_0 is real if $\omega^2 < 2/LC$, as predicted. When $\omega = \sqrt{2/LC}$, we have $Z_0 = 0$. So the circuit looks like the top one shown in Fig. 145, with a short circuit on the right. (You can quickly double check from scratch that the impedance of this circuit is zero when $\omega = \sqrt{2/LC}$.) This zero impedance makes sense, because the resonant frequency of each of the two halves of the circuit shown on the bottom in Fig. 145 is $\omega_0 = 1/\sqrt{L(C/2)} = \sqrt{2/LC}$. Each loop resonates with this frequency, without the need for any applied voltage. Since the impedance is defined by V = IZ, a nonzero current with zero voltage implies zero impedance.





Figure 145

8.35. RC circuit

(a) The total impedance is

$$Z = R - \frac{i}{\omega C} = 2000 \,\Omega - \frac{i}{(377 \,\mathrm{s}^{-1})(10^{-6} \,\mathrm{F})} = (2000 - 2650i) \,\Omega.$$
(578)

The magnitude is $|Z| = \sqrt{2000^2 + 2650^2} = 3320 \,\Omega.$

(b) The rms voltage is V = 120 V, so the rms current is

$$I = \frac{V}{|Z|} = \frac{120 \,\mathrm{V}}{3320 \,\Omega} = 0.036 \,\mathrm{A}. \tag{579}$$

(c) The power dissipated (across just the resistor, of course) is

$$P = I^2 R = (0.036 \,\mathrm{A})^2 (2000 \,\Omega) = 2.6 \,\mathrm{W}.$$
(580)

There is no need for a factor of 1/2 since we are using rms values. We can alternatively use the $P = VI \cos \phi$ expression (where these are the rms values). Here V = 120 V, I = 0.036 A, and $\tan \phi = 2650/2000$ which yields $\cos \phi = 0.60$. These quantities yield P = 2.6 W, as desired.

(d) A voltmeter connected across the resistor will read

$$V_R = IR = (0.036 \,\mathrm{A})(2000 \,\Omega) = 72 \,\mathrm{V}$$
 (rms). (581)

A voltmeter connected across the capacitor will read

$$V_C = \frac{I}{\omega C} = (0.036 \,\mathrm{A})(2650 \,\Omega) = 95 \,\mathrm{V}$$
 (rms). (582)

(e) The amplitudes of the voltages associated with the above rms values are 102 V and 134 V. The voltages across the resistor and capacitor are 90° out of phase, with the resistor ahead of the capacitor. (Remember, in general we have V_L ahead of V_R ahead of V_C .) So the pattern will be an ellipse, as shown in Fig. 146. If the plates are connected in the natural way as shown, then the ellipse is traced out counterclockwise. To see why, consider an instant when the current through the resistor is maximum downward, in which case the right plate of the tube is at a higher potential (so the electrons are deflected that way). The charge on the capacitor is 90° out of phase with the current, so there is no charge on the capacitor at this moment. The voltage across the capacitor is therefore zero, so the electrons are at the point A in the figure.

A quarter cycle later, the top plate of the capacitor will have maximum charge, in which case the top plate of the tube is at a higher potential (so the electrons are deflected that way). And the current is zero at this moment, so the voltage across the resistor is zero. The electrons are therefore at point B in the figure. We see that the curve on the screen passes point B a quarter cycle after point A. So the curve is traced out counterclockwise. On the other hand, if the connections are made in the reverse manner for either of the elements, then the curve would be traced out clockwise. If both connections are reversed, then the trace reverts back to counterclockwise. Without being told which way the connections are made, there is no way to know the direction of the trace.





Figure 146
8.36. High-pass filter

The current I through the resistor is also essentially the current through the inductor, because the circuit on the right is assumed to have a large impedance. The complex $\tilde{V} = \tilde{I}Z$ statement for the voltage between the terminals at A is $V_0 = \tilde{I}(R + i\omega L)$. And the statement for the terminals at B is $\tilde{V}_1 = \tilde{I}(i\omega L)$. Therefore,

$$\frac{\tilde{V}_1}{V_0} = \frac{i\omega L}{R + i\omega L} \implies \left| \frac{\tilde{V}_1}{V_0} \right|^2 = \frac{\omega^2 L^2}{R^2 + \omega^2 L^2} = \frac{1}{1 + \frac{R^2}{\omega^2 L^2}}.$$
(583)

This equals 0.1 when $R^2/\omega^2 L^2 = 9$. At 100 Hz this gives

$$\frac{R}{L} = 3\omega = 3 \cdot 2\pi \cdot 100 \,\mathrm{s}^{-1} \approx 2000 \,\mathrm{s}.$$
(584)

This is satisfied by, for example, $R = 100 \Omega$ and L = 0.05 H. The power is proportional to V^2 . If $\omega \ll R/L$, then we can ignore the "1" in the denominator. So $|\tilde{V}_1/V_0|^2 \approx \omega^2 L^2/R^2 \propto \omega^2$. Halving ω decreases this by a factor of 4.

The physical reason why V_1 decreases with decreasing frequency is the following. For very high frequency, the impedance of the inductor is very large. The impedance of the resistor is negligible in comparison, so essentially all of the V_0 voltage drop occurs across the inductor, which is what V_1 registers. On the other hand, for very low frequency, the impedance of the inductor is very small; it is essentially a short circuit. Therefore, essentially all of the V_0 voltage drop occurs across the resistor. Very little occurs across the inductor which, again, is what V_1 registers.

As in Problem 8.13, adding on another RL loop would square the attenuation effect. The voltage would then be proportional to ω^4 .

8.37. Parallel RLC power

The current I(t) and phase ϕ for the parallel *RLC* circuit are given in Eq. (8.67). Since $\tan \phi$ can be written as $(\omega C - 1/\omega L)/(1/R)$, we have

$$\cos\phi = \frac{1/R}{\sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2}}.$$
(585)

Equation (8.84) therefore gives the average power delivered to the circuit as

$$\overline{P} = \frac{1}{2} \mathcal{E}_0 I_0 \cos \phi$$

$$= \frac{1}{2} \mathcal{E}_0 \cdot \mathcal{E}_0 \sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2} \cdot \frac{1/R}{\sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2}}$$

$$= \frac{1}{2} \frac{\mathcal{E}_0^2}{R}.$$
(586)

The average power dissipated in the resistor is given by Eq. (8.80), where V_0 is the voltage across the resistor. But this voltage is simply \mathcal{E}_0 because we have a parallel circuit. So

$$\overline{P}_R = \frac{1}{2} \frac{\mathcal{E}_0^2}{R} \,, \tag{587}$$

in agreement with Eq. (586).

8.38. Two resistors and a capacitor

(a) The impedance of the capacitor is $Z_C = 1/i\omega C$. But since $\omega = 1/RC$ here, we have $Z_C = -iR$. Using the standard rules for adding impedances in series and parallel, the total impedance of the circuit is

$$Z = \frac{Z_R(Z_R + Z_C)}{Z_R + (Z_R + Z_C)} = \frac{R(R - iR)}{R + (R - iR)} = R\frac{1 - i}{2 - i}.$$
(588)

This can also be written as Z = R(3 - i)/5.

(b) The total complex current is

$$\tilde{I} = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{R} \frac{2-i}{1-i} = \frac{\mathcal{E}_0}{R} \frac{3+i}{2} = \frac{\mathcal{E}_0}{R} \frac{\sqrt{10}}{2} e^{i\phi},$$
(589)

where $\tan \phi = 1/3$. Therefore,

$$I_0 = \frac{\sqrt{10}}{2} \frac{\mathcal{E}_0}{R}$$
 and $\phi = \tan^{-1}(1/3) \approx 18.4^{\circ}.$ (590)

Formally,

$$I(t) = \operatorname{Re}\left[\tilde{I}e^{i\omega t}\right] = \operatorname{Re}\left[\frac{\sqrt{10}}{2}\frac{\mathcal{E}_0}{R}e^{i\phi}e^{i\omega t}\right] = \frac{\sqrt{10}}{2}\frac{\mathcal{E}_0}{R}\cos(\omega t + \phi).$$
(591)

(c) Since $\tan \phi = 1/3$ implies $\cos \phi = 3/\sqrt{10}$, the average power dissipated in the circuit is

$$\frac{1}{2}\mathcal{E}_0 I_0 \cos \phi = \frac{1}{2}\mathcal{E}_0 \left(\frac{\sqrt{10}}{2}\frac{\mathcal{E}_0}{R}\right) \frac{3}{\sqrt{10}} = \frac{3\mathcal{E}_0^2}{4R}.$$
(592)

Alternatively, we can find the power dissipated by finding the amplitude of the voltage across each of the resistors and then using $P_R = (1/2)V_R^2/R$ for each. (The resistors are the only places where power is dissipated.) The complex voltage across the right resistor is simply \mathcal{E}_0 . The complex voltage across the left resistor is $\mathcal{E}_0 Z_R/(Z_R + Z_C) = \mathcal{E}_0/(1-i)$, because the complex voltages across the R and C in series are proportional to their impedances. The magnitudes of these two complex voltages (which are the amplitudes of the two actual voltages) are \mathcal{E}_0 and $\mathcal{E}_0/\sqrt{2}$. Adding the $V_R^2/2R$ powers for each resistor gives $3\mathcal{E}_0^2/4R$, as above.

Chapter 9

Maxwell's equations and E&M waves

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

9.13. Displacement-current flux

The electric field between the plates equals σ/ϵ_0 , so the displacement current is $J_d = \epsilon_0(\partial E/\partial t) = d\sigma/dt$. The flux through S' is therefore $\Phi = J_d A = (d\sigma/dt)A$, where A is the area of each plate. Hence,

$$\Phi = \frac{d\sigma}{dt}A = \frac{d(\sigma A)}{dt} = \frac{dQ}{dt} = I,$$
(593)

as desired. We haven't paid attention to signs, but if the right plate in Fig. 9.4 is positive, and if the capacitor is discharging, then the displacement current points to the right. (The **E** field between the plates points to the left, but it is decreasing, so $\partial \mathbf{E}/\partial t$ points to the right.) The displacement-current flux therefore passes from left to right through S', just as the real-current flux passes from left to right through S. The total flux through the closed volume bounded by S and S' is zero, as it should be, because a closed surface has no boundary, so the line integral of **B** around this (non-existent) boundary is zero.

9.14. Sphere with a hole

Very close to the wire, the magnetic field is $B = \mu_0 I/2\pi r$. Therefore $\int_C \mathbf{B} \cdot d\mathbf{s} = (\mu_0 I/2\pi r)(2\pi r) = \mu_0 I$. On the right-hand side of Maxwell's equation, the term involving \mathbf{J} is zero because no current pieces the surface S (the sphere-minus-hole). To calculate the term involving $\partial \mathbf{E}/\partial t$, we know that the electric field at points on the surface S is $E = Q/4\pi\epsilon_0 R^2$, where Q is the point charge and R is the radius of the sphere. Hence $dE/dt = (dQ/dt)/4\pi\epsilon_0 R^2 = I/4\pi\epsilon_0 R^2$. Integrating this over the surface of the sphere brings in a factor of $4\pi R^2$. Remembering the factor of $\mu_0\epsilon_0$ out front, the right-hand side of Maxwell's equation equals $\mu_0\epsilon_0(I/4\pi\epsilon_0 R^2)(4\pi R^2) + 0 = \mu_0 I$, in agreement with the left-hand side.

9.15. Field inside a discharging capacitor

Written in terms of the displacement current, the integral law reads

$$\int_{C} \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_{S} (\mathbf{J}_{d} + \mathbf{J}) \cdot d\mathbf{a}.$$
(594)

Since $s \ll b$ we can neglect the edge fields, in which case the displacement current \mathbf{J}_{d} is uniformly distributed in the gap. The integral of \mathbf{J}_{d} over the area of the plates equals the conduction current I in the wire (see Exercise 9.13). The fraction of $\int \mathbf{J}_{d} \cdot d\mathbf{a} = I$ that is enclosed in a circle through P, centered on the axis, is $\pi r^{2}/\pi b^{2}$. The integral law applied to this circle therefore gives (with the conduction current $\mathbf{J} = 0$ inside the capacitor)

$$2\pi r B = \mu_0 \left(I \frac{r^2}{b^2} \right) + 0 \implies B = \frac{\mu_0 I r}{2\pi b^2}, \qquad (595)$$

as desired. The similarity of this calculation to the calculation of the **E** field in Fig. 7.16 is the following. If we solve the problem straight from Maxwell's equation, without invoking the definition of the displacement current, we can write (with $\mathbf{J} = 0$ inside the capacitor)

$$\int_{C} \mathbf{B} \cdot d\mathbf{s} = \mu_{0} \epsilon_{0} \int_{S} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} \implies 2\pi r B = \mu_{0} \epsilon_{0} \frac{d\Phi_{\mathrm{E}}}{dt}, \qquad (596)$$

where $\Phi_{\rm E}$ is the flux of the electric field through the given surface. This equation is exactly analogous to Faraday's law of induction, which we used in the example of Fig. 7.16 (among many other places),

$$\int_{C} \mathbf{E} \cdot d\mathbf{s} = -\int_{S} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \implies 2\pi r E = -\frac{d\Phi_{\rm B}}{dt}.$$
(597)

The similarity arises because of the symmetry of the two "curl" Maxwell's equations; and also because there is no current \mathbf{J} of real electric charges inside the capacitor in the present problem, and likewise there is no current of real magnetic charges in Fig. 7.16 (or anywhere else) because magnetic monopoles don't exist (as far as we know).

9.16. Changing flux from a moving charge

A time dt later, the whole field pattern has moved a distance v dt to the right. So all of the electric flux that initially passed through the circle still passes through, but some additional flux now passes through. This additional flux is the flux that passes through the left circle in Fig. 147 but doesn't pass through the right circle. (If you imagine the left circle riding along with the field pattern, it becomes the right (fixed) circle after a time dt.) Equivalently, the additional flux is the flux that passes through the cylindrical ring between the two circles, with radius r and width v dt. The area of this cylinder is $2\pi rv dt$, so the change in electric flux through the fixed right circle is $d\Phi_E = (2\pi rv dt)E\sin\theta$, where the $\sin\theta$ factor comes from the fact that in finding the flux through the cylinder, we are concerned only with the component perpendicular to the x axis. Maxwell's equation is therefore satisfied if the magnitudes E and B satisfy

$$\int \mathbf{B} \cdot d\mathbf{s} = \frac{1}{c^2} \frac{d\Phi_E}{dt} \implies 2\pi r B = \frac{1}{c^2} 2\pi r v E \sin\theta \implies B = \frac{v}{c^2} E \sin\theta.$$
(598)

In view of the given relation $\mathbf{B} = (\mathbf{v}/c^2) \times \mathbf{E}$, this is indeed true. The magnitude is correct because the cross product between \mathbf{v} and \mathbf{E} generates the sin θ . And the sign is correct because the flux increases to the right, which by the right-hand rule corresponds to \mathbf{B} pointing in the counterclockwise direction when viewed from the right. This is consistent with the direction of \mathbf{B} obtained from $\mathbf{B} = (\mathbf{v}/c^2) \times \mathbf{E}$.

9.17. Gaussian conditions

In free space, the two "curl" Maxwell's equations in Gaussian units are

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$
 and $\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$. (599)



Figure 147

The given wave is

$$\mathbf{E} = \hat{\mathbf{z}} E_0 \sin(y - vt) \quad \text{and} \quad \mathbf{B} = \hat{\mathbf{x}} B_0 \sin(y - vt).$$
(600)

The relevant derivatives are:

$$\nabla \times \mathbf{E} = \hat{\mathbf{x}} E_0 \cos(y - vt), \qquad \qquad \frac{\partial \mathbf{E}}{\partial t} = -\hat{\mathbf{z}} v E_0 \cos(y - vt), \\ \nabla \times \mathbf{B} = -\hat{\mathbf{z}} B_0 \cos(y - vt), \qquad \qquad \frac{\partial \mathbf{B}}{\partial t} = -\hat{\mathbf{x}} v B_0 \cos(y - vt). \tag{601}$$

The first of the equations in Eq. (599) yields $E_0 = (v/c)B_0$, and the second yields $B_0 = (v/c)E_0$. Combining these gives $v = \pm c$, and $B_0 = \pm E_0$.

9.18. Associated B field

The wave is traveling in the $-\hat{\mathbf{z}}$ direction, as shown by the sign in (z+ct); if t increases, then z must decrease to keep the same value of (z + ct). **B** is perpendicular to both this direction and to **E**. So **B** must point in the $\pm(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ direction. But since we know that $\mathbf{E} \times \mathbf{B}$ points in the direction of the wave's velocity, which is $-\hat{\mathbf{z}}$, we must pick the "+" sign, as you can quickly verify with the right-hand rule. The magnitude of **B** is 1/c times the magnitude of **E**, so the desired **B** field is

$$\mathbf{B} = (E_0/c)(\hat{\mathbf{x}} - \hat{\mathbf{y}})\sin[(2\pi/\lambda)(z + ct)].$$
(602)

With $E_0 = 20 \text{ V/m}$, we have $B_0 = E_0/c = (20 \text{ V/m})(3 \cdot 10^8 \text{ m/s}) = 6.67 \cdot 10^{-8} \text{ T}$. The amplitudes of the **E** and **B** waves are actually $\sqrt{2}$ times E_0 and B_0/c , respectively, because the magnitude of the $(\hat{\mathbf{x}} \pm \hat{\mathbf{y}})$ vectors is $\sqrt{2}$.

9.19. Find the wave

It is given that $\mathbf{E} \perp \hat{\mathbf{z}}$. And we know that $\mathbf{E} \perp \mathbf{v}$, where $\mathbf{v} \propto -\hat{\mathbf{x}}$ here. So \mathbf{E} must point in the $\pm \hat{\mathbf{y}}$ direction. Let's pick $+\hat{\mathbf{y}}$. The other direction would simply change the sign of E_0 ; the sign is arbitrary, since the trig function switches signs anyway. So we have (a sine would work just as well)

$$\mathbf{E} = \hat{\mathbf{y}} E_0 \cos(kx + \omega t), \tag{603}$$

where $\omega = 2\pi\nu = 6.28 \cdot 10^8 \text{ s}^{-1}$ and $k = \omega/c = 2.09 \text{ m}^{-1}$. The sign inside the cosine is a "+" because the wave is traveling in the negative x direction. Since $\mathbf{E} \times \mathbf{B}$ points in the direction of **v**, which is $-\hat{\mathbf{x}}$, and since $B_0 = E_0/c$, the **B** field must take the form,

$$\mathbf{B} = -\hat{\mathbf{z}}(E_0/c)\cos(kx + \omega t). \tag{604}$$

9.20. Kicked by a wave

Equation (9.28) gives the electric field as

$$\mathbf{E} = \frac{E_0 \hat{\mathbf{y}}}{1 + \frac{(x+ct)^2}{\ell^2}},$$
(605)

where $E_0 = 100 \,\text{kV/m}$ and $\ell = 1$ foot. We are concerned with the field at x = 0. Since $F = dp/dt \implies p = \int F \, dt$, the momentum acquired by the proton during the passage of the pulse is

$$p_y = \int_{-\infty}^{\infty} eE_y dt = eE_0 \int_{-\infty}^{\infty} \frac{dt}{1 + (ct/\ell)^2}$$
$$= \frac{eE_0\ell}{c} \left[\arctan(\infty) - \arctan(-\infty) \right] = \frac{\pi eE_0\ell}{c} .$$
(606)

The proton's final speed is then (using 1 ft = 0.305 m)

$$v_y = \frac{p_y}{m} = \frac{\pi e E_0 \ell}{mc} = \frac{\pi (1.6 \cdot 10^{-19} \,\mathrm{C})(10^5 \,\mathrm{V/m})(0.305 \,\mathrm{m})}{(1.67 \cdot 10^{-27} \,\mathrm{kg})(3 \cdot 10^8 \,\mathrm{m/s})} = 3.1 \cdot 10^4 \,\mathrm{m/s}.$$
(607)

The displacement during the few nanoseconds of acceleration is negligible. So one microsecond later the proton will be located at $y = v_y t = (3.1 \cdot 10^4 \text{ m/s})(10^{-6} \text{ s}) = 0.031 \text{ m}$, or 3.1 cm.

Since $B_0 = E_0/c$, the magnitude of the magnetic force $q\mathbf{v} \times \mathbf{B}$ is smaller than the magnitude of the electric force $q\mathbf{E}$ by the factor $v/c \approx 10^{-4}$. So as mentioned in the statement of the exercise, the magnetic force is indeed negligible.

9.21. Effect of the magnetic field

Before the pulse has completely passed, the proton has acquired some velocity in the $\hat{\mathbf{y}}$ direction. It will therefore experience a force $e\mathbf{v} \times \mathbf{B}$ due to the magnetic field of the wave. Since **B** points in the $-\hat{\mathbf{z}}$ direction, this force points in the $-\hat{\mathbf{x}}$ direction, which is the direction in which the wave is traveling. The force would also be in that direction for a negative particle, because both the q and the v in the $q\mathbf{v} \times \mathbf{B}$ force switch sign. The wave tends to knock the particle along.

Since $F = dp/dt \implies p = \int F dt$, we can say that in order of magnitude, if τ is the duration of the pulse with amplitude E, then $p_y \sim (eE)\tau$, so $v_y \sim eE\tau/m$. The momentum in the x direction due to the magnetic force is $p_x \sim -(ev_yB)\tau = -ev_y(E/c)\tau$. Therefore, $p_x/p_y \sim -v_y/c \sim -eE\tau/mc$. But in order of magnitude, τ equals ℓ/c , so we obtain $p_x/p_y \sim -eE_0\ell/mc^2$. For small angles, this ratio is essentially equal to the angle by which the final velocity has changed. We see that the effect is second order in 1/c. Note that p_x/p_y equals the ratio of two energies; the numerator is the work the electric field would do on the proton over a distance ℓ , and the denominator is the rest energy of the proton.

9.22. Plane-wave pulse

(a) We want to apply the "displacement current" Maxwell equation, $\int \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 d\Phi_E/dt$, to the loop. We'll trace out the line integral in the counterclockwise direction, in which case the right-hand rule defines positive electric-field flux to point out of the page. Let the transverse length of the loop be ℓ , as shown in Fig. 148. Then since B = 0 outside the slab, the left-hand side of the above Maxwell equation is simply $B\ell$. On the right-hand side, the flux increases because there is increasing overlap of the moving slab and the stationary loop. In time dt, the overlap area increases by $\ell(v dt)$. So the rate of area increase is ℓv , which means that $d\Phi_E/dt = E\ell v$. Therefore,

$$\int \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \implies B\ell = \mu_0 \epsilon_0 E\ell v \implies B = \mu_0 \epsilon_0 Ev.$$
(608)

(b) Using the "Faraday" Maxwell equation, the same argument with a loop perpendicular to the page (lying in the horizontal plane) gives

$$\int \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \implies E\ell = -(-B\ell v) \implies E = Bv.$$
(609)

The minus sign in the flux comes from the fact that if the loop is traced out in the counterclockwise direction when viewed from above, the right-hand rule defines



Figure 148

positive magnetic-field flux to point upward, which is opposite to the direction of \mathbf{B} .

Plugging E = Bv into the result in part (a) gives $B = \mu_0 \epsilon_0(Bv)v$, so $v = 1/\sqrt{\mu_0 \epsilon_0}$. This equals c, as we know it should.

9.23. Field in a box

We immediately see that $\nabla \cdot \mathbf{E} = 0$, because E_z has no z dependence. And also $\nabla \cdot \mathbf{B} = 0$, because the $\partial B_x / \partial x$ and $\partial B_y / \partial y$ terms cancel. So two of Maxwell's equations are satisfied. For the other two, we can calculate the curls via the usual determinant method,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial y \\ E_x & E_y & E_z \end{vmatrix}.$$
(610)

You can verify that the various derivatives are

$$\nabla \times \mathbf{E} = kE_0 \left(-\hat{\mathbf{x}} \cos kx \sin ky + \hat{\mathbf{y}} \sin kx \cos ky \right) \cos \omega t,$$

$$\frac{\partial \mathbf{E}}{\partial t} = -\omega \hat{\mathbf{z}} E_0 \cos kx \cos ky \sin \omega t,$$

$$\nabla \times \mathbf{B} = -2k \hat{\mathbf{z}} B_0 \cos kx \cos ky \sin \omega t,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \omega B_0 \left(\hat{\mathbf{x}} \cos kx \sin ky - \hat{\mathbf{y}} \sin kx \cos ky \right) \cos \omega t.$$
(611)

Therefore, $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ gives $kE_0 = \omega B_0$. And $\nabla \times \mathbf{B} = (1/c^2)\partial \mathbf{E}/\partial t$ gives $2kB_0 = \omega E_0/c^2$. These two requirements quickly yield $\omega = \sqrt{2}ck$ and $E_0 = \sqrt{2}cB_0$, as desired. (Technically, $\omega = -\sqrt{2}ck$ and $E_0 = -\sqrt{2}cB_0$ also work, but these relations yield the same wave, as you can verify.)

The fields don't depend on z, so to determine what they look like, let's consider the square cross section of the box in the xy plane. At all times, **E** is zero on the boundary of the box where $(x, y) = (\pm \pi/2k, \pm \pi/2k)$. At a given instant in time, $\cos \omega t$ takes on a specific value, so **E** is proportional to $\hat{\mathbf{z}} \cos kx \cos ky$. This function is maximum at the origin. The plot of $E_z \propto \cos kx \cos ky$ is basically a bump above the xy plane (or a valley below the xy plane at times when $\cos \omega t$ is negative). The bump oscillates up and down according to $\cos \omega t$. The level curves of constant E_z are given by $\cos kx \cos ky = C$. You can show with a Taylor series that these level curves are circles near the origin. So the curves start off as circles and end up as squares. They are shown roughly in Fig. 149. Since **E** has only a z component, it points perpendicular to the page.

B isn't quite as clean, but it's easy to get a handle on its values along the x and y axes, and along the 45° lines, and also along the boundary of the box. Some sample vectors at times when $\sin \omega t = 1$ are shown in Fig. 150. All the vectors oscillate in phase according to $\sin \omega t$.

REMARK: That's all that was required, but we can say a little more about the fields. For small x and y, we can use $\cos \theta \approx 1$ and $\sin \theta \approx \theta$ to obtain $\mathbf{B} \approx kB_0(\hat{\mathbf{x}}y - \hat{\mathbf{y}}x) \sin \omega t$. The field lines associated with this **B** field are circles, because the vector $\mathbf{B} \propto (y, -x)$ is always perpendicular to the radial vector (x, y). Alternatively, since the tangent to the field line is in the direction of **B**, we can separate variables and integrate $dy/dx = B_y/B_x = -x/y$ to obtain $x^2 + y^2 = C$, where C is a constant. The **B** field goes to zero at the origin.



Figure 149



Figure 150

What do the **B** field lines look like for general x and y values? Again, since the tangent to the field line is in the direction of **B**, we have the general relation,

$$\frac{dy}{dx} = \frac{B_y}{B_x} = -\frac{\sin kx \cos ky}{\cos kx \sin ky}.$$
(612)

Separating variables and integrating gives $\ln(\cos ky) = -\ln(\cos kx) + D$, where D is a constant. Exponentiating gives $\cos kx \cos ky = C$, where $C = e^D$ is another constant. Small values of C yield near-squares close to the boundary of the box, and values close to 1 yield the small near-circles close to the origin we found above. Note that the $\cos kx \cos ky = C$ curves of the **B** field lines are also the curves of constant E_z , which we found above and plotted in Fig. 149. This can be traced to the fact that if **E** has only a z component, then $\nabla \times \mathbf{E}$ is perpendicular to ∇E_z , as you can verify.

9.24. Satellite signal

The area covered is $A = \pi (5 \cdot 10^5 \text{ m})^2 \approx 8 \cdot 10^{11} \text{ m}^2$. The power density is therefore $S = P/A = (10^4 \text{ W})/(8 \cdot 10^{11} \text{ m}^2) \approx 10^{-8} \text{ W/m}^2$. So from Eq. (9.37) we have

$$S = \frac{\overline{E^2}}{377\,\Omega} \implies \overline{E^2} = (377\,\Omega)(10^{-8}\,\mathrm{W/m^2}) \implies E_{\mathrm{rms}} \approx 0.002\,\mathrm{V/m}, \quad (613)$$

or 2 millivolts/meter.

9.25. Microwave background radiation

As shown in Section 9.6, the average energy density \mathcal{U} of a sinusoidal electromagnetic wave is $\mathcal{U} = \epsilon_0 E_0^2/2 = \epsilon_0 E_{\rm rms}^2$. So we have

$$E_{\rm rms}^2 = \frac{\mathcal{U}}{\epsilon_0} = \frac{4 \cdot 10^{-14} \,\text{J/m}^3}{8.85 \cdot 10^{-12} \,\frac{\text{s}^2 \,\text{C}^2}{\text{kg m}^3}} = 4.5 \cdot 10^{-3} \,\text{V}^2/\text{m}^2 \implies E_{\rm rms} = 0.067 \,\text{V/m}. \tag{614}$$

If the 1 kilowatt radiated by the transmitter is spread out over a sphere of radius R, then the power density at radius R equals $S = (10^3 \text{ W})/4\pi R^2$. The energy density is then $\mathcal{U} = S/c$. We therefore want

$$\frac{1}{c} \cdot \frac{10^3 \,\mathrm{W}}{4\pi R^2} = 4 \cdot 10^{-14} \,\mathrm{J/m^3} \implies R = 2600 \,\mathrm{m},\tag{615}$$

or 2.6 km. However, the power is undoubtedly emitted in at least a somewhat directed manner, so the distance from an actual radio transmitter would be larger than this.

9.26. An electromagnetic wave

(a) The fields are

$$\mathbf{E} = \hat{\mathbf{y}} E_0 \sin(kx + \omega t), \quad \text{and} \quad \mathbf{B} = -\hat{\mathbf{z}} (E_0/c) \sin(kx + \omega t). \quad (616)$$

We immediately see that $\nabla \cdot \mathbf{E} = 0$ (because the lone *y* component of **E** has no *y* dependence) and $\nabla \cdot \mathbf{B} = 0$ (because the lone *z* component of **B** has no *z* dependence). So two of Maxwell's equations are satisfied. For the other two, you can verify that

$$\nabla \times \mathbf{E} = \hat{\mathbf{z}} k E_0 \cos(kx + \omega t), \qquad \qquad \frac{\partial \mathbf{E}}{\partial t} = \hat{\mathbf{y}} \omega E_0 \cos(kx + \omega t), \qquad (617)$$
$$\nabla \times \mathbf{B} = \hat{\mathbf{y}} k (E_0/c) \cos(kx + \omega t), \qquad \qquad \frac{\partial \mathbf{B}}{\partial t} = -\hat{\mathbf{z}} \omega (E_0/c) \cos(kx + \omega t).$$

ar

Therefore, $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ requires $k = \omega/c$. And (using $\mu_0 \epsilon_0 = 1/c^2$) $\nabla \times \mathbf{B} = (1/c^2)\partial \mathbf{E}/\partial t$ requires $k/c = (1/c^2)\omega$, which again says that $k = \omega/c$.

(b) The wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{2\pi (3 \cdot 10^8 \,\mathrm{m/s})}{10^{10} \,\mathrm{s}^{-1}} = 0.19 \,\mathrm{m}.$$
 (618)

As shown in Section 9.6, the average energy density of a sinusoidal electromagnetic wave is $\epsilon_0 E_0^2/2$, which equals

$$\frac{1}{2}\epsilon_0 E_0^2 = \frac{1}{2} \left(8.85 \cdot 10^{-12} \, \frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg} \,\mathrm{m}^3} \right) (10^3 \,\mathrm{V/m})^2 = 4.4 \cdot 10^{-6} \,\mathrm{J/m^3}. \tag{619}$$

The power density equals the energy density times the speed, so

$$S = \frac{1}{2} \epsilon_0 E_0^2 c = (4.4 \cdot 10^{-6} \,\mathrm{J/m^3})(3 \cdot 10^8 \,\mathrm{m/s}) = 1300 \,\mathrm{J/(m^2s)}.$$
(620)

9.27. Reflected wave

Let E_i be the amplitude of the oscillating electric field of the incident wave, and E_r that of the reflected wave. If half of the incident energy is reflected, then $E_r = E_i/\sqrt{2}$. As in Section 9.5, the incident electric wave is described by Eq. (9.30) with $E_0 \rightarrow E_i$, and the reflected wave is described by Eq. (9.29) with $E_0 \rightarrow E_r$. Using the trig sum formulas, you can check that the sum of these two waves is

$$\mathbf{E} = \hat{\mathbf{z}}(E_{\rm i} + E_{\rm r}) \sin \frac{2\pi y}{\lambda} \cos \frac{2\pi ct}{\lambda} + \hat{\mathbf{z}}(E_{\rm i} - E_{\rm r}) \cos \frac{2\pi y}{\lambda} \sin \frac{2\pi ct}{\lambda} \,. \tag{621}$$

At points where y equals $\lambda/4$, $3\lambda/4$, $5\lambda/4$, etc., the second term is zero, and the $\sin(2\pi y/\lambda)$ factor in the first term equals ± 1 . So **E** oscillates with amplitude $E_i + E_r$. At these points the two oscillating electric fields are, and remain at all times, in phase. (In the setup in Fig. 9.10 the mirror was a perfect conductor, so $E_r = E_i$.)

Similarly, at points where y equals 0, $\lambda/2$, λ , $3\lambda/2$, etc., the first term is zero, and the $\cos(2\pi y/\lambda)$ factor in the second term equals ± 1 . So **E** oscillates with amplitude $E_{\rm i} - E_{\rm r}$. At these points the two oscillating electric fields are, and remain at all times, exactly 180° out of phase. (In Fig. 9.10 these points have **E** = 0 at all times, because $E_{\rm r} = E_{\rm i}$.) Note that at these points **E** reaches its maximum value a quarter cycle before or after the maximum at the points in the previous paragraph, due to the $\sin(2\pi ct/\lambda)$ versus $\cos(2\pi ct/\lambda)$ dependence.

In our case with $E_{\rm r} = E_{\rm i}/\sqrt{2}$, the ratio of maximum amplitude observed to minimum amplitude observed is

$$\frac{E_{\rm i} + E_{\rm r}}{E_{\rm i} - E_{\rm r}} = \frac{1 + 1/\sqrt{2}}{1 - 1/\sqrt{2}} = 5.83.$$
(622)

9.28. Poynting vector and resistance heating

The electric field inside the wire is given by $E = J/\sigma$. Since the curl of **E** is zero, we can draw a thin rectangular loop along the surface to show that the electric field right outside the wire is also $E = J/\sigma$ (and it points in the direction of the current, of course). The magnetic field right outside the wire points tangentially with the usual magnitude of $B = \mu_0 I/2\pi R$, where R is the radius of the wire. **E** and **B** are perpendicular, and you can show with the right-hand rule that the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ points radially into the wire. So the direction is correct; the energy in the wire increases, consistent with the fact that it heats up. The magnitude of **S** equals

$$S = \frac{1}{\mu_0} EB = \frac{1}{\mu_0} \frac{J}{\sigma} \frac{\mu_0 I}{2\pi R} = \frac{JI}{2\pi R\sigma} \,. \tag{623}$$

To obtain the power flux into the wire through the surface, we must multiply by $2\pi R\ell$, where ℓ is the length of a given section of the wire. So the total energy flow per time into a length ℓ of the wire is

$$P_{\ell} = S \cdot 2\pi R\ell = \frac{JI}{2\pi R\sigma} 2\pi R\ell = \frac{JI}{\sigma} \ell = \frac{(I/A)I}{\sigma} \ell = I^2 \frac{\ell}{\sigma A} = I^2 \frac{\rho \ell}{A} = I^2 R, \quad (624)$$

where R is the resistance of the length ℓ of the wire. We have used the fact that the resistivity ρ is given by $\rho = 1/\sigma$. As desired, P_{ℓ} equals the rate of resistance heating in the length ℓ of the wire. P_{ℓ} can also be written as I(IR) = IV, of course, where V is the voltage drop along the length ℓ of the wire.

Alternatively, we never actually had to use the J/σ form of E. A quicker method is:

$$P_{\ell} = S \cdot 2\pi R\ell = \frac{1}{\mu_0} E \frac{\mu_0 I}{2\pi R} \cdot 2\pi R\ell = I E \ell = I V,$$
(625)

because $V = E\ell$.

9.29. Energy flow in a capacitor

We can find the magnetic field inside the capacitor by integrating the $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \partial \mathbf{E} / \partial t$ Maxwell equation over a disk of radius r. Since \mathbf{J} points upward in Fig. 151, the upper plate is positive, so $\partial \mathbf{E} / \partial t$ points downward. We therefore have (using Stokes' theorem)

$$\int \mathbf{B} \cdot d\mathbf{s} = \mu_0 I - \epsilon_0 \mu_0 \frac{\partial E}{\partial t} (\text{area}) \implies B(2\pi r) = \mu_0 I - \epsilon_0 \mu_0 \frac{\partial E}{\partial t} (\pi r^2)$$
$$\implies B = \frac{\mu_0 I}{2\pi r} - \frac{\epsilon_0 \mu_0 r}{2} \frac{\partial E}{\partial t}. \tag{626}$$

It will be helpful to write I in terms of E (or rather $\partial E/\partial t$). We have

$$I = \frac{dQ}{dt} = \frac{d(\sigma A)}{dt} = \frac{d(\epsilon_0 E A)}{dt} = \epsilon_0 (\pi b^2) \frac{dE}{dt}.$$
(627)

Therefore,

$$B = \frac{\epsilon_0 \mu_0}{2} \frac{dE}{dt} \left(\frac{b^2}{r} - r\right).$$
(628)

This is positive since we are concerned only with r values smaller than b. This magnetic field points tangentially around the circle of radius r, counterclockwise when viewed from above, as you can check with the right-hand rule. So it points into the page on the right. Since **E** points downward,¹ you can quickly verify with the right-hand rule that the Poynting vector $\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu_0$ points away from the wire. Energy flows out from the wire into the surrounding space inside the capacitor. Since **E** is perpendicular to **B**, the magnitude of **S** is

$$S = \frac{EB}{\mu_0} = \frac{\epsilon_0}{2} E \frac{dE}{dt} \left(\frac{b^2}{r} - r\right).$$
(629)

The total energy per time (that is, the power) flowing out of a cylinder of radius r is S times the area $2\pi rh$ of the cylinder (where h is the separation between the plates).



Figure 151

 $^{^{1}}$ Very close to the wire, the electric field actually points upward due to surface charges on the wire, but a little farther away the downward field from the capacitor dominates. See the remark at the end of the solution to Problem 9.10.

The power is then

$$P = \left(\frac{\epsilon_0}{2} E \frac{dE}{dt} \left(\frac{b^2}{r} - r\right)\right) 2\pi rh = \epsilon_0 E \frac{dE}{dt} (\pi b^2 - \pi r^2)h = \frac{d}{dt} \left(\frac{\epsilon_0 E^2}{2} (\pi b^2 - \pi r^2)h\right).$$
(630)

This is correctly the rate of change of the energy stored in the electric field, because $\epsilon_0 E_0^2/2$ is energy density and $(\pi b^2 - \pi r^2)h$ is the volume contained between radius r and radius b. If $r \approx 0$, we obtain the statement that the power flowing away from the wire equals the rate of change of the total energy stored in the $\pi b^2 h$ volume between the plates. If r = b, we obtain zero power flow, which is correct. (As usual, we are ignoring edge effects in the capacitor and assuming that the electric field drops abruptly to zero at r = b.) As mentioned at the end of the example in Section 9.6.2, we don't need to worry about the energy stored in the magnetic field.

9.30. Comparing the energy densities

If $E(t) = E_0 \cos \omega t$, then $\partial E/\partial t = -\omega E_0 \sin \omega t$, so the amplitude of the *B* field given in Eq. (9.46) is $B_0 = (\epsilon_0 \mu_0 r/2)(\omega E_0)$. The ratio of the magnetic energy density to the electric energy density is therefore

$$\frac{\frac{B_0^2}{2\mu_0}}{\frac{\epsilon_0 E_0^2}{2}} = \frac{\frac{1}{2\mu_0} \left(\frac{\epsilon_0 \mu_0 r}{2} \,\omega E_0\right)^2}{\frac{\epsilon_0}{2} E_0^2} = \frac{\mu_0 \epsilon_0 r^2 \omega^2}{4} = \left(\frac{\pi r}{cT}\right)^2,\tag{631}$$

where we have used $\omega = 2\pi/T$ and $1/\mu_0\epsilon_0 = c^2$. As desired, this result is small if the period T much larger than r/c, which is (half) the time it takes light to travel across the capacitor disks. As in Problem 9.6, we have ignored the high-order feedback effects between E and B. These effects are negligible if the current doesn't change too quickly.

$9.31.\ {\rm Field}\ {\rm momentum}\ {\rm of}\ {\rm a}\ {\rm moving}\ {\rm charge}$

We know that the electric field points radially with a magnitude essentially equal to the Coulomb value of $E = q/4\pi\epsilon_0 r^2$. And Eq. (6.81) gives the magnetic field as $\mathbf{B} = (\mathbf{v}/c^2) \times \mathbf{E}$, so **B** points out of the page at the point shown in Fig.152 (We're using **v** here as the velocity of the charged particle, not the velocity of the primed frame in Eq. (6.81).) With θ defined as shown, the magnitude of **B** is $(v/c^2)(q/4\pi\epsilon_0 r^2)\sin\theta = \mu_0 qv\sin\theta/4\pi r^2$, where we have used $1/\epsilon_0 c^2 = \mu_0$.

Problem 9.11 tells us that the momentum density is $\tilde{\mathbf{p}} = \mathbf{S}/c^2 = (\mathbf{E} \times \mathbf{B})/\mu_0 c^2$. At the point shown in Fig. 152, $\mathbf{E} \times \mathbf{B}$ points down and to the right. When we integrate $\tilde{\mathbf{p}}$ over all space, only the rightward component survives; you can check that the transverse components associated with the angles θ and $\pi - \theta$ cancel (and likewise for the angles θ and $-\theta$). So this brings in another factor of $\sin \theta$. The magnitude of the total momentum is therefore

$$p = \int \frac{S}{c^2} \sin \theta \, dv = \int \frac{1}{c^2} \frac{1}{\mu_0} EB \sin \theta \, dv$$

$$= \int_a^\infty \int_0^\pi \frac{1}{c^2} \frac{1}{\mu_0} \frac{q}{4\pi\epsilon_0 r^2} \frac{\mu_0 qv \sin \theta}{4\pi r^2} \sin \theta (2\pi r \sin \theta) (r \, d\theta) \, dr$$

$$= \frac{q^2 v}{8\pi\epsilon_0 c^2} \int_a^\infty \frac{dr}{r^2} \int_0^\pi \sin^3 \theta \, d\theta$$

$$= \frac{1}{c^2} \frac{4}{3} \frac{q^2}{8\pi\epsilon_0 a} v, \qquad (632)$$



Figure 152

as desired. We evaluated the trig integral here by writing $\sin^3 \theta$ as $\sin \theta (1 - \cos^2 \theta)$. Alternatively, you can use the integral table in Appendix K.

9.32. A Lorentz invariant

In terms of the parallel and perpendicular components, we have

$$E'^{2} - c^{2}B'^{2} = (E_{\parallel}'^{2} + E_{\perp}'^{2}) - c^{2}(B_{\parallel}'^{2} + B_{\perp}'^{2}) = (E_{\parallel}^{2} - c^{2}B_{\parallel}^{2}) + (\mathbf{E}_{\perp}' \cdot \mathbf{E}_{\perp}' - c^{2}\mathbf{B}_{\perp}' \cdot \mathbf{B}_{\perp}'),$$
(633)

where we have used $E'_{\parallel} = E_{\parallel}$ and $B'_{\parallel} = B_{\parallel}$.

We must now deal with the " \perp " terms. Our goal is to show that they combine to form $E_{\perp}^2 - c^2 B_{\perp}^2$. Using the expression for \mathbf{E}_{\perp}' in Eq. (6.76), the $\mathbf{E}_{\perp}' \cdot \mathbf{E}_{\perp}'$ term becomes

$$\gamma^2 \left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp} \right) \cdot \left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp} \right) = \gamma^2 \left[\mathbf{E}_{\perp}^2 + 2\mathbf{E}_{\perp} \cdot \left(\mathbf{v} \times \mathbf{B}_{\perp} \right) + \left(\mathbf{v} \times \mathbf{B}_{\perp} \right)^2 \right].$$
(634)

Since \mathbf{B}_{\perp} is perpendicular to \mathbf{v} by definition, we have $(\mathbf{v} \times \mathbf{B}_{\perp})^2 = v^2 B_{\perp}^2$. Hence

$$\mathbf{E}'_{\perp} \cdot \mathbf{E}'_{\perp} = \gamma^2 \Big[E_{\perp}^2 + 2\mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{B}_{\perp}) + v^2 B_{\perp}^2 \Big].$$
(635)

In a similar manner we find

$$-c^{2}\mathbf{B}_{\perp}' \cdot \mathbf{B}_{\perp}' = c^{2}\gamma^{2} \Big[-B_{\perp}^{2} + 2(1/c^{2})\mathbf{B}_{\perp} \cdot (\mathbf{v} \times \mathbf{E}_{\perp}) - (v^{2}/c^{4})E_{\perp}^{2} \Big].$$
(636)

When we add the previous two equations, the two middle terms will cancel if $\mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{B}_{\perp}) = -\mathbf{B}_{\perp} \cdot (\mathbf{v} \times \mathbf{E}_{\perp})$. This is indeed true, because each of these "dot-cross" products can be transformed into the other by cyclicly permuting the vectors (which doesn't change anything) and then reversing the order of the cross product (which brings in a minus sign). The sum of Eqs. (635) and (636) is therefore

$$\mathbf{E}'_{\perp} \cdot \mathbf{E}'_{\perp} - c^{2} \mathbf{B}'_{\perp} \cdot \mathbf{B}'_{\perp} = \gamma^{2} (1 - v^{2}/c^{2}) E_{\perp}^{2} - c^{2} \gamma^{2} (1 - v^{2}/c^{2}) B_{\perp}^{2}
= E_{\perp}^{2} - c^{2} B_{\perp}^{2}.$$
(637)

Substituting this into Eq. (633) gives

$$E^{\prime 2} - c^2 B^{\prime 2} = (E_{\parallel}^2 - c^2 B_{\parallel}^2) + (E_{\perp}^2 - c^2 B_{\perp}^2)$$

$$= (E_{\parallel}^2 + E_{\perp}^2) - c^2 (B_{\parallel}^2 + B_{\perp}^2)$$

$$= E^2 - c^2 B^2, \qquad (638)$$

as desired. Alternatively, it doesn't take too long to solve this exercise by explicitly writing out the squares of all the components of \mathbf{E} and \mathbf{B} in Eq. (6.74).

Chapter 10

Electric fields in matter

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

10.15. Densities on a capacitor

The charge density σ_2 on the right part of each plate is κ times the charge density σ_1 on the left part. So

$$\sigma_1(b-x)a + \sigma_2 xa = Q \implies \sigma_1(b-x)a + (\kappa \sigma_1)xa = Q.$$
(639)

The two charge densities, σ_1 and $\sigma_2 = \kappa \sigma_1$, are therefore given by

$$\sigma_1 = \frac{Q/a}{b + (\kappa - 1)x}, \qquad \sigma_2 = \frac{\kappa Q/a}{b + (\kappa - 1)x}.$$
(640)

Since $\kappa > 1$, both of these densities decrease as x increases. It is possible for both densities to decrease while the total charge remains at the given value Q, because the charge in the right region increases (while the charge in the left region decreases), but it does so at a slower rate than the area increases; so the density decreases. We would have a paradox if the areas stayed the same.

An analogy: 10 people each have the same amount of money. 20 other people each have the same amount, but it is smaller than what the first 10 have. One of these 20 people takes some money from each of the first 10, and also from each of the other 19, so that she now has the same amount as the first 10. The total amount of money held by all 30 people is still the same, but the average amounts in the two groups (now with 11 and 19 people) have both decreased.

10.16. Leyden jar

Assume that the jar is cylindrical, with the height being twice the diameter d (the result will depend somewhat on the proportions assumed). Then the volume is $\pi (d/2)^2 \cdot (2d)$. Setting this equal to 10^{-3} m³ gives d = 0.086 m. The area of the capacitor (assuming it has no top) is $A = \pi (d/2)^2 + \pi d(2d) = 9\pi d^2/4 = 0.052$ m². So the capacitance is

$$C = \frac{\kappa \epsilon_0 A}{s} = \frac{(4) \left(8.85 \cdot 10^{-12} \frac{s^2 C^2}{\lg m^3}\right) (0.052 \,\mathrm{m}^2)}{0.002 \,\mathrm{m}} = 9.2 \cdot 10^{-10} \,\mathrm{F}.$$
 (641)

If we had chosen the height to instead be four times the diameter, then the capacitance would be about 20% larger. As long as the jar isn't too squat (in which case it would

be better called a tray) or too tall (in which case it would be better called a tube), the dependence of the capacitance on the exact dimensions is fairly weak. (If the height is h = nd, then you can show that the capacitance is proportional to $(n + 1/4)/n^{2/3}$.) The capacitance of a sphere is $4\pi\epsilon_0 r$, so a sphere will have a capacitance of $9.2 \cdot 10^{-10}$ F if r = 8.3 m. The diameter is then 16.6 m, or about 54 feet.

10.17. Maximum energy storage

The maximum field is 14 kilovolts/mil, which in volts/meter equals

$$E_{\rm max} = \frac{1.4 \cdot 10^4 \,\rm V}{2.54 \cdot 10^{-5} \,\rm m} = 5.5 \cdot 10^8 \,\rm V/m.$$
(642)

The capacitance of the Mylar-filled capacitor is $\kappa \epsilon_0 A/s$. The energy stored is still $C\phi^2/2$, so the maximum possible energy density is

$$\frac{\text{energy}}{\text{volume}} = \frac{1}{2}C\phi^2 \frac{1}{V} = \frac{1}{2}\frac{\kappa\epsilon_0 A}{s}(Es)^2 \frac{1}{As} = \frac{1}{2}\kappa\epsilon_0 E^2$$

$$= \frac{1}{2}(3.25) \left(8.85 \cdot 10^{-12} \frac{s^2 C^2}{\text{kg m}^3}\right) (5.5 \cdot 10^8 \text{ V/m})^2 = 4.4 \cdot 10^6 \text{ J/m}^3.$$
(643)

The maximum energy per kilogram of Mylar is therefore

$$\frac{4.4 \cdot 10^6 \,\mathrm{J/m^3}}{1400 \,\mathrm{kg/m^3}} = 3100 \,\mathrm{J/kg}.$$
(644)

To determine how high the capacitor could lift itself, let the entire mass of the capacitor be m. Then 3m/4 of this is Mylar, so conservation of energy gives

$$E = mgh \implies (3100 \,\mathrm{J/kg})(3m/4) = mgh \implies h = \frac{(3/4)((3100 \,\mathrm{J/kg})}{9.8 \,\mathrm{m/s^2}} = 240 \,\mathrm{m}.$$

(645)

The D cell in Exercise 4.41 had an energy storage of $1.8 \cdot 10^5 \text{ J/kg}$, which is about 60 times as much as the Mylar capacitor. However, the capacitor can deliver all the stored energy in less than a microsecond!

10.18. Partially filled capacitors

The second capacitor in the figure consists of two capacitors in series; you can imagine the boundary between them to be two plates with charge Q and -Q superposed. Both of these capacitors have plate separation s/2 and area A, so the capacitances are (with the two halves labeled by "v" for vacuum and "d" for dielectric) $C_v = \epsilon_0 A/(s/2)$ and $C_d = \kappa \epsilon_0 A/(s/2)$. Since $C_0 = \epsilon_0 A/s$, we have $C_v = 2C_0$ and $C_d = 2\kappa C_0$. Problem 3.18 gives the rule for adding capacitors in series, so the desired capacitance is (with "S" for series)

$$\frac{1}{C_{\rm S}} = \frac{1}{C_{\rm v}} + \frac{1}{C_{\rm d}} = \frac{1}{2C_0} + \frac{1}{2\kappa C_0} \implies C_{\rm S} = \frac{2\kappa}{\kappa + 1} C_0.$$
(646)

The third capacitor in the figure consists of two capacitors in parallel. They both have plate separation s and area A/2, so the capacitances are $C_v = \epsilon_0(A/2)/s$ and $C_d = \kappa \epsilon_0(A/2)/s$. These can be written as $C_v = C_0/2$ and $C_d = \kappa C_0/2$. Problem 3.18 gives the rule for adding capacitors in parallel, so the desired capacitance is (with "P" for parallel)

$$C_{\rm P} = C_{\rm v} + C_{\rm d} = \frac{C_0}{2} + \frac{\kappa C_0}{2} = \frac{1+\kappa}{2} C_0.$$
 (647)

value of κ (greater than 1, of course), both $C_{\rm S}$ and $C_{\rm P}$ are larger than C_0 . This makes sense because the effect of a dielectric is to partially cancel the existing charge, which means that more charge must be added if the same potential is to be maintained.

If $\kappa = \infty$, then $C_{\rm S} = 2C_0$ and $C_{\rm P} = \infty$. The former makes sense because the dielectric is actually a conductor in this case, so we effectively have a vacuum capacitor with separation s/2. The latter makes sense because the capacitance of a conductor is infinite, since any charge you dump on it will be neutralized by the shifting of charges within the conductor. So the left half of the third capacitor has infinite capacitance.

10.19. Capacitor roll

Let w and s stand for the width and thickness of the materials. Then the various constants in the problem are (after converting to meters) $\kappa = 2.3$, $w_{\rm p} = 5.72 \cdot 10^{-2}$ m, $s_{\rm p} = 2.54 \cdot 10^{-5}$ m, $w_{\rm a} = 5.08 \cdot 10^{-2}$ m, and $s_{\rm a} = 1.27 \cdot 10^{-5}$ m. We want the capacitance to be $C = 5 \cdot 10^{-8}$ F. The capacitance of a parallel-plate capacitor (with width w, length ℓ , and separation s) in the presence of a dielectric is

$$C = \frac{\kappa \epsilon_0 A}{s} = \frac{\kappa \epsilon_0 w \ell}{s} \implies \ell = \frac{Cs}{\kappa \epsilon_0 w}.$$
(648)

If we have the tape stretched out straight, with the polyethylene sandwiched between two strips of aluminum, then we need the length of this linear aluminum capacitor to be

$$\ell = \frac{Cs_{\rm p}}{\kappa\epsilon_0 w_{\rm a}} = \frac{(5\cdot10^{-8}\,{\rm F})(2.54\cdot10^{-5}\,{\rm m})}{(2.3)\left(8.85\cdot10^{-12}\,\frac{{\rm s}^2\,{\rm C}^2}{{\rm kg\,m^3}}\right)(5.08\cdot10^{-2}\,{\rm m})} = 1.23\,{\rm m}.$$
(649)

If we simply rolled up the capacitor into a spiral, then one aluminum strip would touch the other. This would ruin the capacitor, because we need the two strips to have opposite charge. So we need to add on a second layer of polyethylene, as shown in Fig. 153. (We don't need a third layer of polyethylene on bottom.) So it appears that the total length of each kind of tape should be 2(1.23 m) = 2.46 m. However, from Problem 3.21 and Exercise 3.57, we effectively have twice as much area in the capacitor, because the two sides of each strip of aluminum tape act like two different sheets. So ℓ only needs to be half of the above 1.23 m. Hence $\ell = 0.61 \text{ m}$, and the total length of each kind of tape is 2(0.61 m) = 1.23 m.

To calculate the diameter of the rolled-up capacitor, note that the area (in the plane of the page) of the capacitor in Fig. 153 is $(2s_{\rm p} + 2s_{\rm a})(\ell) = (7.62 \cdot 10^{-5} \,\mathrm{m})(0.61 \,\mathrm{m}) = 4.7 \cdot 10^{-5} \,\mathrm{m}^2$. This area doesn't change when we roll up the capacitor, so the radius of the roll is given by

$$\pi r^2 = 4.7 \cdot 10^{-5} \,\mathrm{m}^2 \implies r = 3.9 \cdot 10^{-3} \,\mathrm{m}.$$
 (650)

The diameter is therefore a little less than 0.8 cm. If you missed the above factor of 1/2 in ℓ , the diameter would be larger by a factor of $\sqrt{2}$, but it would still be in the right ballpark.

10.20. Work in a dipole field

From Eq. (10.15) the potential energy, per unit charge, in the field of a dipole is $\phi = p \cos \theta / 4\pi \epsilon_0 r^2$. Hence

$$\phi_A = \frac{p}{4\pi\epsilon_0 a^2}$$
 and $\phi_B = \frac{p(1/\sqrt{2})}{4\pi\epsilon_0 (a/\sqrt{2})^2} = \frac{\sqrt{2}p}{4\pi\epsilon_0 a^2}$. (651)

The work done per unit charge is therefore $\phi_B - \phi_A = (\sqrt{2} - 1)(p/4\pi\epsilon_0 a^2)$.

polyethylene

Figure 153

10.21. A few dipole moments

- (a) Let the origin be located at the -2q charge (although the choice doesn't affect the result, because the net charge is zero). Then **p** points downward with magnitude $p = 2 \cdot q(\sqrt{3} d/2) = \sqrt{3} qd$.
- (b) p = 0, by symmetry.
- (c) Let the origin be located at the upper left corner (although, again, the choice doesn't matter). Then $p_x = qd + q(2d) = 3qd$, and $p_y = (-q)(-d) + (2q)(-d) = -qd$. So p has magnitude $\sqrt{10} qd$ and points diagonally rightward and downward at an angle of $\tan \theta = -1/3 \Longrightarrow \theta = -18.4^{\circ}$ below the horizontal.

10.22. Fringing field from a capacitor

The capacitor can be thought of as a large number of dipoles situated next to each other, and these dipoles all produce essentially the same field at a given point far away. We can therefore treat the capacitor like one effective dipole. The charge on it is $Q = CV = (2.5 \cdot 10^{-10} \text{ F})(2000 \text{ V}) = 5 \cdot 10^{-7} \text{ C}$, so the dipole moment is $p = Qs = (5 \cdot 10^{-7} \text{ C})(0.015 \text{ m}) = 7.5 \cdot 10^{-9} \text{ C m}.$

(a) From Eq. (10.18) the field at a point 3 m away in the plane of the plates is

$$E = \frac{p}{4\pi\epsilon_0 r^3} = \frac{7.5 \cdot 10^{-9} \,\mathrm{Cm}}{4\pi (8.85 \cdot 10^{-12} \,\frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg \,m^3}})(3 \,\mathrm{m})^3} = 2.5 \,\mathrm{V/m}.$$
 (652)

If the upper plate is the positively charged one, this field points downward.

(b) From Eq. (10.18) the field at a point in the direction perpendicular to the plates is $E = p/2\pi\epsilon_0 r^3$. This is just twice the field in part (a), so at 3 m away it equals 5 V/m. If the upper plate is the positively charged one, this field points upward (both above and below the capacitor).

We should check that our far-field dipole approximation is in fact valid. The capacitance of a parallel plate capacitor is $C = \epsilon_0 A/s$, so the area is

$$A = \frac{sC}{\epsilon_0} = \frac{(0.015 \,\mathrm{m})(250 \cdot 10^{-12} \,\mathrm{F})}{8.85 \cdot 10^{-12} \,\frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg} \,\mathrm{m}^3}} = 0.42 \,\mathrm{m}^2.$$
(653)

If the plates are square, then they are about 0.65 m on a side. The largest distance from the center to a point on the plates is therefore $(0.65 \text{ m})/\sqrt{2} = 0.46 \text{ m}$. The given distance of 3 m is reasonably large compared with this, so our dipole approximation is a fairly good one. However, if s or C were increased enough, then the length scale of the capacitor would be on the order of 3 m.

10.23. Dipole field plus uniform field

We want the field of the dipole to point in the $-\hat{\mathbf{y}}$ direction with magnitude $1.5 \cdot 10^5 \,\mathrm{V/m}$. So we are interested in two points in the yz plane like the ones shown in Fig. 154. Since we need $E_z = 0$, Eq. (10.17) tells us that $3\cos^2\theta - 1 = 0$. Hence $\cos\theta = \pm 1/\sqrt{3}$, and then $\sin\theta = \pm \sqrt{2/3}$. We want the points with opposite signs in these two trig relations (the other two points yield fields in the $+\hat{\mathbf{y}}$ direction). From Eq. (10.17) we have $E_y = 3p\sin\theta\cos\theta/4\pi\epsilon_0 r^3$. Therefore,

$$r^{3} = \frac{3p\sin\theta\cos\theta}{4\pi\epsilon_{0}E_{y}} = \frac{3(6\cdot10^{-10}\,\mathrm{C\,m})(\pm\sqrt{2}/\sqrt{3})(\mp1/\sqrt{3})}{4\pi\left(8.85\cdot10^{-12}\,\frac{\mathrm{s}^{2}\,\mathrm{C}^{2}}{\mathrm{kg\,m^{3}}}\right)(-1.5\cdot10^{5}\,\mathrm{V/m})}.$$
(654)





This yields r = 0.037 m. The coordinates of the lower right point are then

$$(y, z) = (r \sin \theta, -r \cos \theta) = (0.030, -0.021) \,\mathrm{m}.$$
 (655)

The upper left point has the negative of these coordinates.

10.24. Field lines

Let the dipole point in the z direction. If $r = r_0 \sin^2 \theta$, then

$$x = r \sin \theta = r_0 \sin^3 \theta,$$

$$z = r \cos \theta = r_0 \sin^2 \theta \cos \theta.$$
(656)

Therefore,

$$\frac{dx}{d\theta} = 3r_0 \sin^2 \theta \cos \theta,
\frac{dz}{d\theta} = r_0 (2\sin\theta\cos^2\theta - \sin^3\theta) = r_0 \sin\theta (2\cos^2\theta - \sin^2\theta)
= r_0 \sin\theta (3\cos^2\theta - 1).$$
(657)

The ratio of these derivatives gives the slope of the tangent to the $r = r_0 \sin^2 \theta$ curve as

$$\frac{dz}{dx} = \frac{3\cos^2\theta - 1}{3\sin\theta\cos\theta} \,. \tag{658}$$

This is the same as the ratio E_z/E_x as given by Eq. (10.17). The tangent to the $r = r_0 \sin^2 \theta$ curve therefore points in the same direction as the **E** field, as we wanted to show.

Alternatively, we can work with polar coordinates, as we did in Section 2.7.2. With respect to the local $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}}$ basis, the slope of the field-line curve is

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{r_0 \sin^2 \theta} \cdot 2r_0 \sin \theta \cos \theta = \frac{2\cos \theta}{\sin \theta}.$$
(659)

But from Eq. (10.18) this equals E_r/E_{θ} . So again, the tangent to the $r = r_0 \sin^2 \theta$ curve points in the same direction as the **E** field.

10.25. Average dipole field on a sphere

From Fig. 10.6, we quickly see that the average E_x value is zero, because for every field line pointing one way, there is another field line with the opposite E_x value. This holds for any vertical plane containing the z axis, so it holds for E_y too. Hence the averages of both E_x and E_y over the whole surface of a sphere are zero.

It isn't obvious from Fig. 10.6 that the average E_z value is zero. We can't use a quick symmetry argument, so we need to actually integrate E_z over a spherical shell. The area element of a horizontal ring on the sphere is $da = (2\pi r \sin \theta)(r d\theta)$. Using the form of E_z given in Eq. (10.17), and ignoring the r's and all the other constant factors, we have

$$E_z^{\text{avg}} \propto \int_0^\pi (3\cos^2\theta - 1)\sin\theta \,d\theta = (-\cos^3\theta + \cos\theta) \Big|_0^\pi = 0, \tag{660}$$

as desired.

10.26. Quadrupole for a square

The $y \equiv x_2$ coordinate of all four of the given charges is zero, so the quadrupole matrix from Problem 10.6 becomes

$$\mathbf{Q} = \sum_{\text{all charges}} q_i \begin{pmatrix} 3x_1^2 - r^2 & 0 & 3x_1x_3 \\ 0 & -r^2 & 0 \\ 3x_3x_1 & 0 & 3x_3^2 - r^2 \end{pmatrix}.$$
 (661)

We haven't bothered to tack on the index i (which labels each of the four charges) to the coordinates, and we've dropped the primes on the coordinates. All four charges have either x_1 or x_3 equal to zero, so the x_1x_3 terms in the matrix vanish. Also, all of the charges have the same value of r^2 , so the sum of r^2 over all the charges is zero, because the sum of the charges itself is zero. We are therefore left with only the $3x_1^2$ and $3x_3^2$ terms. Each of these picks up a contribution of $3a^2$ from each of two charges, so the complete quadrupole matrix is

$$\mathbf{Q} = ea^2 \begin{pmatrix} -6 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 6 \end{pmatrix}.$$
 (662)

The monopole and dipole moments are zero, so Eq. (12.469) gives the potential at position \mathbf{r} as $(1/4\pi\epsilon_0)(\hat{\mathbf{r}}\cdot\mathbf{Q}\hat{\mathbf{r}}/2r^3)$. If $\hat{\mathbf{r}} = (0,0,1)$, then we quickly find $\hat{\mathbf{r}}\cdot\mathbf{Q}\hat{\mathbf{r}} = 6ea^2$, so $\phi = 3ea^2/4\pi\epsilon_0r^3$, as desired. If $\hat{\mathbf{r}} = (1,0,1)/\sqrt{2}$, then $\hat{\mathbf{r}}\cdot\mathbf{Q}\hat{\mathbf{r}} = 0$, so $\phi = 0$. This makes sense, because the given point is equidistant from the upper +e and right -e charges, yielding zero potential. Likewise for the lower +e and left -e charges.

10.27. Pascal's triangle and the multipole expansion

- (a) If we make a copy of one of the configurations, negate all of its charges, shift it one unit to the left relative to the original, and then add its charges to the original, then we end up with the next configuration in the series. This is demonstrated in Fig. 155.
- (b) For the octupole, the potential at a point P on the axis is (ignoring the factor of $q/4\pi\epsilon_0$)

$$\phi_P = \frac{1}{r} - \frac{3}{r+a} + \frac{3}{r+2a} - \frac{1}{r+3a} = \frac{1}{r} \left(1 - \frac{3}{1+a/r} + \frac{3}{1+2a/r} - \frac{1}{1+3a/r} \right).$$
(663)

We can expand this with the Taylor series $1/(1+\epsilon) \approx 1-\epsilon+\epsilon^2-\epsilon^3$. With $\epsilon \equiv a/r$ we have

$$\phi_P \approx \frac{1}{r} \Big[1 - 3(1 - \epsilon + \epsilon^2 - \epsilon^3) \\ + 3(1 - 2\epsilon + 2^2\epsilon^2 - 2^3\epsilon^3) \\ - 1(1 - 3\epsilon + 3^2\epsilon^2 - 3^3\epsilon^3) \Big].$$
(664)

Collecting terms with the same power of ϵ , the sum in brackets can be written as

$$1 + \epsilon^{0} (-3 \cdot 1^{0} + 3 \cdot 2^{0} - 1 \cdot 3^{0}) - \epsilon^{1} (-3 \cdot 1^{1} + 3 \cdot 2^{1} - 1 \cdot 3^{1}) + \epsilon^{2} (-3 \cdot 1^{2} + 3 \cdot 2^{2} - 1 \cdot 3^{2}) - \epsilon^{3} (-3 \cdot 1^{3} + 3 \cdot 2^{3} - 1 \cdot 3^{3}).$$
(665)





You can quickly check that only the ϵ^3 term has a nonzero coefficient; the sum equals $6\epsilon^3$. Remembering the 1/r out front in Eq. (664), bringing back in the factor of $q/4\pi\epsilon_0$, and using $\epsilon \equiv a/r$, the potential at P is

$$\phi_P \approx \frac{q}{4\pi\epsilon_0} \frac{1}{r} (6\epsilon^3) = \frac{q}{4\pi\epsilon_0} \frac{6a^3}{r^4} \,. \tag{666}$$

As promised, this is proportional to $1/r^4$. And the units are (charge)/[$\epsilon_0 \cdot (\text{length})$], which are correct.

You can see that each line in Eq. (665) takes the form of $(-\epsilon)^m \sum_{k=0}^3 (-1)^k {3 \choose k} k^m$, for m = 0, 1, 2, 3. (The first line takes this form if we let 0^0 equal 1.) The sum is nonzero only for m = 3.

To prove the theorem mentioned in the problem, start with the " $(1-x)^N = [binomial expansion]$ " equation, and then repeat the process m times of alternately taking derivatives and multiplying by x (which will generate the mth power of the k's in the above sum), and then set x = 1. The left-hand side will vanish as long as we haven't taken more than N - 1 derivatives. If we take N derivatives, the left-hand side (and hence the right-hand side) will equal $(-1)^N N!$. It's easy to see how all of this works out if you do the procedure for, say, N = 4.

10.28. Force on a dipole

Let the middle dipole be located at the origin. Intuitively, the downward field at the origin arising from the left dipole is slightly stronger at the (negative) left end of the middle dipole than at its (positive) right end. The left end therefore feels a larger force upward than the right end feels downward. So there is a net upward force due to the field of the left dipole. Similar reasoning shows that there is a net rightward force due to the field of the right dipole. So the total force on the middle dipole is upward and rightward.

Let's be quantitative. From Eq. (10.26) the x component of the force on a dipole is $F_x = \mathbf{p} \cdot \nabla E_x$, and likewise for the other components. Let's first look at the y force due to the field from the left dipole. This field is $E_y = -p/4\pi\epsilon_0(b+x)^3$ at points on the x axis near the origin. The gradient of this has only an x component, and it is

$$\left. \frac{\partial E_y}{\partial x} \right|_{x=0} = \frac{3p}{4\pi\epsilon_0 b^4} \,. \tag{667}$$

Therefore,

$$F_y = \mathbf{p} \cdot \nabla E_y = p_x (\nabla E_y)_x = p \frac{3p}{4\pi\epsilon_0 b^4} = \frac{3p^2}{4\pi\epsilon_0 b^4} \,. \tag{668}$$

Now consider the x force due to the field from the right dipole. This field is $E_x = 2p/4\pi\epsilon_0(b-x)^3$ at points on the x axis near the origin. The gradient of this has only an x component, and it is

$$\left. \frac{\partial E_x}{\partial x} \right|_{x=0} = \frac{6p}{4\pi\epsilon_0 b^4} \,. \tag{669}$$

Therefore,

$$F_x = \mathbf{p} \cdot \nabla E_x = p_x (\nabla E_x)_x = p \frac{6p}{4\pi\epsilon_0 b^4} = \frac{6p^2}{4\pi\epsilon_0 b^4}.$$
 (670)

Since $F_y/F_x = 1/2$, the field points up to the right at an angle $\tan \theta = 1/2 \implies \theta = 26.6^{\circ}$ with respect to the x axis. The magnitude is $F = 3\sqrt{5}p^2/4\pi\epsilon_0 b^4 = (6.71)p^2/4\pi\epsilon_0 b^4$.

Alternatively, you could work out the force from scratch, by letting the dipole consist of two charges $\pm q$ at positions $x = \pm \ell/2$, with $q\ell = p$. If you explicitly calculate the forces on the two charges due to the left and right dipoles, to leading order in ℓ , you will end up with the above values of F_x and F_y . The differences in the forces on the two charges will effectively give the above gradients of the E's.

10.29. Energy of dipole pairs

(a) If we bring in the right dipole from infinity, with it pointing *leftward*, this requires no work, because the leftward displacement is always perpendicular to the vertical field from the left dipole. But when we rotate the right dipole 90° to the desired orientation, this requires pE worth of work; see Eq. (10.22). Since $E = p/4\pi\epsilon_0 d^3$, the work required is $W = p^2/4\pi\epsilon_0 d^3$.

Note that if we bring in the right dipole from infinity, with it pointing *upward*, then we cannot say that the leftward displacement is always perpendicular to the field from the left dipole, because this field is *not* vertical at points off the x axis. If we imagine the right dipole to consist of two point charges slightly above and slightly below the x axis, then at the location of these charges, the field from the left dipole has a slight horizontal component (rightward above the x axis, and leftward below). Positive work must be done to move each charge against this field. You can be quantitative about this if you want.

- (b) From reasoning similar to that in part (a), the work required is $W = -p^2/4\pi\epsilon_0 d^3$.
- (c) If we bring in the right dipole from infinity, with it pointing upward, this requires no work, because the works for the charges on the two ends of the dipole cancel. But when we rotate it 90° to the desired orientation, this requires -pE worth of work, from Eq. (10.22). Since $E = 2p/4\pi\epsilon_0 d^3$ along the axis of the left dipole, the work required is $W = -2p^2/4\pi\epsilon_0 d^3$.

(d) From reasoning similar to that in part (c), the work required is $W = 2p^2/4\pi\epsilon_0 d^3$. The task of Problem 10.3 is to directly calculate the above potential energies by looking at the point charges that make up the dipoles.

10.30. Polarized hydrogen

Since volume is proportional to r^3 , the negative charge inside radius Δz is $q = -e(\Delta z/a)^3$. Gauss's law therefore gives the field due to the inner part of the electron cloud as

$$\int \mathbf{E}_e \cdot d\mathbf{a} = \frac{q}{\epsilon_0} \implies E_e \cdot 4\pi (\Delta z)^2 = \frac{-e(\Delta z)^3}{\epsilon_0 a^3} \implies E_e = \frac{-e\Delta z}{4\pi\epsilon_0 a^3}.$$
 (671)

This field pulls the proton downward. In equilibrium, it must be equal and opposite to the applied field E that pushes the proton upward. Hence Δz is given by

$$\Delta z = \frac{4\pi\epsilon_0 E a^3}{e} \,, \tag{672}$$

which agrees with Eq. (10.27). The hydrogen atom won't actually remain spherically symmetric, but that won't affect the rough size of Δz .

10.31. Mutually induced dipoles

Consider the setup shown in Fig. 156. The field at B due to p_A is $2p_A/4\pi\epsilon_0 r^3$. Hence the induced dipole at B is $p_B = \alpha(2p_A/4\pi\epsilon_0 r^3)$. In a similar manner we find $p_A = \alpha(2p_B/4\pi\epsilon_0 r^3)$. Substituting p_B from the first of these relations into the second gives $p_A = 4\alpha^2 p_A/(4\pi\epsilon_0)^2 r^6$. This is satisfied by $p_A = 0$, or by any value of p_A provided that

$$r^6 = \frac{4\alpha^2}{(4\pi\epsilon_0)^2} \implies r = \left(\frac{\alpha}{2\pi\epsilon_0}\right)^{1/3} \equiv r_c,$$
 (673)

where the "c" stands for critical (or cutoff). If $r > r_c$, and if both dipole moments are nonzero at a given instant, they will decay to zero. But if $r < r_c$, and if both dipole moments are nonzero at a given instant, they will increase until limited by the nonlinearity of polarizability.

Atomic polarizabilities $\alpha/4\pi\epsilon_0$ are typically, in order of magnitude, an atomic volume; see Section 10.5. So $r_c = (2 \cdot \alpha/4\pi\epsilon_0)^{1/3}$ is on the order of an atomic radius. The object we are concerned with therefore might look something like what is shown in Fig. 157. Whether the lowest state of this system is a spontaneously polarized structure cannot be decided by considering only the interactions of dipoles. Ordinarily the lowest state of two similar atoms would be symmetrical with $\mathbf{p}_A + \mathbf{p}_B = 0$. But we cannot exclude the possibility that the symmetry is "spontaneously broken."

10.32. Hydration

In a water molecule, the side with the two hydrogen atoms is positively charged, and the side with the oxygen atom is negatively charged. So if the given ion is negative, the hydrogen side will be closer to the ion.

The field of a dipole along its axis equals $2p/4\pi\epsilon_0 r^3$, so the force on the negative ion will be attractive with magnitude $2ep/4\pi\epsilon_0 r^3$. The work required to move the ion to infinity by applying an equal and oppositive force is therefore

$$W = \int_{r_0}^{\infty} \frac{2ep \, dr}{4\pi\epsilon_0 r^3} = \frac{ep}{4\pi\epsilon_0 r_0^2} \,. \tag{674}$$



Figure 156



Figure 157

Alternatively, from Eq. (10.15) we know that the initial energy of the negative ion is $(-e)\phi = (-e)(p/4\pi\epsilon_0 r_0^2)$. The work that must be done in bringing the ion to infinity where $\phi = 0$ is the negative of this, in agreement with the above result. With $p = 6.13 \cdot 10^{-30}$ C-m and $r_0 = 1.5 \cdot 10^{-10}$ m, we find

$$W = \frac{(1.6 \cdot 10^{-19} \,\mathrm{C})(6.13 \cdot 10^{-30} \,\mathrm{C} \cdot\mathrm{m})}{4\pi \left(8.85 \cdot 10^{-12} \,\frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg \,m^3}}\right)(1.5 \cdot 10^{-10} \,\mathrm{m})^2} = 3.9 \cdot 10^{-19} \,\mathrm{J}.$$
(675)

10.33. Field from hydrogen chloride

From Eq. (10.18) the magnitude of the field is $p/2\pi\epsilon_0 r^3$ at points on the z axis, and $p/4\pi\epsilon_0 r^3$ at points on the y axis. From Fig. 10.14, **p** points in the direction from the Cl to the H, which is downward here. So the field points downward on the z axis and upward on the y axis. An angstrom equals 10^{-10} m, so the magnitude of the field at z = 10 angstroms is

$$E = \frac{p}{2\pi\epsilon_0 z^3} = \frac{3.43 \cdot 10^{-30} \,\mathrm{C\,m}}{2\pi \left(8.85 \cdot 10^{-12} \,\frac{\mathrm{s}^2 \,\mathrm{C}^2}{\mathrm{kg\,m^3}}\right) (10^{-9} \,\mathrm{m})^3} = 6.2 \cdot 10^7 \,\mathrm{V/m.}$$
(676)

And the magnitude at y = 10 angstroms is just half of this, or $3.1 \cdot 10^7$ V/m.

10.34. Hydrogen chloride dipole moment

If the electron distribution is spherically symmetric around the chlorine nucleus, then we effectively have one proton and one electron separated by 1.28 angstroms. The dipole moment is therefore

$$p = qd = (1.6 \cdot 10^{-19} \,\mathrm{C})(1.28 \cdot 10^{-10} \,\mathrm{m}) = 2.05 \cdot 10^{-29} \,\mathrm{Cm},$$
 (677)

which is about 6 times the actual dipole moment, $3.43 \cdot 10^{-30}$ C m.

To determine where the charge really is, we can treat the entire group of electrons like a point charge of -18e located at their "center of gravity" (defined to be the origin with respect to which $\int \mathbf{r} \rho \, dv = 0$). So we have the distribution shown in Fig. 158. Let x be the distance from the chlorine nucleus to the electron center. Then with the chlorine nucleus as our origin, we want x to satisfy

$$(17e)(0) + (-18e)x + (e)(1.28 \cdot 10^{-10} \,\mathrm{m}) = 3.43 \cdot 10^{-30} \,\mathrm{Cm}$$

$$\Rightarrow \quad x = \frac{1.28 \cdot 10^{-10} \,\mathrm{m}}{18} - \frac{3.43 \cdot 10^{-30} \,\mathrm{Cm}}{18(1.6 \cdot 10^{-19} \,\mathrm{C})} = 5.9 \cdot 10^{-12} \,\mathrm{m}, \tag{678}$$

or 0.059 angstroms, which is about 1/22 of the way from the chlorine nucleus to the proton. If it were 1/18 of the way (at 0.071 angstroms), which is the location of the center of the positive charge, then p would be zero. This would be the case if the electron from the hydrogen atom remained spherically symmetric around the hydrogen nucleus (the proton).

10.35. Some electric susceptibilities

=

The susceptibility is given by $\chi = \kappa - 1$, so $\chi = CNp^2/\epsilon_0 kT$ yields $C = (\kappa - 1)\epsilon_0 kT/Np^2$. The κ values for water and methanol in Table 10.1 are given for room temperature (20°C), whereas the κ for ammonia is given for -34° C. The values of kT at these temperatures are $4.0 \cdot 10^{-21}$ J and $3.3 \cdot 10^{-21}$ J, respectively. Knowing the values of κ , kT, N, and p (and ϵ_0 , of course), we can compute $C = (\kappa - 1)\epsilon_0 kT/Np^2$. So we just need to find N for the various substances.



Figure 158

A mole of something with molecular weight w has a mass of w grams. Equivalently, since the proton mass is $1.67 \cdot 10^{-24}$ g, it takes $1/1.67 \cdot 10^{-24} = 6 \cdot 10^{23}$ protons to make a gram. This number is essentially Avogadro's number.

Water has a molecular weight of 18, so the number of water molecules in 1 gram is $(6 \cdot 10^{23}/\text{mole})/(18 \text{ g/mole}) = 3.33 \cdot 10^{22} \text{ g}^{-1}$. The number of molecules per cm³ is then obtained by multiplying by the mass density, $\rho = 1.00 \text{ g/cm}^3$ (which doesn't change the number in the case of water). Finally, to obtain the number of molecules per m³, N, we must multiply by 10^6 . The resulting N's for the three substances are shown in the table below. The dipole moments, p, from Fig. 10.14 are also listed. The calculated values of $C = (\kappa - 1)\epsilon_0 kT/Np^2$ are listed in the righthand column.

	κ	kT	N	p	C
H_2O	80	$4.0 \cdot 10^{-21} \mathrm{J}$	$3.3 \cdot 10^{28} \mathrm{m}^{-3}$	$6.13 \cdot 10^{-30} \mathrm{C}\text{-m}$	2.3
NH_3	23	$3.3 \cdot 10^{-21} \mathrm{J}$	$2.9 \cdot 10^{28} \mathrm{m}^{-3}$	$4.76 \cdot 10^{-30} \mathrm{C}{-\mathrm{m}}$	1.0
CH_3OH	34	$4.0 \cdot 10^{-21} \mathrm{J}$	$2.5 \cdot 10^{28} \mathrm{m}^{-3}$	$5.66 \cdot 10^{-30} \mathrm{C}\text{-m}$	1.5

10.36. Discontinuity in E_{\perp}

The internal field is $\mathbf{E} = -\mathbf{P}/3\epsilon_0$, so $E_{\perp}^{\text{in}} = P\cos\theta/3\epsilon_0$. (As a double check, the factor here is indeed $\cos\theta$ because the field at the north pole is $P/3\epsilon_0$, pointing downward.) This component points *inward* in the upper hemisphere (and outward in the lower hemisphere), because \mathbf{E} points downward. The perpendicular external field is the radial field from a dipole, $E_{\perp}^{\text{out}} = E_r = p_0 \cos\theta/2\pi\epsilon_0 r_0^3$. This component points *outward* in the upper hemisphere (and inward in the lower hemisphere). The effective dipole moment p_0 equals $(4\pi r_0^3/3)P$. Hence $E_{\perp}^{\text{out}} = 2P\cos\theta/3\epsilon_0$. Due to the different inward/outward directions of the vectors, the discontinuity in E_{\perp} is $2P\cos\theta/3\epsilon_0 - (-P\cos\theta/3\epsilon_0) = P\cos\theta/\epsilon_0 = P_{\perp}/\epsilon_0$, as desired.

10.37. E at the center of a polarized sphere

Consider a horizontal ring at an angle θ down from the top of the sphere, with angular span $d\theta$. The area of this ring is $2\pi(R\sin\theta)(R\,d\theta)$. Since the density is $\sigma = P\cos\theta$, the charge in the ring is $q = 2\pi P R^2 \sin\theta\cos\theta\,d\theta$. A little bit of charge dq in a ring in the upper hemisphere creates a diagonally downward field of $dq/4\pi\epsilon_0 R^2$ at the center of the sphere. But by symmetry we are concerned only with the vertical component, which brings in a factor of $\cos\theta$. Integrating over all the dq's in a ring simply gives the total charge q in the ring. The net field from the ring therefore points downward with magnitude

$$\frac{2\pi P R^2 \sin\theta \cos\theta \,d\theta}{4\pi\epsilon_0 R^2} \cos\theta = \frac{P \sin\theta \cos^2\theta \,d\theta}{2\epsilon_0} \,. \tag{679}$$

You can verify that this expression is valid for rings in the lower hemisphere too; all contributions to the field point downward. Integrating over θ from 0 to π gives a total magnitude of

$$E = \int_0^\pi \frac{P\sin\theta\cos^2\theta\,d\theta}{2\epsilon_0} = -\frac{P\cos^3\theta}{6\epsilon_0}\Big|_0^\pi = \frac{P}{3\epsilon_0}\,,\tag{680}$$

as desired. The direction is downward.

10.38. Uniform field via superposition

(a) Applying Gauss's law to a sphere of radius r inside the given sphere yields

$$4\pi r^2 E = \frac{q}{\epsilon_0} \implies 4\pi r^2 E = \frac{(4\pi r^3/3)\rho}{\epsilon_0} \implies E = \frac{\rho r}{3\epsilon_0}.$$
 (681)





The field points radially, so we have $\mathbf{E} = (\rho/3\epsilon_0)\mathbf{r}$.

(b) Let **s** be the vector from C_2 to C_1 in Fig. 159. The field at an arbitrary point P inside both distributions is (using $\mathbf{r}_1 - \mathbf{r}_2 = -\mathbf{s}$)

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\rho \mathbf{r}_1}{3\epsilon_0} + \frac{(-\rho)\mathbf{r}_2}{3\epsilon_0} = \frac{\rho}{3\epsilon_0}(\mathbf{r}_1 - \mathbf{r}_2) = -\frac{\rho \mathbf{s}}{3\epsilon_0}.$$
 (682)

The vector **s** is the displacement of the positive charge distribution with respect to the negative, so the polarization density is $\mathbf{P} = \rho \mathbf{s}$. (This is true because if the dipoles consist of charges q separated by a distance s, then the magnitude of **P** is given by $P = Np = Nqs = \rho s$.) This is the numerator of the result in Eq. (682), so we have

$$\mathbf{E} = -\frac{\rho \mathbf{s}}{3\epsilon_0} = -\frac{\mathbf{P}}{3\epsilon_0}\,,\tag{683}$$

in agreement with Eq. (10.47).

(c) For a cylindrical distribution, applying Gauss's law to an internal cylinder of radius r and length ℓ gives

$$2\pi r\ell E = \frac{q}{\epsilon_0} \implies 2\pi r\ell E = \frac{(\pi r^2 \ell)\rho}{\epsilon_0} \implies E = \frac{\rho r}{2\epsilon_0}.$$
 (684)

The field points radially, so we have $\mathbf{E} = (\rho/2\epsilon_0)\mathbf{r}$. From the same reasoning as in part (b) (the picture looks exactly the same, when viewing along the axis), the field at an arbitrary point P inside both distributions is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\rho \mathbf{r}_1}{2\epsilon_0} + \frac{(-\rho)\mathbf{r}_2}{2\epsilon_0} = \frac{\rho}{2\epsilon_0}(\mathbf{r}_1 - \mathbf{r}_2) = -\frac{\rho \mathbf{s}}{2\epsilon_0}.$$
 (685)

We again have $\mathbf{P} = \rho \mathbf{s}$, so the field inside a cylinder with uniform transverse polarization is $\mathbf{E} = -\mathbf{P}/2\epsilon_0$.

REMARK: A similar result holds if we kick things down another dimension and consider a slab. For a slab, applying Gauss's law to an internal box centered on the center-plane of the slab, with thickness 2x and end-face area A, gives

$$E(2A) = \frac{q}{\epsilon_0} \implies 2EA = \frac{(2x \cdot A)\rho}{\epsilon_0} \implies E = \frac{\rho x}{\epsilon_0}.$$
 (686)

The field points perpendicular to the faces of the slab, so we have $\mathbf{E} = (\rho/\epsilon_0)\mathbf{x}$. The same reasoning holds again, with two slightly displaced slabs. It's even easier in this case, because everything takes place in one dimension:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\rho \mathbf{x}_1}{2\epsilon_0} + \frac{(-\rho)\mathbf{x}_2}{\epsilon_0} = \frac{\rho}{\epsilon_0}(\mathbf{x}_1 - \mathbf{x}_2) = -\frac{\rho \mathbf{s}}{\epsilon_0}.$$
 (687)

We again have $\mathbf{P} = \rho \mathbf{s}$, so the desired field is $\mathbf{E} = -\mathbf{P}/\epsilon_0$. The only difference in the **E**'s for the various objects we have considered is the numerical factor (3, 2, or 1) in the denominator.

10.39. Conducting-sphere limit

Equation (10.54) gives the polarization of a dielectric sphere as $P = 3\epsilon_0 E_0(\kappa - 1)/(\kappa + 2)$. In the $\kappa \to \infty$ limit, this becomes $P = 3\epsilon_0 E_0$. Let's check that this produces an external field that is perpendicular to the conductor, which must be the case for a perfect conductor. The total dipole moment of the sphere is $p = (4\pi a^3/3)P = 4\pi\epsilon_0 a^3 E_0$. And the sphere does indeed act like a dipole, as far as the external field is

concerned, from the reasoning near the beginning of Section 10.9. From Eq. (10.18) the tangential field of a dipole is $p \sin \theta / 4\pi \epsilon_0 a^3$, which gives $E_0 \sin \theta$ here. But this is exactly what is needed to cancel the tangential component of the original uniform field E_0 (you can check that the direction is correct). The tangential component of the total external field is therefore zero, as desired.

The field strength inside the dielectric sphere, which from Eq. (10.53) is $3E_0/(2 + \kappa)$, goes to zero as $\kappa \to \infty$. The sphere therefore becomes an equipotential, which is correct for a conducting sphere. A sketch of the field lines outside the sphere is shown in Fig. 160.

Using the above value of p, we see that the polarizability α , defined by $\mathbf{p} = \alpha \mathbf{E}_0$, of a perfectly conducting sphere equals $4\pi\epsilon_0 a^3$. The quantity that we normally work with, $\alpha/4\pi\epsilon_0$, therefore equals a^3 for our conducting sphere of radius a. Since $\alpha/4\pi\epsilon_0 = 0.66 \cdot 10^{-30} \text{ m}^3$ for hydrogen, a conducting sphere of equal polarizability has a radius of $(0.66 \cdot 10^{-30} \text{ m}^3)^{1/3} = 8.7 \cdot 10^{-11} \text{ m}$. This is very close to the Bohr radius, $5.3 \cdot 10^{-11} \text{ m}$.

10.40. Continuity of D

The **E** field due to the slab is the same as the **E** field due to two capacitor plates with surface charge densities $\pm P$. Both **E** and **P** are zero outside the slab, so the external **D** is likewise zero. Our task is therefore to show that **D** is zero inside the slab. And indeed, $\mathbf{E} = -\mathbf{P}/\epsilon_0$ (this is the field between two plates with densities $\pm P$), so $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (-\mathbf{P}/\epsilon_0) + \mathbf{P} = 0$.

10.41. **Discontinuity in** D_{\parallel}

Inside the sphere, we have $\mathbf{E} = -\mathbf{P}/3\epsilon_0$, so the displacement vector is $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0(-\mathbf{P}/3\epsilon_0) + \mathbf{P} = 2\mathbf{P}/3$. The tangential component of this is

$$D_{\parallel}^{\rm in} \equiv D_{\theta}^{\rm in} = -\frac{2P\sin\theta}{3} \,. \tag{688}$$

The minus sign here comes from the fact that **P** points toward the north pole, whereas the positive $\hat{\theta}$ direction is defined to point away from the north pole.

Outside the sphere, **E** is the field due to a dipole with $\mathbf{p}_0 = (4\pi R^3/3)\mathbf{P}$. From Eq. (10.18) the tangential component of the dipole field is $E_\theta = p_0 \sin \theta / 4\pi \epsilon_0 R^3$. In terms of *P* this becomes $E_\theta = P \sin \theta / 3\epsilon_0$. Since $\mathbf{P} = 0$ outside the sphere, the external **D** is obtained by simply multiplying the external **E** by ϵ_0 . Therefore

$$D_{\parallel}^{\text{out}} \equiv D_{\theta}^{\text{out}} = \frac{P \sin \theta}{3} \,. \tag{689}$$

Comparing this with the $D_{\parallel}^{\text{in}}$ in Eq. (688), we see that D_{\parallel} has a discontinuity of $P \sin \theta$. D_{\parallel} increases by $P \sin \theta$ in going from inside to outside. This makes sense, because we know that E_{θ} is continuous across the boundary, so the discontinuity in the tangential component of $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$ is simply the discontinuity in the tangential component of \mathbf{P} .

10.42. Energy density in a dielectric

With a dielectric present, the capacitance of a parallel-plate capacitor is $C = \kappa \epsilon_0 A/s \equiv \epsilon A/s$. The energy stored is still $C\phi^2/2$, because it equals $Q\phi/2$ for all the same reasons as in the vacuum case (imagine a battery doing work in transferring charge from one plate to the other). So the energy density is

$$\frac{\text{energy}}{\text{volume}} = \frac{1}{2}C\phi^2 \frac{1}{V} = \frac{1}{2}\frac{\epsilon A}{s}(Es)^2 \frac{1}{As} = \frac{\epsilon E^2}{2},$$
(690)



as desired. Since $\epsilon \equiv \kappa \epsilon_0$, this energy density is κ times the $\epsilon_0 E^2/2$ energy density without the dielectric. Basically, since C is κ times larger, so is the energy, and hence the energy density. To see physically why the energy is larger, consider the case of induced dipole moments, discussed in Section 10.5. The stretched atoms and molecules are effectively little springs that are stretched, so they store potential energy. This makes the total energy larger than it would be for the same equivalent charge on/at the capacitor plates (free charge plus bound-charge layer).

In an electromagnetic wave in a dielectric, the energy density of the magnetic field is still $B^2/2\mu_0$. (It would be $B^2/2\mu$ in a magnetized material, but we're assuming that the material here is only electrically polarizable.) But from Eq. (10.83) the amplitudes of the *E* and *B* fields are related by $B_0 = \sqrt{\mu_0 \epsilon} E_0$. So $B^2/2\mu_0 = \epsilon E^2/2$. The *E* and *B* energy densities are therefore equal, just as they are in vacuum.

10.43. Reflected wave

The incident, transmitted, and reflected waves are, respectively (using $\mathbf{E} \times \mathbf{B} \propto \mathbf{v}$ to find the direction of the **B**'s),

$$\mathbf{E}_{i} = \hat{\mathbf{z}} E_{i} \sin(ky - \omega t), \qquad \mathbf{B}_{i} = \hat{\mathbf{x}} B_{i} \sin(ky - \omega t), \\
 \mathbf{E}_{r} = \hat{\mathbf{z}} E_{r} \sin(ky + \omega t), \qquad \mathbf{B}_{r} = -\hat{\mathbf{x}} B_{r} \sin(ky + \omega t), \\
 \mathbf{E}_{t} = \hat{\mathbf{z}} E_{t} \sin(k'y - \omega t), \qquad \mathbf{B}_{t} = \hat{\mathbf{x}} B_{t} \sin(k'y - \omega t). \tag{691}$$

The total wave in the empty space y < 0 is the sum of the incident and reflected waves.

Let's apply continuity of **E** and **B** at y = 0. After setting y = 0, we can cancel all the $\sin \omega t$ terms (which means that our results will hold for all t), but we must be careful about the extra minus sign in $\sin(-\omega t)$. We obtain

$$-E_{\rm i} + E_{\rm r} = -E_{\rm t}$$
 and $-B_{\rm i} - B_{\rm r} = -B_{\rm t}.$ (692)

We also have

$$E_{\rm i} = cB_{\rm i}, \qquad E_{\rm r} = cB_{\rm r}, \qquad E_{\rm t} = (c/n)B_{\rm t}.$$
 (693)

Given the "i" quantities, we have four equations in four unknowns (the "r" and "t" quantities). Eliminating the B's quickly turns Eq. (692) into

$$E_{\rm i} - E_{\rm r} = E_{\rm t}$$
 and $E_{\rm i} + E_{\rm r} = nE_{\rm t}$. (694)

Solving these yields

$$E_{\rm r} = \frac{n-1}{n+1}E_{\rm i}$$
 and $E_{\rm t} = \frac{2}{n+1}E_{\rm i}$. (695)

So $E_{\rm r}/E_{\rm i} = (n-1)/(n+1)$, and $E_{\rm t}/E_{\rm i} = 2/(n+1)$.

The energy is proportional to the square of E, so the faction of the incident energy that is reflected, with n = 1.6, is

$$\frac{E_{\rm r}^2}{E_{\rm i}^2} = \left(\frac{n-1}{n+1}\right)^2 = 0.053. \tag{696}$$

Chapter 11

Magnetic fields in matter

Solutions manual for *Electricity and Magnetism, 3rd edition*, E. Purcell, D. Morin. morin@physics.harvard.edu (Version 1, January 2013)

11.12. Earth dipole

(a) Equation (11.15) gives the field at position R along the axis of a dipole as $B_r = \mu_0 m/2\pi R^3$, so

$$m = \frac{2\pi R^3 B_r}{\mu_0} = \frac{2\pi (6.4 \cdot 10^6 \,\mathrm{m})^3 (6.2 \cdot 10^{-5} \,\mathrm{T})}{4\pi \cdot 10^{-7} \,\frac{\mathrm{kg} \,\mathrm{m}}{C^2}} = 8.1 \cdot 10^{22} \,\mathrm{J/T}.$$
 (697)

(b) Equation (6.53) gives the field at position z on the axis of a current ring as $B_z = \mu_0 I b^2 / 2(z^2 + b^2)^{3/2}$. If R is the radius of the earth, then we have z = R and $b \approx R/2$, so in terms of R the field is $B_z = \mu_0 I / (5^{3/2} R)$. Therefore,

$$I = \frac{5^{3/2} RB_z}{\mu_0} = \frac{5^{3/2} (6.4 \cdot 10^6 \text{ m}) (6.2 \cdot 10^{-5} \text{ T})}{4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^2}} = 3.5 \cdot 10^9 \text{ A.}$$
(698)

If we instead treat the current ring as a dipole with moment $m = 8.1 \cdot 10^{22}$ J/T, then we have

$$m = I(\pi b^2) \implies I = \frac{m}{\pi (R/2)^2} = \frac{4(8.1 \cdot 10^{22} \text{ J/T})}{\pi (6.4 \cdot 10^6 \text{ m})^2} = 2.5 \cdot 10^9 \text{ A},$$
 (699)

which is a so-so approximation to the correct result of $3.5 \cdot 10^9$ A.

11.13. Disk dipole

Let's divide the disk into rings and then add up the magnetic moments of all the rings. The surface current density at radius r is σv , where $v = \omega r$. This is true because $\sigma \ell(v dt)$ is the amount of charge that crosses a transverse segment with length ℓ in a time dt. So the charge per time per unit transverse length (that is, the surface current density) equals $\sigma \ell(v dt)/(\ell dt) = \sigma v$.

The current in a given ring with radius r and thickness dr is therefore $I_r = (\sigma v)dr = \sigma \omega r \, dr$. The magnetic moment of this ring is then $I_r(\pi r^2) = \pi \sigma \omega r^3 \, dr$. Integrating from r = 0 to r = R gives the total magnetic moment of the disk as $\pi \sigma \omega R^4/4$.

Note that this result can be written as $\omega(\pi R^2 \sigma)R^2/4 = \omega QR^2/4$, where Q is the total charge on the disk. If all of this charge were instead located on the rim, then since one revolution takes a time of $2\pi/\omega$, the magnetic moment would be $I\pi R^2 = (Q/(2\pi/\omega))\pi R^2 = \omega QR^2/2$, which is twice the moment of the disk.

11.14. Sphere dipole

Let the axis of rotation be vertical. Consider a strip located at angle θ , with width $d\theta$, as shown in Fig. 161. The speed of any point in this strip is $v = \omega r = \omega(R \sin \theta)$. The effective linear charge density of the strip, $d\lambda$, equals σ times the width, so $d\lambda = \sigma(R d\theta)$. The current dI in the strip is then

$$dI = v \, d\lambda = (\omega R \sin \theta) (\sigma R \, d\theta) = \omega \sigma R^2 \sin \theta \, d\theta = \frac{\omega Q \sin \theta \, d\theta}{4\pi} \,, \tag{700}$$

where we have used $\sigma = Q/4\pi R^2$. Alternatively, you can find dI by multiplying the total charge in the ring, which is $\sigma(2\pi R \sin \theta)(R d\theta)$, by the number of revolutions per second, which is $\omega/2\pi$.

The magnetic moment of the ring is

$$dm = (dI)\pi(R\sin\theta)^2 = \frac{\omega QR^2 \sin^3\theta \, d\theta}{4}.$$
(701)

Integrating this over the whole sphere to obtain the total magnetic moment gives (using the integral table in Appendix K or writing $\sin^3 \theta$ as $\sin \theta (1 - \cos^2 \theta)$)

$$m = \frac{\omega Q R^2}{4} \int_0^\pi \sin^3 \theta \, d\theta = \frac{\omega Q R^2}{4} \left(-\cos\theta + \frac{\cos^3\theta}{3} \right) \Big|_0^\pi = \frac{\omega Q R^2}{4} \cdot \frac{4}{3} = \frac{\omega Q R^2}{3} \,.$$
(702)

Note that if all of the charge Q were located on the equator, then the current would be $I = (\omega/2\pi)Q$, and the magnetic moment would be $m = I\pi R^2 = (\omega Q/2\pi)(\pi R^2) = \omega QR^2/2$. The *m* for a spinning shell is therefore 2/3 of the *m* for a spinning ring with the same radius and total charge. Also, from Exercise 11.13 the *m* for a spinning shell is 4/3 times the *m* for a spinning disk with the same radius and total charge. It makes sense that this factor is larger than 1, because if the shell is projected onto the equatorial disk, the density near the rim is larger than the density at the center.

11.15. A solenoid as a dipole

To estimate roughly the magnetic dipole moment of the solenoid, let us suppose that it is equivalent to a point dipole that would produce, 20 cm away on its axis, a field strength B_z equal to that at the end of the solenoid, namely 1.8 T. This is reasonable because the magnetic field configuration near the end of the solenoid and beyond looks not very different from a dipole field. On this assumption, Eq. (11.15) gives

$$B = \frac{\mu_0 m}{2\pi r^3} \implies m = \frac{2\pi r^3 B}{\mu_0} = \frac{2\pi (0.2 \text{ m})^3 (1.8 \text{ T})}{4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^2}} = 7.2 \cdot 10^4 \text{ J/T.}$$
(703)

However, there is actually no need to know this value of m, because we are interested only in a rough estimate of the field at the location of the complaining physicists. All we care about is that the field decreases like $1/r^3$; the exact nature of the radial and tangential components isn't critical. The physicists are 100 feet away, which is about 30 meters. This is 150 times the 0.2-meter distance at which the field is 18,000 gauss, so the desired field is smaller by a factor of 150^3 . It is therefore roughly equal



Figure 161

to $18,000/150^3 \approx 5 \cdot 10^{-3}$ gauss. (If you want to find the two components, you can simply plug the above value of m into Eq. (11.15), with $\theta = \tan^{-1}(4/3) \approx 53^{\circ}$.)

This field is about 100 times smaller than the earth's field of about 0.5 gauss. If it were perfectly steady it would not be noticed. But if it were frequently switched on and off, it might cause trouble.

11.16. Dipole in a uniform field

Let the magnetic dipole point upward, which means that the uniform field \mathbf{B}_0 points downward. From Eq. (11.15) the radial component of the dipole field is $(\mu_0 m/2\pi r^3)\cos\theta$. And the radial component of the uniform field is $-B_0 \cos \theta$. So the total radial component on the surface of a sphere with radius r is $B_r = (\mu_0 m/2\pi r^3 - B_0)\cos\theta$. This equals zero everywhere on a sphere with radius $r = (\mu_0 m/2\pi B_0)^{1/3}$. Since $B_r = 0$ on this sphere, no field lines pass through the sphere, so the internal field lines must look something like those shown in Fig. 162.

On the equator, we need to add \mathbf{B}_0 to the field from the dipole. For the dipole field, we are concerned with the tangential component, which from Eq. (11.15) points downward with magnitude $\mu_0 m/4\pi r^3$ on the equator. Using $r = (\mu_0 m/2\pi B_0)^{1/3}$, this equals $B_0/2$. Adding this to the downward uniform field B_0 gives a total field of $3B_0/2$ pointing downward.

If we create the identical field outside the sphere by replacing the dipole by the appropriate distribution of surface current on the sphere, the field inside will be zero. This is true because a field line can't end on the spherical surface (there are no magnetic monopoles, so magnetic field lines can't end) or pass through it, so it would have to be a closed loop inside. But that region is now empty of current, so we must have $\int \mathbf{B} \cdot d\mathbf{s} = 0$ around any path. A closed field-line loop would make this integral be nonzero. So there must be no closed loops, and hence no magnetic field at all.

11.17. Trapezoid dipole

The left and right sides of the trapezoid each produce zero field at P, because they point directly at P so the cross product in the Biot-Savart law is zero. For the top and bottom sides, the angle in the Biot-Savart law is essentially 90° for the whole length, so the magnitude of the Biot-Savart contribution is $\mu_0 I \ell / 4\pi R^2$, where ℓ is the length of the particular side and R is the distance to the side.

Let d be half the height of the trapezoid; this is essentially equal to a/2, but it will be easier to work with d. The top and bottom sides are distances r - d and r + d from P. If the bottom side has length b, then from similar triangles the top side has length b(r-d)/(r+d).

At point P, the top side produces a magnetic field into the page, and the bottom side produces a field out of the page. So the net field into the page at P is

$$B = B_{\text{top}} - B_{\text{bottom}} = \frac{\mu_0 I}{4\pi} \left(\frac{b(r-d)/(r+d)}{(r-d)^2} - \frac{b}{(r+d)^2} \right)$$
$$= \frac{\mu_0 I}{4\pi} \frac{b}{r+d} \left(\frac{1}{r-d} - \frac{1}{r+d} \right) = \frac{\mu_0 I}{4\pi} \frac{b}{r+d} \frac{2d}{r^2 - d^2} \approx \frac{\mu_0 I}{4\pi} \frac{ba}{r^3}, \quad (704)$$

where we have used $2d \approx a$, and where we have ignored the d's in the denominator because they are small compared with r. Since the area of the trapezoid is essentially equal to ba, we have $B \approx \mu_0 m / 4\pi r^3$, in agreement with the B_θ in Eq. (11.15) when $\theta = 90^{\circ}$.



Note that the above result is first order in b. You can show that the approximations we made (setting the Biot-Savart angle equal to 90°, using $2d \approx a$) involve errors that are second order in b, so we were justified in ignoring them.

11.18. Field somewhat close to a solenoid

Consider a single loop of the solenoid at its middle. From Eq. (11.15), this little dipole creates a field at P equal to $\mu_0 m/4\pi\ell^3 = \mu_0(\pi R^2 I)/4\pi\ell^3$. Up to numerical factors, all the other loops create this same field too; the only differences are some trig factors of order 1. So if there are N loops in all, the total field at P behaves like

$$B \sim N \frac{\mu_0(\pi R^2 I)}{4\pi \ell^3} = \mu_0(N/\ell) I \frac{R^2}{4\ell^2} \,. \tag{705}$$

Ignoring the factor of 1/4, and recalling that $\mu_0(N/\ell)I \equiv \mu_0 nI$ is the field B_0 inside the solenoid, we obtain $B \sim B_0 R^2/\ell^2$, as desired.

11.19. Using reciprocity

Let the dipole $\mathbf{m}(t)$ consist of a ring of area **a** carrying current $I_2(t) = I_2 \cos \omega t$. So $\mathbf{m}(t) = (I_2 \cos \omega t) \mathbf{a} \Longrightarrow \mathbf{m}_0 = I_2 \mathbf{a}$. If a current $I_1(t)$ in C_1 causes a field $\mathbf{B}_1(t)$ at the location of the dipole ring, then the flux through the ring is $\Phi_{\text{ring}}(t) = \mathbf{B}_1(t) \cdot \mathbf{a}$. The coefficient of mutual inductance is therefore $M = \Phi_{\text{ring}}/I_1 = (\mathbf{B}_1 \cdot \mathbf{a})/I_1$. (We've suppressed the t arguments here.) Due to the reciprocity theorem, this is also the M going the other way. The emf in circuit C_1 is therefore given by $\mathcal{E}_1(t) = -M \, dI_2(t)/dt$, where $I_2(t)$ is the current in the dipole ring. Hence the emf in C_1 is

$$\mathcal{E}_{1}(t) = -\frac{\mathbf{B}_{1} \cdot \mathbf{a}}{I_{1}} \frac{d(I_{2} \cos \omega t)}{dt} = -\frac{\mathbf{B}_{1} \cdot \mathbf{a}}{I_{1}} (-\omega I_{2} \sin \omega t)$$
$$= \frac{\omega}{I_{1}} \mathbf{B}_{1} \cdot (I_{2} \mathbf{a}) \sin \omega t = \frac{\omega}{I_{1}} \mathbf{B}_{1} \cdot \mathbf{m}_{0} \sin \omega t \equiv \mathcal{E}_{1} \sin \omega t, \quad (706)$$

with $\mathcal{E}_1 = (\omega/I_1)\mathbf{B}_1 \cdot \mathbf{m}_0$, as desired. I_1 and \mathbf{B}_1 are technically functions of time. But since they have the same time dependence, we can take them to be the amplitudes (a vector amplitude in the \mathbf{B}_1 case).

11.20. Force between a wire and a loop

J

At an arbitrary point on the wire, the magnetic field from the square-loop dipole has both an upward vertical (z) component and a horizontal component along the wire. But the latter produces no force on the current in the wire, so we care only about the z component. Since the current in the wire in Fig. 6.47 points into the page, the right-hand rule gives the magnetic force on the wire as pointing to the right.

Equation (11.14) gives the z component of the dipole field, with m equal to $m = I_2 \ell^2$. The rightward force on a little piece dx of the wire equals $I_1 B_z dx$. With θ measured away from the vertical axis, dx is given by the usual expression, $dx = z d\theta / \cos^2 \theta$. (See the reasoning in the paragraph following Eq. (1.37).) Also, the distance r to the little piece is $r = z / \cos \theta$. Integrating over the entire infinite wire, we find the total rightward force on it to be

$$F = \int_{-\infty}^{\infty} I_1 B_z \, dx = \int_{-\infty}^{\infty} I_1 \left(\frac{\mu_0 I_2 \ell^2}{4\pi} \frac{3\cos^2 \theta - 1}{r^3} \right) dx$$
$$= \frac{\mu_0 I_1 I_2 \ell^2}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{3\cos^2 \theta - 1}{(z/\cos \theta)^3} \frac{z \, d\theta}{\cos^2 \theta}$$
$$= \frac{\mu_0 I_1 I_2 \ell^2}{4\pi z^2} \int_{-\pi/2}^{\pi/2} (3\cos^3 \theta - \cos \theta) \, d\theta = \frac{\mu_0 I_1 I_2 \ell^2}{2\pi z^2} \,. \tag{707}$$

11.21. Dipoles on a chessboard

(a) Equation (11.23) gives the force on a dipole as $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$. The applied force is the negative of this, so the associated work equals the line integral of $-\nabla(\mathbf{m} \cdot \mathbf{B})$. The line integral of the gradient of a quantity is simply the change in that quantity. So the work required to move a dipole to infinity equals the change in $-\mathbf{m} \cdot \mathbf{B}$, which is $0 - (-\mathbf{m} \cdot \mathbf{B}_0) = \mathbf{m} \cdot \mathbf{B}_0$, because the field is zero at infinity. \mathbf{B}_0 here is the field (due to all the other dipoles) at the initial location of a dipole on a particular square. This result is consistent with the fact that the energy of a dipole is given by $-\mathbf{m} \cdot \mathbf{B}$.

Due to the opposite directions of the dipoles on the white and black squares, the **B** fields from the four nearest neighbors (or two or three, if the dipole is near the edge) point *parallel* to a given dipole **m**. So we expect that the sum of all 63 of the $\mathbf{m} \cdot \mathbf{B}_0$ contributions to the work will be positive. That is, we expect all of the dipoles to be bound.

Eq. (11.15) gives the magnetic field due to a dipole, in the plane of the dipole, as $\mu_0 m/4\pi r^3$. The distance r between the various squares is found from the Pythagorean theorem. For a given point labeled by the coordinates (a, b), where a and b each run from 1 to 8, the sum of the $\mathbf{m} \cdot \mathbf{B}_0$ contributions to the work equals

$$-\frac{\mu_0 m^2}{4\pi s^3} \sum_{i=1}^8 \sum_{j=1}^8 \frac{(-1)^{a-i} (-1)^{b-j}}{\left((a-i)^2 + (b-j)^2\right)^{3/2}},$$
(708)

with the caveat that the term with (i, j) = (a, b) is excluded from the sum. You can check that the negative sign out front makes the overall sign correct. There must be a way to cleanly exclude the (i, j) = (a, b) term in *Mathematica*, but I can't figure out what it is. At any rate, this program gets the job done (the values of a and b can be changed):

```
a = 3; b = 2;
NSum[-(-1)^(a-i)(-1)^(b-j)/((a-i)^2+(b-j)^2)^(3/2), {i,1,a-1}, {j,1,8}]+
NSum[-(-1)^(a-i)(-1)^(b-j)/((a-i)^2+(b-j)^2)^(3/2), {i,a+1,8}, {j,1,8}]+
NSum[-(-1)^(a-i)(-1)^(b-j)/((a-i)^2+(b-j)^2)^(3/2), {i,a,a}, {j,1,b-1}]+
NSum[-(-1)^(a-i)(-1)^(b-j)/((a-i)^2+(b-j)^2)^(3/2), {i,a,a}, {j,b+1,8}]
```

Up to a factor of $\mu_0 m^2/4\pi s^3$, the amounts of work for the various squares on the chessboard are shown in Fig. 163. These entries are repeated in other parts of the board; there are only ten independent entries. The most tightly bound dipoles are the ones on the "bishop's pawn" and equivalent squares; there are eight such squares. However, these are only negligibly more bound than the other interior dipoles. The binding energies of the 36 interior dipoles differ from one another by less than 1%.

(b) The field from a magnetic dipole takes the same form as the field from an electric dipole. And the force on a dipole does also, because there aren't any outside currents in the setup; see the discussion following Eq. (11.24). So the magnetic force on each of our magnetic dipoles is conservative, just as the electric force on an electric dipole is. So we can use the same reasoning we used back in Chapters 1 and 2 to say that the total work required to remove all of the dipoles far from



Figure 163

each other equals one half of the sum of all the individual works. The factor of 1/2 gets rid of the double counting in the energy. From Fig. 163 you can show that the total work required is 77.67, in units of $\mu_0 m^2/4\pi s^3$.

11.22. Potential momentum

We quickly see that $\nabla \times \mathbf{A}$ gives the desired **B** field in the negative z direction because

$$\nabla \times \mathbf{A} = \frac{B}{2} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix} = -B\hat{\mathbf{z}}.$$
 (709)

If B = 0 and $v = v_0$ initially, then **L** is initially $Mv_0r\hat{\mathbf{z}}$. When the field is turned on, Eq. (11.37) gives the increase in the speed as $\Delta v = qrB/2M$. So the change in $\mathbf{r} \times (M\mathbf{v})$ is $M \Delta v r\hat{\mathbf{z}} = (qr^2B/2)\hat{\mathbf{z}}$. But the change in $\mathbf{r} \times (q\mathbf{A})$ is

$$\frac{qB}{2} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & y & 0 \\ y & -x & 0 \end{vmatrix} = -\frac{qB}{2} (x^2 + y^2) \hat{\mathbf{z}} = -\frac{qBr^2}{2} \hat{\mathbf{z}}.$$
(710)

So the total change in $\mathbf{r} \times (M\mathbf{v} + q\mathbf{A})$ is zero, as desired.

11.23. Energy of a dipole configuration

If you consider the little loops of current that make up the dipoles, you can quickly verify that neither of the dipoles experiences a force along the dotted line (which we'll call the x axis). Therefore, no work is done in bringing the dipoles toward each other, along the x axis, to the configuration in Fig. 11.39(b). Alternatively, if we look at, say, \mathbf{m}_1 , then Eq. (11.23) tells us that the force is $\nabla(\mathbf{m} \cdot \mathbf{B}) = \nabla(m_x B_x) = m_x \nabla B_x$. But ∇B_x has no x component because the B_x due to \mathbf{m}_2 is identically zero along the x axis. ∇B_x does have a nonzero y component (where y is the direction of \mathbf{m}_2), because $\partial B_x / \partial y \neq 0$. So there is a force in the y direction. But this force is irrelevant in finding the work done in moving \mathbf{m}_1 along the x axis. Alternatively again, there is no need to even mention forces. The energy of a dipole is $\mathbf{m} \cdot \mathbf{B}$, so if \mathbf{m} is always perpendicular to \mathbf{B} , as it is in Fig. 11.39(b), then the energy is always zero. So no work is done.

The work required to rotate a dipole is the integral of the torque with respect to angle. If θ is defined relative to the x axis as in Fig. 11.39(a), then the magnitude of the torque on \mathbf{m}_1 is $N = |\mathbf{m}_1 \times \mathbf{B}_2| = m_1 B_2 \cos \theta$, because \mathbf{B}_2 is perpendicular to the x axis. (We could try to keep track of the signs, but it's easier to just work with the magnitudes and then put in the correct sign by hand at the end.) The integral of this from 0 to θ_1 is $m_1 B_2 \sin \theta_1$. From Eq. (11.15), B_2 equals $\mu_0 m_2/4\pi r^3$. So the work done by the external torque agency is $\mu_0 m_1 m_2 \sin \theta_1/4\pi r^3$. The sign is correct because the work is positive; we are making \mathbf{m}_1 be more antiparallel to \mathbf{B}_2 .

Now let's rotate \mathbf{m}_2 through an angle of $\pi/2 - \theta_2$ to its final position. We can treat \mathbf{m}_1 as the sum of two separate dipoles: $\hat{\mathbf{x}}m_1 \cos \theta_1$ and $\hat{\mathbf{y}}m_1 \sin \theta_1$. At the position of \mathbf{m}_2 , the former produces a field $B_x = \mu_0(m_1 \cos \theta_1)/2\pi r^3$, and the latter produces a field $B_y = -\mu_0(m_1 \sin \theta_1)/4\pi r^3$.

If α is the variable angle of \mathbf{m}_2 with respect to the y axis, then the torque associated with B_x involves the factor $\cos \alpha$. The integral of this gives a factor of $\sin \alpha$. But since α starts at zero and ends at $\pi/2 - \theta_2$, this factor becomes $\cos \theta_2$. So the work done is $-m_2(\mu_0 m_1 \cos \theta_1/2\pi r^3) \cos \theta_2$. We have put in the minus sign because the required work is negative; we are making \mathbf{m}_2 be more parallel to $B_x \hat{\mathbf{x}}$. Similarly, with the B_y field, if α is defined in the same way, the torque involves the factor $\sin \alpha$. The integral of this gives a factor of $-\cos \alpha$. But since α starts at zero and ends at $\pi/2 - \theta_2$, this yields a factor of $(1 - \sin \theta_2)$ So the work done is $m_2(\mu_0 m_1 \sin \theta_1/4\pi r^3)(-1 + \sin \theta_2)$. We have put in a minus sign because the required work is negative; we are making \mathbf{m}_2 be more parallel to the negative $B_y \hat{\mathbf{y}}$ field.

The total required work done (that is, the potential energy of the system) is the sum of the above three results:

$$W = \frac{\mu_0 m_1 m_2}{4\pi r^3} \left[\sin \theta_1 - 2\cos \theta_1 \cos \theta_2 + (-\sin \theta_1 + \sin \theta_1 \sin \theta_2) \right]$$
$$= \frac{\mu_0 m_1 m_2}{4\pi r^3} \left(-2\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \right), \tag{711}$$

as desired. If $\theta_1 = 0$ and $\theta_2 = 90^\circ$, then we have the configuration in Fig. 11.39(b), and W is correctly zero. Note that if $\theta_1 = \theta_2 \equiv \theta$, then the potential energy is positive if $\tan \theta > \sqrt{2}$ and negative if $\tan \theta < \sqrt{2}$. This cutoff angle is the angle for which the field of one dipole is perpendicular to the other dipole at its location. This makes sense, because if the dipoles are bought together in this configuration, then $\mathbf{m} \cdot \mathbf{B}$ is always zero. So the energy is always zero, and no work is required.

11.24. Octahedron energy

The connecting lines between the 15 pairs of dipoles in the octahedron fall into four groups:

- 8 lines are tilted at 45° with respect to the dipoles; these are the lines connected to either of the two vertices on the z axis. The associated angles θ_1 and θ_2 in Exercise 11.23 are both equal to 45° . And the distance r equals b.
- 4 lines form a square in the xy plane. The θ 's are 90°, and r = b.
- 2 lines are diagonals of the square in the xy plane. The θ 's are 90°, and $r = \sqrt{2}b$.
- 1 line is vertical along the z axis. The θ 's are 0° , and $r = \sqrt{2}b$.

Using the $U = (\mu_0 m_1 m_2 / 4\pi r^3)(\sin \theta_1 \sin \theta_2 - 2\cos \theta_1 \cos \theta_2)$ result from Exercise 11.23, the potential energy is

$$U = \frac{\mu_0 m^2}{4\pi b^3} \left(8 \left(\frac{1}{2} - 2 \cdot \frac{1}{2} \right) + 4 \left(1 - 2 \cdot 0 \right) + 2 \frac{1}{2^{3/2}} \left(1 - 2 \cdot 0 \right) + 1 \frac{1}{2^{3/2}} \left(0 - 2 \cdot 1 \right) \right)$$

= 0. (712)

REMARK: It isn't obvious why the potential energy should be zero. However, as you can check, it is also zero in the analogous cases of a tetrahedron, a cube with the dipoles aligned parallel to an edge, and a cube with the dipoles aligned parallel to a long diagonal (this case involves some tricky angles). So it is reasonable to conjecture that the potential energy is zero in the case of any platonic solid, for any (common) orientation of the dipoles. Unfortunately, I can't think of a general proof. Note that our setup with magnetic dipoles is equivalent to one with electric dipoles, because the forces and torques take the same form in electric and magnetic dipoles (since there are no external currents involved; see the discussion following Eq. (11.24)). An extreme case is the limit of an infinite number of infinitesimal electric dipoles lying on the surface of a sphere (so we effectively have two opposite shells of charge near each other). You can show that the potential energy of this system is zero. There must be a general proof for the platonic solids, and perhaps other configurations with sufficient symmetry...

11.25. Rotating a bacterium

The dipole moment of one of the crystals is

$$m = M_0 V = (4.8 \cdot 10^5 \,\mathrm{J/Tm^3})(5 \cdot 10^{-8} \,\mathrm{m})^3 = 6 \cdot 10^{-17} \,\mathrm{J/T}.$$
 (713)

From the discussion in Section 11.6, the work required to rotate the dipole by 90° is mB (assuming that it starts aligned with the field). If we have 10 crystals, and if we take the earth's magnetic field to be 0.5 gauss, then the required work is

$$W = 10mB = 10(6 \cdot 10^{-17} \text{ J/T})(5 \cdot 10^{-5} \text{ T}) = 3 \cdot 10^{-20} \text{ J}.$$
 (714)

At room temperature, kT equals $4 \cdot 10^{-21}$ J, so W is roughly 10 times kT. A bacterium therefore maintains close alignment with the earth's field; the thermal fluctuations have only a small effect.

11.26. Electric vs. magnetic dipole moments

The magnetic and electric forces behave like qvB and qE. From the expressions for the dipole fields in Eq. (10.18) and Eq. (11.15), these forces will be in the ratio of $v(\mu_0 m)$ to p/ϵ_0 . (Since we're doing things roughly, we'll ignore any factors of order 1, including trig factors.) Therefore (using $\epsilon_0\mu_0 = 1/c^2$),

$$\frac{F_{\rm m}}{F_{\rm e}} = \frac{v\epsilon_0\mu_0m}{p} = \frac{(m/c)}{p} \cdot \frac{v}{c} \,. \tag{715}$$

With the given values of p, m, and v, this becomes

$$\frac{F_{\rm m}}{F_{\rm e}} = \frac{(10^{-23}\,{\rm A}\,{\rm m}^2)/(3\cdot10^8\,{\rm m/s})}{10^{-30}\,{\rm C}\,{\rm m}} \cdot \frac{1}{100} \approx 3\cdot10^{-4}.$$
(716)

Using 10^{-29} C m for p would make the result even smaller. And even if $v \approx c$, the ratio will still be small.

11.27. Diamagnetic susceptibility of water

The magnetic susceptibility is given by $\mathbf{M} = \chi_m \mathbf{B}/\mu_0$. (The $\mathbf{M} = \chi_m \mathbf{H}$ definition would give essentially the same result, because χ_m will turn out to be very small; see Exercise 11.38.) The magnetic dipole moment of a given volume V is $\mathbf{m} = \mathbf{M}V = \chi_m \mathbf{B}V/\mu_0$. The force on a dipole is therefore

$$F = m \frac{\partial B_z}{\partial z} = \frac{\chi_m B_z V}{\mu_0} \frac{\partial B_z}{\partial z} \,. \tag{717}$$

From Table 11.1 we have (being careful with the signs; we'll take upward to be positive for all quantities) $B_z = 1.8$ tesla, $\partial B_z / \partial z = -17$ tesla/m, and F = 0.22 newtons. The volume taken up by 1 kilogram of water (which is what the data in Table 11.1 are given for, even though an actual sample would of course be much smaller) is 10^{-3} m³, so Eq. (717) gives

$$\chi_m = \frac{\mu_0 F}{B_z V(\partial B_z / \partial z)} = \frac{\left(4\pi \cdot 10^{-7} \,\frac{\text{kg m}}{\text{C}^2}\right)(0.22 \,\text{N})}{(1.8 \,\text{T})(10^{-3} \,\text{m}^3)(-17 \,\text{T/m})} = -9.0 \cdot 10^{-6}.$$
 (718)

11.28. Paramagnetic susceptibility of water

(a) We have $\chi = \mu_0 N m^2 / kT$. There are about $6 \cdot 10^{23}$ nuclei in a gram of anything, because the mass of a nucleon is about $1.67 \cdot 10^{-24}$ g, while the mass of an electron is negligible in comparison. So a cubic meter of water, which has a mass of 10^6 grams, contains $6 \cdot 10^{29}$ protons. In water, 2 out of 18 nuclei are hydrogen protons, so the number of protons per cubic meter is $N = (2/18)(6 \cdot 10^{29} \text{ m}^{-3}) =$ $6.7 \cdot 10^{28} \text{ m}^{-3}$.

The proton magnetic moment is 1/700 of the electron magnetic moment, or $(9.3 \cdot 10^{-24} \text{ A m}^2)/700 = 1.3 \cdot 10^{-26} \text{ A m}^2$. At room temperature, $kT = 4 \cdot 10^{-21} \text{ J}$. So

$$\chi = \frac{\mu_0 N m^2}{kT} = \frac{(4\pi \cdot 10^{-7} \,\frac{\text{kg m}}{\text{C}^2})(6.7 \cdot 10^{28} \,\text{m}^{-3})(1.3 \cdot 10^{-26} \,\text{A} \,\text{m}^2)^2}{4 \cdot 10^{-21} \,\text{J}} = 3.6 \cdot 10^{-9}.$$
(719)

(b) The magnetization is given by $M = \chi B/\mu_0$. (The $M = \chi_m H$ definition would give essentially the same result, because the above χ is very small; see Exercise 11.38.) The magnetic moment in a volume V is m = MV. $V = 10^{-3} \text{ m}^3$ here, so

$$m = \frac{\chi VB}{\mu_0} = \frac{(3.6 \cdot 10^{-9})(10^{-3} \,\mathrm{m}^3)(1.5 \,\mathrm{T})}{4\pi \cdot 10^{-7} \,\mathrm{kg} \,\mathrm{m/C^2}} = 4.3 \cdot 10^{-6} \,\mathrm{A} \,\mathrm{m^2}.$$
(720)

You should check that these units work out.

(c) If the flask were cubical, the cross section would be 10 cm by 10 cm, or 100 cm^2 . So let's say the area is $a = 50 \text{ cm}^2$. The magnetic moment is m = Ia, so

$$I = \frac{m}{a} = \frac{4.3 \cdot 10^{-6} \text{ A m}^2}{5 \cdot 10^{-3} \text{ m}^2} \approx 9 \cdot 10^{-4} \text{ A},$$
(721)

or 900 microamps.

11.29. Work on a paramagnetic material

If a dipole **m** is located on the axis of a solenoid, parallel to **B**, then the magnetic force on it is $F = m \partial B / \partial z$. In terms of the specific susceptibility χ , the magnetic moment of a material of mass \tilde{m} (there are only so many ways to write the letter "m") is $\mathbf{m} = \chi \mathbf{B} \tilde{m} / \mu_0$. (You can show that the units of the specific susceptibility χ are $m^3/\text{kg.}$) The force on the mass \tilde{m} is then $F = (\chi B \tilde{m} / \mu_0) \partial B / \partial z$. The work done against this force is

$$W = -\int F \, dz = -\frac{\chi \tilde{m}}{\mu_0} \int B \frac{\partial B}{\partial z} \, dz = -\frac{\chi \tilde{m}}{2\mu_0} \int d(B^2). \tag{722}$$

If the material is moved along the axis from a point where the field is B_1 to a point where the field is B_2 , the work done is $(\chi \tilde{m}/2\mu_0)(B_1^2-B_2^2)$. If B_2 is negligible compared with B_1 , then $W = \chi B_1^2 \tilde{m}/2\mu_0$. So the work per kilogram is $\chi B_1^2/2\mu_0$, as desired.

From above, we have $\chi = \mu_0 F/(\tilde{m}B \partial B/\partial z)$, so we can eliminate χ from the result for W and write

$$W = \frac{\tilde{m}}{2\mu_0} \chi B^2 = \frac{\tilde{m}}{2\mu_0} \left(\frac{\mu_0 F}{\tilde{m} B \partial B / \partial z} \right) B^2 = \frac{FB}{2 \partial B / \partial z} \,. \tag{723}$$

From Table 11.1, the data for 1 kg of liquid oxygen are F = 75 N, B = 1.8 T, and $\partial B/\partial z = 17$ T/m. So the work for 1 kg is

$$W = \frac{(75\,\mathrm{N})(1.8\,\mathrm{T})}{2(17\,\mathrm{T/m})} = 4\,\mathrm{J}.$$
(724)

The work for 1 gram is 1/1000 of this, or $4 \cdot 10^{-3}$ J.

11.30. Greatest force in a solenoid

If a dipole **m** is located on the axis of a solenoid, parallel to **B**, then the force on it is $F = m \partial B/\partial z$. Per unit volume of the sample, we also have $\mathbf{m} = \chi \mathbf{B}/\mu_0$. The force per unit volume is therefore $F = (\chi/\mu_0) B \partial B/\partial z$. So our goal is to find B and $\partial B/\partial z$ and then maximize their product by setting the derivative equal to zero:

$$0 = \frac{\partial F}{\partial z} \propto \frac{\partial}{\partial z} \left(B \frac{\partial B}{\partial z} \right) = B \frac{\partial^2 B}{\partial z^2} + \left(\frac{\partial B}{\partial z} \right)^2.$$
(725)

Equation (6.56) gives the field in the interior of a finite solenoid as $B = (\mu_0 n I/2)(\cos \theta_1 - \cos \theta_2)$, where the angles are shown in Fig. 6.16. In the present case we have $\theta_2 = \pi$ and $\cos \theta_1 = z/\sqrt{z^2 + r_0^2}$, where z is the distance from the end (with positive z being inside the solenoid). So Eq. (6.56) gives

$$B = \frac{\mu_0 nI}{2} \left(1 + \frac{z}{\sqrt{z^2 + r_0^2}} \right) \implies \frac{\partial B}{\partial z} = \frac{\mu_0 nI}{2} \frac{r_0^2}{(z^2 + r_0^2)^{3/2}}$$
$$\implies \frac{\partial^2 B}{\partial z^2} = \frac{\mu_0 nI}{2} \frac{-3r_0^2 z}{(z^2 + r_0^2)^{5/2}}.$$
 (726)

Equation (725) then becomes

$$0 = \left(1 + \frac{z}{\sqrt{z^2 + r_0^2}}\right) \left(\frac{-3r_0^2 z}{(z^2 + r_0^2)^{5/2}}\right) + \left(\frac{r_0^2}{(z^2 + r_0^2)^{3/2}}\right)^2$$

$$\implies 0 = \left(\sqrt{z^2 + r_0^2} + z\right) (-3z) + r_0^2$$

$$\Rightarrow 3z\sqrt{z^2 + r_0^2} = r_0^2 - 3z^2.$$
(727)

Squaring yields

==

$$9z^{4} + 9z^{2}r_{0}^{2} = r_{0}^{4} - 6r_{0}^{2}z^{2} + 9z^{4} \implies 15z^{2} = r_{0}^{2} \implies z = r_{0}/\sqrt{15}.$$
 (728)

We have chosen the positive root because the negative root was introduced in the squaring operation and isn't a solution to Eq. (727). The desired point therefore lies slightly *inside* the end of the solenoid.

11.31. Boundary conditions for B

From Problem 11.8(a) the external field is the field of a dipole with strength $m = (4\pi R^3/3)M$. So from Eq. (11.15) the components are

$$B_r^{\text{out}} = \frac{\mu_0 m \cos \theta}{2\pi R^3} = \frac{2\mu_0 M \cos \theta}{3} ,$$

$$B_\theta^{\text{out}} = \frac{\mu_0 m \sin \theta}{4\pi R^3} = \frac{\mu_0 M \sin \theta}{3} .$$
(729)
From Problem 11.8(b) the internal field is $B = (2/3)\mu_0 M$ upward (assuming **M** points upward). The radial component of this is $B_r^{\text{in}} = (2/3)\mu_0 M \cos\theta$, and the tangential component is $B_{\theta}^{\text{in}} = -(2/3)\mu_0 M \sin\theta$. The negative sign comes from the fact that the field points toward the top of the sphere, which is the direction of decreasing θ .

As predicted, B_r is continuous across the surface, and B_{θ} has a discontinuity of $\mu_0 M \sin \theta$. This gives us the desired result of $\mu_0 \mathcal{J}_{\theta}$ provided that the surface current density \mathcal{J}_{θ} equals $M \sin \theta$. And it does indeed, from the reasoning in the example at the end of Section 11.8; the component of **M** parallel to the surface is $M_{\parallel} = M \sin \theta$.

11.32. B at the center of a solid rotating sphere

From Problem 11.7 we know that the magnetic field due to a spinning *shell* with radius r and uniform surface charge density σ is the same as the magnetic field due to a sphere with uniform magnetization $M_r = \sigma \omega r$ (both inside and outside the shell). And then from Problem 11.8 we know that the internal field of a magnetized sphere is $B = 2\mu_0 M_r/3 = 2\mu_0 \sigma \omega r/3$. (Alternatively, we could have just invoked the result from Problem 6.11 to obtain this field. But the idea here was to parallel the solution to Problem 11.9.)

We can consider the solid spinning sphere to be the superposition of many spinning shells with uniform surface charge density $\sigma = \rho dr$. The center of the sphere is inside all of the shells, so we can use the above form of B for every shell. The total field at the center is therefore

$$B = \int_0^R \frac{2\mu_0(\rho \, dr)\omega r}{3} = \frac{\mu_0 \rho \omega R^2}{3} \,. \tag{730}$$

In terms of the total charge $Q = (4\pi R^3/3)\rho$, this result can be written as $B = \mu_0 \omega Q/4\pi R$. This field is 5/2 as large as the field at the north pole; see Problem 11.9.

11.33. Spheres of frozen magnetization

(a) From Problem 11.8(a), we know that the external magnetic field of a uniformly magnetized sphere with radius R is the same as the field of a magnetic dipole **m** located at the center, with magnitude $m = (4\pi R^3/3)M$. From Eq. (11.15) the field at points on the axis of the dipole is radial and has magnitude $B_r = \mu_0 m/2\pi R^3$. Writing m in terms of M gives

$$B_r = \frac{2}{3}\mu_0 M = \frac{2}{3} \left(4\pi \cdot 10^{-7} \,\frac{\text{kg m}}{\text{C}^2} \right) \left(7.5 \cdot 10^5 \frac{\text{J}}{\text{T m}^3} \right) = 0.628 \,\text{T}.$$
 (731)

You should check that the units work out correctly. Note that this result is independent of R.

- (b) From Eq. (11.15) the field at points on the equator of the sphere is tangential and has magnitude $B_{\theta} = \mu_0 m / 4\pi R^3$. This is half of the above B_r , which yields 0.314 T.
- (c) The force acting on each of the two uniformly polarized spheres must be the same as if the other sphere were replaced by a point dipole m at its center, because that leaves the external field unchanged. So we need to find the force between two point dipoles separated by 2R. From Eq. (11.20) this force is $F = m \partial B_z / \partial z$, where $\pm \hat{z}$ are the directions of the dipoles. Since $B_z = \mu_0 m / 2\pi z^3$, we have $\partial B_z / \partial z = 3\mu_0 m / 2\pi z^4$ (in magnitude). So the attractive force between

the dipoles is

$$F = \frac{3\mu_0 m^2}{2\pi (2R)^4} = \frac{3\mu_0 (4\pi R^3 M/3)^2}{32\pi R^4} = \frac{\pi \mu_0 R^2 M^2}{6}$$
$$= \frac{\pi}{6} \left(4\pi \cdot 10^{-7} \frac{\text{kg m}}{\text{C}^2} \right) (0.01 \text{ m})^2 \left(7.5 \cdot 10^5 \frac{\text{J}}{\text{T} \text{ m}^3} \right)^2$$
$$= 37 \text{ N} \approx 8.3 \text{ pounds.}$$
(732)

The units are easier to see if you write the units of M as C/(ms).

11.34. Muon deflection

Since $U = \gamma mc^2$ (we'll use U for the energy) we have

$$\gamma = \frac{U}{mc^2} = \frac{10^{10} \text{ eV}}{2 \cdot 10^8 \text{ eV}} = 50.$$
(733)

So $\beta \equiv v/c$ is essentially equal to 1. The magnitude of the momentum is

$$p = \gamma m v = \gamma m \beta c = \beta (\gamma m c^2) / c \approx U / c.$$
(734)

The magnetization is

$$M = Nm = (1.5 \cdot 10^{29} \,\mathrm{m}^{-3})(9.3 \cdot 10^{-24} \,\mathrm{J/T}) = 1.4 \cdot 10^{6} \,\mathrm{J/(T \,m^{3})}.$$
(735)

From Eq. (11.55) the bound surface current on each face of the plate is $\mathcal{J} = M$. But we know from Section 6.6 that the magnetic field between two such sheets (with the surface currents pointing in opposite directions) is $\mu_0 \mathcal{J}$. So the magnetic field inside the plate equals

$$B = \mu_0 \mathcal{J} = \mu_0 M = (4\pi \cdot 10^{-7})(1.4 \cdot 10^6 \,\text{J/m}^3\text{T}) = 1.76 \,\text{T}.$$
 (736)

Equivalently, since there are no free currents, we have $\mathbf{H} = 0$, which means $\mathbf{B} = \mu_0 \mathbf{M}$. The transverse force is $F_{\perp} = evB$, so the transverse momentum gained is $p_{\perp} = evBt = eB(vt) = eBs$, where s is the thickness of the plate. The angle of deflection is therefore (using the definition of an eV)

$$\theta = \frac{p_{\perp}}{p} = \frac{eBs}{U/c} = \frac{eBsc}{U} = \frac{eBsc}{10^{10} \cdot e(1\,\mathrm{V})} = \frac{Bsc}{10^{10}(1\,\mathrm{V})}$$
$$= \frac{(1.76\,\mathrm{T})(0.2\,\mathrm{m})(3\cdot10^8\,\mathrm{m/s})}{10^{10}(1\,\mathrm{V})} = 0.011, \tag{737}$$

which is a shade more than 0.6°. As long as $\beta \approx 1$, the angle of deflection is inversely proportional to the energy U. More generally, it is inversely proportional to βU (assuming the angle is small).

11.35. Volume integral of near field

The product of the near-region volume and the field strength in that volume is proportional to $s^3 \cdot Q/s^2 = sQ = p$. But we are assuming that p is held constant. Therefore $\int \mathbf{E} \, dv$ over the near region remains constant as s shrinks down. See Problem 10.5 for some quantitative aspects of this setup.

In the case of a current loop with magnetic dipole moment $m = I\pi r^2$, the nearregion volume behaves like r^3 , and the near field behaves like I/r (using the form of the field from a wire). So the product of the volume and the field strength is $r^3 \cdot I/r = Ir^2 = m/\pi$. And we are assuming that *m* is held constant. Therefore $\int \mathbf{B} dv$ over the near region remains constant as *r* shrinks down.

The average of, say, **E** over a macroscopic volume V equals $(\int \mathbf{E} dv)/V$. There will be finite contributions to the numerator from the near fields of all the dipoles in the volume, no matter how small they are. So the near fields will necessarily contribute to the average of **E**. Likewise for **B**.

11.36. Equilibrium orientations

First consider the more general case of the isosceles triangle in Fig. 164. If the dipoles \mathbf{m}_1 and \mathbf{m}_2 make the same angle θ shown, then the sum of their fields at the top vertex is horizontal. This follows from the form of the dipole field, but it also follows from a symmetry argument: Start with just \mathbf{m}_1 and \mathbf{B}_1 , and then rotate the setup 180° around the vertical axis to produce the \mathbf{m}'_1 and \mathbf{B}'_1 vectors shown in Fig. 165. Then negate \mathbf{m}'_1 to obtain \mathbf{m}_2 and \mathbf{B}_2 . The \mathbf{B}_1 and \mathbf{B}_2 fields make the same angles above and below the horizontal, so the total \mathbf{B} field points horizontally.



Figure 165

In the case of an equilateral triangle, we see that if the dipoles are arranged as shown in Fig. 166(a), then each dipole will point in the direction of the field produced by the other two. Since a dipole has the least energy when it points in the direction of the field it lies in, this is the stable equilibrium configuration that the dipoles will assume. Similarly in the case of other N-gons, we obtain the situations shown in Fig. 166(b,c). The same symmetry argument that we used above can be used to show that each dipole points in the direction of the field due to all the other dipoles.

11.37. B inside a magnetized sphere

Since $\mathbf{H} \equiv \mathbf{B}/\mu_0 - \mathbf{M}$, we have $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. The two magnetic equations can therefore be written as

$$\nabla \cdot (\mathbf{H} + \mathbf{M}) = 0$$
 and $\nabla \times \mathbf{H} = 0.$ (738)

These take exactly the same form as the electric equations,

$$\nabla \cdot (\mathbf{E} + \mathbf{P}/\epsilon_0) = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = 0,$$
 (739)

with the correspondence being

$$\mathbf{H} \longleftrightarrow \mathbf{E}$$
 and $\mathbf{M} \longleftrightarrow \mathbf{P}/\epsilon_0$. (740)

Now, the solution for \mathbf{E} in the polarized sphere is completely determined by the polarization \mathbf{P} , along with the two equations in Eq. (739). Likewise, the solution









Figure 166

for **B** in the magnetized sphere is completely determined by the magnetization **M**, along with the two equations in Eq. (738). So to obtain the solution in the latter case, we can simply replace the letters in the solution in the former case, via Eq. (740). The solution $\mathbf{E} = -\mathbf{P}/3\epsilon_0$ for the polarized sphere therefore becomes the solution $\mathbf{H} = -\mathbf{M}/3$ for the magnetized sphere. And since $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, the desired **B** field inside the sphere is given by

$$\mathbf{B} = \mu_0 \left(-\frac{\mathbf{M}}{3} + \mathbf{M} \right) = \frac{2\mu_0 \mathbf{M}}{3}, \qquad (741)$$

in agreement with the result from Problem 11.8.

11.38. Two susceptibilities

From Eq. (11.52) we have $\mathbf{M} = \chi'_m \mathbf{B}/\mu_0$. And from Eq. (11.72) we have $\mathbf{M} = \chi_m \mathbf{H}$. Equating these two expressions for \mathbf{M} , and using $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, gives

$$\chi'_{m}\mathbf{B}/\mu_{0} = \chi_{m}(\mathbf{B}/\mu_{0} - \mathbf{M})$$
$$= \chi_{m}(\mathbf{B}/\mu_{0} - \chi'_{m}\mathbf{B}/\mu_{0}).$$
(742)

Canceling the \mathbf{B}/μ_0 gives

$$\chi'_m = \chi_m (1 - \chi'_m) \implies \chi_m = \chi'_m / (1 - \chi'_m),$$
(743)

as desired. In cases where $\chi'_m \ll 1$, the denominator here is essentially equal to 1, so we have $\chi_m \approx \chi'_m$.

11.39. Magnetic moment of a rock

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(a) Ignore the rock for a moment and imagine that there is a current I in the coils. Using the expression in Eq. (6.53) for the field due to a single ring, we find that the field at the location of the rock due to the current in the coils is

$$B = 2(1500) \frac{\mu_0 I r^2}{2(r^2 + z^2)^{3/2}}$$

$$\Rightarrow \frac{B}{I} = \frac{1500(4\pi \cdot 10^{-7} \,\mathrm{kg} \,\mathrm{m/C^2})(0.06 \,\mathrm{m})^2}{((0.06 \,\mathrm{m})^2 + (0.03 \,\mathrm{m})^2)^{3/2}} = 0.0225 \,\mathrm{T/A}.$$
(744)

Now let's reintroduce the rock. Its angular frequency is $\omega = 2\pi (1740/(60 \text{ s})) = 182 \text{ s}^{-1}$. The vertical component of its dipole moment produces zero net flux through the coils. But the horizontal component does, and this component oscillates sinusoidally. So we effectively have the setup presented in Exercise 11.19, with \mathbf{m}_0 pointing along the axis of the coils. So \mathbf{m}_0 is parallel to \mathbf{B}_1 in the $\mathcal{E}_1 = (\omega/I_1)\mathbf{B}_1 \cdot \mathbf{m}_0$ result from that exercise. The magnetic moment of the rock is therefore given by

$$m_0 = \frac{\mathcal{E}}{\omega} \cdot \frac{1}{B/I} = \frac{10^{-3} \,\mathrm{V}}{182 \,\mathrm{s}^{-1}} \cdot \frac{1}{0.0225 \,\mathrm{T/A}} = 2.4 \cdot 10^{-4} \,\mathrm{J/T}.$$
 (745)

(b) The electron magnetic moment is about 10^{-23} J/T, so the above magnetic moment corresponds to $(2.4 \cdot 10^{-4})/10^{-23} = 2.4 \cdot 10^{19}$ electrons. Each iron atom can contribute 2 electron spins, so we need $1.2 \cdot 10^{19}$ atoms. There are 56 nucleons in an iron atom, so each atom has a mass of about $9 \cdot 10^{-26}$ kg. The minimum mass is therefore $(1.2 \cdot 10^{19})(9 \cdot 10^{-26}$ kg) $\approx 10^{-6}$ kg, or 0.001 g.

11.40. Deflecting high-energy particles

(a) From the *B*-*H* curve in Fig. 11.41(d), we find that B = 1.6 T corresponds to $H \approx 5000 \text{ A/m}$. In the 20 cm gap, $H = (1.6 \text{ T})/\mu_0 = 1.3 \cdot 10^6 \text{ A/m}$. As a rough estimate, the curve *bcdea* has a length of about 300 cm. So

$$\int \mathbf{H} \cdot d\mathbf{l} = \int_{\text{gap}} \mathbf{H} \cdot d\mathbf{l} + \int_{\text{iron}} \mathbf{H} \cdot d\mathbf{l}$$

= $(1.3 \cdot 10^6 \text{ A/m})(0.2 \text{ m}) + (5000 \text{ A/m})(3 \text{ m})$
= $(260,000 + 15,000) \text{ A}$
= $275,000 \text{ A}.$ (746)

We see that, as mentioned in the statement of the exercise, the integral of **H** is dominated by the gap contribution. From Eq. (11.70) this integral equals NI, where N is the number of turns in the two coils, which is twice the number of turns in each coil. So 275,000 A is the desired number of ampere turns (a fancy name for the total current passing through the loop).

(b) Each coil is roughly rectangular with sides of 300 and 100 cm. So the length of one full turn is about L = 8 m. If the total cross section of copper is 1500 cm^2 (in both coils), and if we have N turns (in both coils), then the cross section of each wire is $(0.15 \text{ m}^2)/N$. The total resistance in both coils is then

$$R = \frac{\rho N L}{A} = \frac{\rho N(8 \,\mathrm{m})}{(0.15 \,\mathrm{m}^2)/N} = N^2 \rho(53 \,\mathrm{m}^{-1}).$$
(747)

So the power is (using the above result, NI = 275,000 A)

$$P = I^2 R = I^2 N^2 \rho(53 \,\mathrm{m}^{-1}) = (275,000 \,\mathrm{A})^2 (2 \cdot 10^{-8} \,\mathrm{ohm} \cdot \mathrm{m})(53 \,\mathrm{m}^{-1}) = 8 \cdot 10^4 \,\mathrm{J/s}.$$
(748)

This is independent of N and I individually, because the product NI takes on a given value.

(c) Another expression for the power is P = IV. If V = 400 V, then

$$I = \frac{P}{V} = \frac{8 \cdot 10^4 \,\mathrm{J/s}}{400 \,\mathrm{V}} = 200 \,\mathrm{A}.$$
 (749)

Therefore,

$$N = \frac{NI}{I} = \frac{275,000 \,\mathrm{A}}{200 \,\mathrm{A}} \approx 1400,\tag{750}$$

or 700 turns per coil. The cross-sectional area of the wire is $(1500 \text{ cm}^2)/1400 \approx 1.1 \text{ cm}^2$. If the voltage source were instead, say, 800 V, then the current would be 100 A, so the number of turns would be 2800, and the cross-sectional area would be roughly 0.5 cm^2 .

CHAPTER 11. MAGNETIC FIELDS IN MATTER

Appendix F

F.1. Divergence using two systems

- (a) In Cartesian coordinates we quickly find $\nabla \cdot \mathbf{A} = 2$. In cylindrical coordinates we have $\nabla \cdot \mathbf{A} = (1/r) \partial (rA_r) / \partial r = (1/r) \partial (r^2) / \partial r = 2$, as desired.
- (b) In Cartesian coordinates we quickly find $\nabla \cdot \mathbf{A} = 3$. In cylindrical coordinates we can write \mathbf{A} as

$$\mathbf{A} = x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}}$$

= $r\cos\theta(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta) + 2r\sin\theta(\hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta)$
= $\hat{\mathbf{r}}r(\cos^2\theta + 2\sin^2\theta) + \hat{\boldsymbol{\theta}}r\sin\theta\cos\theta.$ (751)

You can check that this agrees with what you would obtain by projecting **A** onto the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ unit vectors. Taking the divergence of **A** gives

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial (rA_r)}{\partial r} + \frac{1}{r} \frac{\partial A_{\theta}}{\partial \theta}$$

= $\frac{1}{r} \frac{\partial (r \cdot r(\cos^2 \theta + 2\sin^2 \theta))}{\partial r} + \frac{1}{r} \frac{\partial (r\sin \theta \cos \theta)}{\partial \theta}$
= $(2\cos^2 \theta + 4\sin^2 \theta) + (\cos^2 \theta - \sin^2 \theta)$
= $3\cos^2 \theta + 3\sin^2 \theta = 3,$ (752)

as desired.

F.2. Cylindrical divergence

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In Fig. 167, the first-order change in a function f in going from point A to point B is $\Delta f = (\partial f / \partial x) \Delta x$. But we can also imagine going from A to B in two steps along the radial and tangential segments shown, via point C. So we can also write Δf as $(\partial f / \partial r) \Delta r + (\partial f / \partial \theta) \Delta \theta$. Therefore,

$$\Delta f = \frac{\partial f}{\partial r} \Delta r + \frac{1}{r} \frac{\partial f}{\partial \theta} (r \Delta \theta)$$

$$= \frac{\partial f}{\partial r} (\Delta x \cos \theta) + \frac{1}{r} \frac{\partial f}{\partial \theta} (-\Delta x \sin \theta)$$

$$\Rightarrow \frac{\Delta f}{\Delta x} = \frac{\partial f}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial f}{\partial \theta} \sin \theta.$$
(753)

But in the limit where A and B are close together, the left-hand side is just $\partial f/\partial x$. Since this equation holds for an arbitrary function f, we can erase the f, which yields the first equation in Eq. (F.28). Similarly, breaking a vertical Δy segment into the $\Delta r = \Delta y \sin \theta$ and $r\Delta \theta = \Delta y \cos \theta$ pieces yields the second equation in Eq. (F.28).



Figure 167





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We can use the same type of reasoning to find the relation between the Cartesian and cylindrical components of a vector. In Fig. 168, A_x is the horizontal component of a vector \mathbf{A} . But we can also imagine breaking up $A_x \hat{\mathbf{x}}$ into its radial and tangential components. The horizontal component of the radial vector $A_r \hat{\mathbf{r}}$ is $A_r \cos \theta$, and the horizontal component of the tangential vector $A_{\theta} \hat{\theta}$ is $-A_{\theta} \sin \theta$ (note that A_{θ} is negative here). So we must have $A_x = A_r \cos \theta - A_{\theta} \sin \theta$, in agreement with the first equation in Eq. (F.29). Likewise, looking at the vertical component A_y gives $A_y = A_r \sin \theta + A_{\theta} \cos \theta$.

Alternatively, we can use the expressions for the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ unit vectors given in the statement of Exercise F.1.

$$\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}}$$

= $A_r (\hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta) + A_\theta (-\hat{\mathbf{x}} \sin \theta + \hat{\mathbf{y}} \cos \theta)$
= $\hat{\mathbf{x}} (A_r \cos \theta - A_\theta \sin \theta) + \hat{\mathbf{y}} (A_r \sin \theta + A_\theta \cos \theta).$ (754)

The x and y components that we read off from this equation agree with those in Eq. (F.29).

Now we must calculate $\nabla \cdot \mathbf{A} = \partial A_x / \partial x + \partial A_y / \partial y + \partial A_z / \partial z$. Using the above expressions for the partial derivatives and the components, and ignoring the z term, we have (with $c \equiv \cos \theta$ and $s \equiv \sin \theta$)

$$\nabla \cdot \mathbf{A} = \left(\cos\theta \frac{\partial}{\partial r} - \sin\theta \frac{1}{r} \frac{\partial}{\partial \theta}\right) \left(A_r \cos\theta - A_\theta \sin\theta\right) \\
+ \left(\sin\theta \frac{\partial}{\partial r} + \cos\theta \frac{1}{r} \frac{\partial}{\partial \theta}\right) \left(A_r \sin\theta + A_\theta \cos\theta\right) \\
= \left(c^2 \frac{\partial A_r}{\partial r} - cs \frac{\partial A_\theta}{\partial r}\right) + \left(-sc \frac{1}{r} \frac{\partial A_r}{\partial \theta} + s^2 \frac{1}{r} A_r + s^2 \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + sc \frac{1}{r} A_\theta\right) \\
+ \left(s^2 \frac{\partial A_r}{\partial r} + sc \frac{\partial A_\theta}{\partial r}\right) + \left(cs \frac{1}{r} \frac{\partial A_r}{\partial \theta} + c^2 \frac{1}{r} A_r + c^2 \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} - cs \frac{1}{r} A_\theta\right) \\
= \frac{\partial A_r}{\partial r} + \frac{1}{r} A_r + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta}, \qquad (755)$$

in agreement with the expression in Eq. (F.2) (with the first term expanded out). The z term is the same.

F.3. General expression for divergence

Consider a small "box" in space, analogous to the one in Fig. F.2. The faces can be grouped in three pairs, with each pair being perpendicular to a particular coordinate axis. Consider one of the faces perpendicular to $\hat{\mathbf{x}}_1$. Let the center of this face be located at (x_1, x_2, x_3) . Its area is $[f_2(x_1) dx_2][f_3(x_1) dx_3]$, where we have suppressed the x_2 and x_3 arguments of the f factors. The flux of a vector \mathbf{A} inward through this face is therefore $A_1(x_1) \cdot f_2(x_1) dx_2 \cdot f_3(x_1) dx_3$. Similarly, the flux of \mathbf{A} outward through the opposite face, whose center is located at the point $(x_1 + dx_1, x_2, x_3)$, is

$$A_1(x_1 + dx_1) \cdot f_2(x_1 + dx_1) \, dx_2 \cdot f_3(x_1 + dx_1) \, dx_3. \tag{756}$$

This flux is different due to the facts that both the value of A_1 and the area of the face at $(x_1 + dx_1, x_2, x_3)$ are different (in general) from what they are at (x_1, x_2, x_3) . The volume of the box is $f_1 dx_1 \cdot f_2 dx_2 \cdot f_3 dx_3$, so the net contribution from these two sides to the divergence is

$$=\frac{A_{1}(x_{1}+dx_{1})\cdot f_{2}(x_{1}+dx_{1})\,dx_{2}\cdot f_{3}(x_{1}+dx_{1})\,dx_{3}-A_{1}(x_{1})\cdot f_{2}(x_{1})\,dx_{2}\cdot f_{3}(x_{1})\,dx_{3}}{f_{1}\,dx_{1}\cdot f_{2}\,dx_{2}\cdot f_{3}\,dx_{3}}$$

$$=\frac{1}{f_{1}f_{2}f_{3}}\frac{A_{1}(x_{1}+dx_{1})\cdot f_{2}(x_{1}+dx_{1})\cdot f_{3}(x_{1}+dx_{1})-A_{1}(x_{1})\cdot f_{2}(x_{1})\cdot f_{3}(x_{1})}{dx_{1}}\,.$$
 (757)

By the definition of the partial derivative, this is simply equal to

$$\frac{1}{f_1 f_2 f_3} \frac{\partial (A_1 f_2 f_3)}{\partial x_1} , \qquad (758)$$

which agrees with the first term in Eq. (F.31). The other two pairs of faces work out in exactly the same manner.

In spherical coordinates, f_1, f_2, f_3 are equal to $1, r, r \sin \theta$. So the given expression becomes (with the indices 1, 2, 3 changed to r, θ, ϕ)

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2 \sin \theta} \left(\frac{\partial (r^2 \sin \theta A_r)}{\partial r} + \frac{\partial (r \sin \theta A_\theta)}{\partial \theta} + \frac{\partial (r A_\phi)}{\partial \phi} \right)$$
$$= \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}, \qquad (759)$$

which agrees with the expression in Eq. (F.3).

F.4. Laplacian using two systems

- (a) In Cartesian coordinates we quickly find $\nabla^2 f = 2 + 2 = 4$. In cylindrical coordinates we have $\nabla^2 f = (1/r) \partial (r \partial f / \partial r) / \partial r = (1/r) \partial (r \cdot 2r) / \partial r = 4$, as desired.
- (b) In Cartesian coordinates we have $\nabla^2 f = 12x^2 + 12y^2 = 12r^2$. In cylindrical coordinates we can write $x = r \sin \theta$ and $y = r \cos \theta$, so $x^4 + y^4 = r^4 (\sin^4 \theta + \cos^4 \theta)$. We need to calculate

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} \,. \tag{760}$$

The first term is quickly found to be $16r^2(\sin^4\theta + \cos^4\theta)$. The second term equals (letting $\sin\theta \to s$ and $\cos\theta \to c$)

$$r^{2}\frac{\partial(4s^{3}c - 4c^{3}s)}{\partial\theta} = r^{2}(12s^{2}c^{2} - 4s^{4} + 12c^{2}s^{2} - 4c^{4}) = 24r^{2}s^{2}c^{2} - 4r^{2}(s^{4} + c^{4}).$$
(761)

Putting it all together gives

$$\nabla^2 f = 12r^2(s^4 + c^4) + 24r^2s^2c^2 = 12(s^2 + c^2)^2 = 12r^2,$$
(762)

as desired.

F.5. "Sphere" averages in one and two dimensions

In one and two dimensions, the procedure leading up to Eq. (F.22) is essentially the same. In 2D, the area in Eq. (F.19) is the area $2\pi rz$ of a cylinder, so we have

$$\frac{df_{\text{avg},r}}{dr} = \frac{1}{2\pi rz} \int \nabla^2 f \, dV. \tag{763}$$

In the special case where the cylinder is very small, we can write the volume integral $\int \nabla^2 f \, dV$ as $(\pi r^2 z) (\nabla^2 f)_{\text{center}}$. Therefore,

$$\frac{df_{\text{avg},r}}{dr} = \frac{1}{2\pi rz} (\pi r^2 z) (\nabla^2 f)_{\text{center}} = \frac{r}{2} (\nabla^2 f)_{\text{center}}.$$
(764)

Since $(\nabla^2 f)_{\text{center}}$ is a constant, we can integrate with respect to r to obtain

$$f_{\text{avg},r} = f_{\text{center}} + \frac{r^2}{4} (\nabla^2 f)_{\text{center}} \,. \tag{765}$$

Similarly, in 1D the area in Eq. (F.19) is the area 2yz of the two faces of the slab. So we have

$$\frac{df_{\text{avg},\pm x}}{dx} = \frac{1}{2yz} \int \nabla^2 f \, dV. \tag{766}$$

In the special case where the slab is very small, we can write the volume integral $\int \nabla^2 f \, dV$ as $(2xyz)(\nabla^2 f)_{\text{center}}$, assuming that the slab extends from -x to x. Therefore,

$$\frac{df_{\text{avg},\pm x}}{dx} = \frac{1}{2yz} (2xyz) (\nabla^2 f)_{\text{center}} = x (\nabla^2 f)_{\text{center}}.$$
(767)

Integrating with respect to x gives

$$f_{\text{avg},\pm x} = f_{\text{center}} + \frac{x^2}{2} (\nabla^2 f)_{\text{center}} \,. \tag{768}$$

Note that the denominator in the second term in the results in Eqs. (768), (765), and (F.25) equals 3d, where d is the dimension of the space.

The result in Eq. (768) makes sense, because in 1D, $\nabla^2 f$ is simply $d^2 f/dx^2$, so the result is consistent with what we obtain by adding together the Taylor series,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \cdots,$$
(769)

evaluated at x and -x, and dividing by 2. The first-order terms cancel, while the zeroth- and second-order terms add, yielding Eq. (768).

F.6. Average over a cube

To second order, the Taylor series in three Cartesian coordinates, expanded around the origin, looks like

$$f(x, y, z) = f_0 + xf_x + yf_y + zf_z + xyf_{xy} + xzf_{xz} + yzf_{yz} + \frac{x^2}{2}f_{xx} + \frac{y^2}{2}f_{yy} + \frac{z^2}{2}f_{zz}.$$
(770)

The subscripts denote the coordinates that the partial derivatives are taken with respect to. All partial derivatives are evaluated at the origin. To check that this is indeed the correct expansion, we can take various partial derivatives of both sides and then evaluate both sides at the origin. For example, taking $\partial^2/\partial x \,\partial y$ of both sides and then setting (x, y, z) = (0, 0, 0) generates an equality; all terms except the f_{xy} term vanish on the right-hand side. (Remember that the partial derivatives are constants.) This procedure shows that at the origin, the right-hand side has the correct value of

the function and its first two derivatives. So it must in fact be the correct function (at least to second order).

When we take the average of the above expression over the surface of the given cube, the x, y, and z terms vanish, because for every point with a given value of x, there is a point with the value -x. For the same reason, the xy, xz, and yz terms vanish. Our task therefore reduces to finding the average value of, say, x^2 over the surface of the cube. Two of the faces have $x = \pm \ell$, so the average value of x^2 over these faces is simply ℓ^2 . For the other four faces, x ranges from $-\ell$ to ℓ , so the average of x^2 is given by the integral $(\int_{-\ell}^{\ell} x^2 dx)/2\ell$, which equals $\ell^2/3$. The average of x^2 over all six faces is therefore

$$\overline{x^2} = \frac{1}{6} \left(2 \cdot \ell^2 + 4 \cdot \frac{\ell^2}{3} \right) = \frac{5\ell^2}{9} \,. \tag{771}$$

The same result holds for the averages of y^2 and z^2 , so from Eq. (770) the average of f over the entire surface of the cube is

$$f_{\text{avg}} = f_{\text{center}} + \frac{1}{2} \frac{5\ell^2}{9} (f_{xx} + f_{yy} + f_{zz}) = f_{\text{center}} + \frac{5\ell^2}{18} (\nabla^2 f)_{\text{center}},$$
(772)

as desired. A sphere with radius ℓ lies completely inside the given cube of side 2ℓ . From Eq. (F.25), this sphere would have a 1/6 in place of the 5/18. Consistent with this, we have 5/18 > 1/6; the average value of f over the surface of the cube is larger than the average value over the surface of the sphere (assuming $(\nabla^2 f)_{\text{center}}$ is positive). Also, a sphere with radius $\sqrt{3}\ell$ lies completely outside the given cube of side 2ℓ (it touches its corers). This sphere would have a $(\sqrt{3})^2/6$ in place of the 5/18. Consistent with this, we have 3/6 > 5/18.

CHAPTER 11. MAGNETIC FIELDS IN MATTER

Appendix H

H.1. Ratio of energies

From Eq. (H.7) the power radiated is $P = e^2 a^2/6\pi\epsilon_0 c^3$. The total energy radiated is Pt. Therefore, since a = v/t, the desired ratio of energies is

$$\frac{\frac{e^2(v/t)^2}{6\pi\epsilon_0 c^3} \cdot t}{\frac{mv^2}{2}} = \frac{e^2}{3\pi\epsilon_0 mc^3 t} = \frac{4}{3} \frac{e^2}{4\pi\epsilon_0 mc^2} \frac{1}{ct} = \frac{4}{3} \frac{r_0}{ct} \,. \tag{773}$$

Note that ct is larger than vt, which is twice as large as the stopping distance vt/2. So unless the electron stops within a distance that is of order r_0 , the radiated energy will be negligible compared with the initial kinetic energy.

H.2. Simple harmonic motion

(a) If the position is given by $x = A \cos \omega t$, then the acceleration is $a(t) = d^2 x/dt^2 = -A\omega^2 \cos \omega t$. The average of $\cos^2 \omega t$ over a period is 1/2, so the average of a^2 is $A^2 \omega^4/2$. From Eq. (H.7) the average power radiated is then

$$\overline{P} = \frac{e^2 (A^2 \omega^4 / 2)}{6\pi\epsilon_0 c^3} = \frac{e^2 A^2 \omega^4}{12\pi\epsilon_0 c^3}.$$
(774)

(b) The velocity of the electron is $v(t) = -A\omega \sin \omega t$, so the initial speed is $v = A\omega$. The initial energy of the oscillator is therefore $U = mv^2/2 = mA^2\omega^2/2$. As time goes on, the amplitude will decrease. But in terms of the amplitude at any instant, U and \overline{P} are given by the above expressions. They are therefore always related by $\overline{P} = (e^2\omega^2/6\pi\epsilon_0 mc^3)U$. So with $6\pi\epsilon_0 mc^3/e^2\omega^2 \equiv T$, we have

$$\frac{dU}{dt} = -\overline{P} \implies \frac{dU}{dt} = -\frac{U}{T} \implies \int_{U_0}^U \frac{dU'}{U'} = -\int_0^t \frac{dt'}{T}$$
$$\implies \ln\left(\frac{U}{U_0}\right) = -\frac{t}{T} \implies U(t) = U_0 e^{-t/T}.$$
(775)

So U falls to 1/e of its initial value after a time $T = 6\pi\epsilon_0 mc^3/e^2\omega^2$. There was actually no need to separate variables and integrate as we did, because we know that the solution to the differential equation, $dU/dt \propto -U$, is simply an exponential.

Note that ωT , which is the number of radians of oscillation during the time T, can be written as

$$\omega T = \frac{6\pi\epsilon_0 mc^3}{e^2 \omega} = \frac{c}{\omega} \frac{6\pi\epsilon_0 mc^2}{e^2} = \frac{\lambda}{2\pi} \frac{3}{2r_0} = \frac{3}{4\pi} \frac{\lambda}{r_0}, \qquad (776)$$

where λ is the wavelength of the emitted light (which satisfied $\lambda \nu = c$), and r_0 is the classical electron radius, $r_0 = e^2/4\pi\epsilon_0 mc^2$. So if λ is much larger than r_0 , then many oscillations will occur before the amplitude decays significantly.

H.3. Thompson scattering

If the maximum acceleration is $E_0 e/m$, the acceleration as a function of time looks like $a(t) = (E_0 e/m) \cos \omega t$. Since the average value of $\cos^2 \omega t$ is 1/2, the average value of a^2 is $E_0^2 e^2/2m^2$. From Eq. (H.7) the average power radiated is therefore

$$\overline{P} = \frac{e^2 (E_0^2 e^2 / 2m^2)}{6\pi\epsilon_0 c^3} = \frac{E_0^2 e^4}{12\pi\epsilon_0 m^2 c^3} \,. \tag{777}$$

Dividing this result by the power density (power per unit area), $\epsilon_0 E_0^2 c/2$, we find that the area that receives an amount of power \overline{P} is

$$\sigma = \frac{e^4}{6\pi\epsilon_0^2 m^2 c^4} \,. \tag{778}$$

This is the scattering cross section. In terms of the classical electron radius, $r_0 = e^2/4\pi\epsilon_0 mc^2$, we can write σ as $\sigma = (8\pi/3)r_0^2$. Since $r_0 = 2.8 \cdot 10^{-15}$ m, we have $\sigma = 6.6 \cdot 10^{-29}$ m². As far as energy absorption goes, the electron looks like it takes up this much area, from the wave's point of view.

H.4. Synchrotron radiation

Let the electron's velocity in the lab frame be **v**. Consider the inertial frame F' moving along with the electron at a given instant. Using the Lorentz transformations, the electric field in F' is $\mathbf{E}'_{\perp} = \gamma \mathbf{v} \times \mathbf{B}_{\perp}$. (\mathbf{B}_{\perp} here is simply the **B** field in the lab frame.) This electric field causes the electron to accelerate in frame F' (where its initial velocity was zero) with an acceleration of $a = eE_{\perp}/m$. From Eq. (H.7) this acceleration causes the electron to radiate energy in F' at a rate

$$P' = \frac{e^2 (eE_{\perp}/m)^2}{6\pi\epsilon_0 c^3} = \frac{\gamma^2 e^4 v^2 B^2}{6\pi\epsilon_0 m^2 c^3} \approx \frac{\gamma^2 e^4 B^2}{6\pi\epsilon_0 m^2 c},$$
(779)

where we have used the fact that $\beta \approx c$, since we are told that the electron is highly relativistic.

We now claim that the power is the same in both the frame F' and the lab frame F, in which case transforming back to the lab frame doesn't change the answer. This claim is true because power is energy per time, and both energy U and time t transform the same way under a Lorentz transformation; they are both the fourth (or first, depending on the convention) component of a 4-vector. More precisely, in this particular case the x-t Lorentz transformation gives $\Delta t = \gamma \Delta t'$, because $\Delta x' = 0$ in F'. And the p-ELorentz transformation (E here is energy, not electric field) gives $E = \gamma E'$, because p' = 0 in F'. This then implies $\Delta E = \gamma \Delta E'$. Therefore, $\Delta E / \Delta t = \Delta E' / \Delta t' \implies$ P = P'.

As time goes on, we will need to continually pick new inertial frames F' that move along with the electron. In any one of these frames the power equals the P' we found above, so the power in the lab frame takes on the constant value P = P'.

Appendix J

J.1. Emf from a proton

At an instant when the magnetic moment is perpendicular to the plane of the coil, the *B* field at points in the plane of the coil is perpendicular to the plane and has magnitude $\mu_0 m/4\pi r^3$. As mentioned in the caption of Fig. J.2, the flux through the coil is determined by the field *outside* the coil. Demonstrating this fact was the task of Exercise 7.37. In short, every field line of the dipole passes through the tiny region inside the dipole's current loop (or whatever is going on inside a quantum mechanical spin). The lines that then loop around and pass back through the inside of the coil form a closed loop inside the coil and hence produce no net flux. The lines that loop around outside the coil have an uncanceled flux through the inside of the coil. See Figure 137. Since there are four turns in the given coil, the maximum flux through the coil has magnitude

$$\Phi_{\rm max} = 4 \int_a^\infty \frac{\mu_0 m}{4\pi r^3} 2\pi r \, dr = \frac{2\mu_0 m}{a} \,. \tag{780}$$

If $\mathbf{m}(t)$ precesses sinusoidally, then $\Phi(t)$ is likewise a sinusoidal function. Therefore $\Phi(t) = (2\mu_0 m/a) \cos \omega_{\rm p} t$, and the induced emf is $\mathcal{E}(t) = -d\Phi/dt = (2\mu_0 m\omega_{\rm p}/a) \sin \omega_{\rm p} t$. The amplitude of the emf is then

$$\mathcal{E}_{0} = \frac{2\mu_{0}m\omega_{\rm p}}{a} = \frac{2(4\pi \cdot 10^{-7} \,\frac{\text{kg m}}{\text{C}^{2}})(1.411 \cdot 10^{-26} \,\text{J/T})\omega_{\rm p}}{a} \\ = (3.55 \cdot 10^{-32} \,\text{V m s})\frac{\omega_{\rm p}}{a}.$$
(781)

Alternatively, you can use the result from Exercise 11.19. Since the magnetic field at the center of a ring is $\mu_0 I/2a$, the result $\mathcal{E}_1 = (\omega/I_1)\mathbf{B}_1 \cdot \mathbf{m}_0$ from that exercise becomes $\mathcal{E} = \omega_p(\mu_0/2a)m$. Multiplying by 4 due to the four turns yields the above result.

J.2. Emf from a bottle

(a) There are essentially $6 \cdot 10^{23}$ nucleons in a gram of anything. This number (Avogadro's number) is the inverse of the proton mass $1.67 \cdot 10^{-24}$ g (the neutron mass is nearly the same). Since 2/18 of the nucleons in water are the protons in the hydrogen atoms, the number of protons in 200 cm³ of water (which is the same as 200 g) is

$$N = \frac{2}{18} (200 \,\mathrm{g}) (6 \cdot 10^{23} \,\mathrm{g}^{-1}) = 1.33 \cdot 10^{25}.$$
 (782)

In order of magnitude, the fractional excess of magnetic moments pointing along the $B_0 = 0.1 \text{ T}$ magnetic field is

$$f = \frac{mB_0}{kT} = \frac{(1.41 \cdot 10^{-26} \,\mathrm{J/T})(0.1 \,\mathrm{T})}{4 \cdot 10^{-21} \,\mathrm{J}} = 3.5 \cdot 10^{-7}.$$
 (783)

So we expect the net magnetic moment in the sample to be roughly

$$m_{\rm net} = fNm = (3.5 \cdot 10^{-7})(1.33 \cdot 10^{25})(1.41 \cdot 10^{-26} \,\mathrm{J/T}) = 6.6 \cdot 10^{-8} \,\mathrm{J/T}.$$
 (784)

If you want to be a little more precise, you can use the Boltzmann distribution. Let N/2 + n protons point in the direction of \mathbf{B}_0 , and N/2 - n point in the opposite direction. The difference in energy of these two states is $2mB_0$. So in thermal equilibrium we have

$$\frac{N/2 - n}{N/2 + n} = e^{-2mB_0/kT} \implies \frac{1 - 2n/N}{1 + 2n/N} \approx 1 - \frac{2mB_0}{kT}$$
$$\implies 1 - \frac{4n}{N} \approx 1 - \frac{2mB_0}{kT} \implies n \approx \frac{mB_0}{2kT} N(785)$$

The net magnetic moment is then

$$m_{\rm net} = nm - n(-m) = 2nm = \frac{mB_0}{kT}Nm.$$
 (786)

This agrees with the $m_{\rm net} = fNm$ result we obtained above, even though that was just an order-of-magnitude calculation.

(b) From page 822 in the text, the proton spin precesses at a frequency of 4258 revolutions per second in a field of 1 gauss. Since this frequency is proportional to *B* (it equals mB/J), the angular frequency in a field of 0.4 gauss is $\omega_{\rm p} = 2\pi \cdot 0.4 \cdot 4258 \,\mathrm{s}^{-1} = 1.07 \cdot 10^4 \,\mathrm{s}^{-1}$. For a rough estimate of the signal voltage, assume that the protons are all near the center of a ring coil. Then the only modification to Exercise J.1 we need to make is to multiply the result by 500/4, since we now have 500 turns instead of 4. So we find

$$\mathcal{E}_{0} = \frac{500}{4} \frac{2\mu_{0} m_{\text{net}} \omega_{\text{p}}}{a} = \frac{500}{4} \frac{2(4\pi \cdot 10^{-7} \,\frac{\text{kg m}}{\text{C}^{2}})(6.6 \cdot 10^{-8} \,\text{J/T})(1.07 \cdot 10^{4} \,\text{s}^{-1})}{0.04 \,\text{m}}$$
$$= 5.5 \cdot 10^{-6} \,\text{V}.$$
(787)

Actually, the coil in Fig. J.3 is more like a solenoid than a ring, and the water almost fills it. But the result from Exercise 11.19, combined with the fact that the field inside the solenoid is somewhat uniform, implies that the actual \mathcal{E}_0 won't be too much different from the one we calculated.