Mechanics IV: Oscillations

Chapter 4 of Morin covers oscillations, as does chapter 10 of Kleppner and Kolenkow, and chapter 10 of Wang and Ricardo, volume 1. For a deeper treatment that covers normal modes in more detail, see chapters 1 through 6 of French. Jaan Kalda also has short articles on using Lagrangian-like techniques and the adiabatic theorem. For some fun discussion, see chapters I-21 through I-25, II-19, and II-38 of the Feynman lectures. There is a total of **85** points.

1 Small Oscillations

Idea 1

If an object obeys a linear force law, then its motion is simple harmonic. To compute the frequency, one must the restoring force per unit displacement. More generally, if the force an object experiences can be expanded in a Taylor series with a nonzero linear restoring term, the motion is approximately simple harmonic for small displacements. (However, don't forget that there are also situations where oscillations are not even approximately simple harmonic, no matter how small the displacements are.)

Idea 2

A useful generalization of Newton's second law is given by generalized coordinates. Let q be any number that describes the state of the system, not necessarily a Cartesian coordinate. Suppose the energy of a system can be decomposed into two parts, a potential energy that depends only on q and a kinetic energy that depends only on $\dot{q}$,

$$K = K(\dot{q}), \quad V = V(q).$$

Then since energy is conserved, $d(K+V)/dt = 0$, the chain rule gives

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} = -\frac{\partial V}{\partial q}.$$

We call the left-hand side the rate of change of a "generalized momentum", and the right-hand side a "generalized force". When q is a Cartesian coordinate, this recovers the usual $F = ma$.

Idea 3

Generalized coordinates are really useful for problems that involve complicated objects but only have one relevant degree of freedom, which is especially true for oscillations problems. For instance, if the kinetic and potential energy have the form

$$K = \frac{1}{2} m_{\text{eff}} \dot{q}^2, \quad V = \frac{1}{2} k_{\text{eff}} q^2$$

then the oscillation frequency is always

$$\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}.$$

Note that q need not have units of position, m_{eff} need not have units of mass, and so on. When $V(q)$ is a more general function, we can expand it about a minimum q_{min}, so that $k_{\text{eff}} = V''(q_{\text{min}})$. This technique allows us to avoid dealing with possibly complicated constraint forces.

Idea 4: Normal Modes

A system with N degrees of freedom has N normal modes when displaced from equilibrium. In a normal mode, the positions of the particles are of the form $x_i(t) = A_i \cos(\omega t + \phi_i)$. That is, all particles oscillate with the same frequency. Normal modes can be either guessed physically, or found using linear algebra as explained in section 4.5 of Morin.

The general motion of the system is a superposition of these normal modes. So to compute the time evolution of the system, it's useful to decompose the initial conditions into normal modes, because they all evolve independently by linearity.

Idea 5

When a problem contains two widely separate timescales, such as a fast oscillation superposed on a slow overall motion, one can solve for the fast motion while neglecting the slow motion, then solve for the slow motion by replacing the fast motion with an appropriate average.

Idea 6: Adiabatic Theorem

If a particle performs a periodic motion in one dimension in a potential that changes very slowly, then the "adiabatic invariant"

$$I = \oint p \, dx$$

is conserved. This is the area of the orbit in phase space, an abstract space whose axes are position and momentum.