Mechanics III: Dynamics

Chapters 3 and 5 of Morin cover dynamics, energy, and momentum. Alternatively, see chapters 2 and 3 of Kleppner and Kolenkow, or chapters 4 and 6 of Wang and Ricardo, volume 1. For fun, see chapters I-9 through I-14 of the Feynman lectures. There is a total of **82** points.

1 Blocks, Pulleys, and Ramps

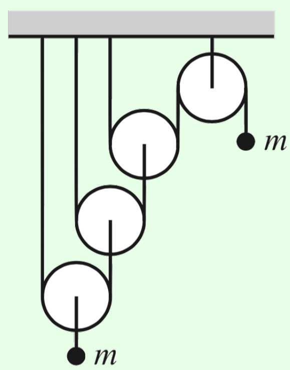
Idea 1

To solve dynamics problems with constraints, it's easiest to first write the constraint in terms of coordinates (e.g. "conservation of string" for pulleys, or stationarity of the CM for an isolated system), then differentiate to get constraints on the velocity and acceleration.

Questions of this type are generally straightforward, as long as you write down the correct equations. The trickiest part is often solving the equations, which can get messy.

Example 1: Morin 3.30

Find the acceleration of the masses in the Atwood's machine shown below.



Neglect friction, and treat all pulleys are massless.

Solution

Let x and x' be the amounts by which the left and right mass have moved down, and number the pulleys 1 through 4 from left to right, and the strings 1 through 3 from left to right. Pulley 4 is stationary, so conservation of string 3 means that pulley 3 moves up by $x'/2$. Next, conservation of string 2 means that pulley 2 moves up by $x'/4$. Finally, conservation of string 1 implies that pulley 1 moves up by $x'/8$, so our final conservation of string constraint is

$$x = -\frac{x'}{8}$$

which upon applying the derivative twice gives

$$a = -\frac{a'}{8}.$$

Now consider the tensions T_i in the strings. We know that

$$a = g - \frac{2T_1}{m}, \quad a' = g - \frac{2T_3}{m}.$$

Since pulley 3 is massless, the forces on it must balance, so $T_2 = 2T_3$. Similarly $T_1 = 2T_2$, so $T_1 = 4T_3$. We hence have a system of three equations in three unknowns (T_1, a, and a'), which can be solved straightforwardly to give

$$a' = \frac{56}{65}\,g, \quad a = -\frac{7}{65}\,g.$$

2 Momentum

Idea 2

The momentum of a system is

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i = M\mathbf{v}_{\text{CM}}.$$

In particular, the total external force on the system is $M\mathbf{a}_{\text{CM}}$, and if there are no external forces, the center of mass moves at constant velocity.

Example 2

A massless rope passes over a frictionless pulley. A monkey hangs on one side, while a bunch of bananas with exactly the same weight hangs from the other side. When the monkey tries to climb up the rope, what happens?

Solution

Remarkably, the answer doesn't depend on how the monkey climbs, whether slowly or quickly, or symmetrically or not! The total vertical force on the monkey is $T - mg$, so the acceleration of the center of mass of the monkey is $T/m - g$. But since the tension is uniform through a massless rope, the acceleration of the bananas is also $T/m - g$. Therefore, the monkey and bananas rise at the same rate, and meet each other at the pulley.

Now here's a question for you: compared to climbing up a rope fixed to the ceiling, climbing up to the pulley takes twice as much work, because the bananas are raised too. But in both cases, isn't the monkey applying the same force through the same distance? Where does the extra work come from? (The answer involves the ideas at the end of this problem set.)

Example 3: KK 3.14 / INPhO 2014.5

Two men, each with mass m, stand on a railway flatcar of mass M initially at rest. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction. Find the final velocities of the flatcar if they jump off at the same time, and if they jump off one at a time. Generalize to the case of $N \gg 1$ men, with a total mass of m_{tot}.

Solution

In the first case, by conservation of momentum, we have

$$Mv + 2m(v - u) = 0$$

where v is the final velocity of the flatcar, so

$$v = \frac{2mu}{M + 2m}.$$

In the second case, by a similar argument, we find that after the first man jumps,

$$v_1 = \frac{mu}{M + 2m}.$$

Now transform to the frame moving with the flatcar. When the second man jumps, he imparts a further velocity $v_2 = mu/(M + m)$ to the flatcar by another similar argument. The final velocity of the flatcar relative to the ground is then

$$v = v_1 + v_2 = mu \left(\frac{1}{M + 2m} + \frac{1}{M + m} \right).$$

It might be a bit disturbing that the final speeds and hence energies of the flatcar are different, even though the men are doing the same thing (i.e. expending the same amount of energy in their legs to jump) in both cases.

The reason for the difference is that in the second case, the second man to jump ends up with less energy, since the velocity he gets from jumping is partially cancelled by the existing velocity v_1. So the extra energy that goes into the flatcar corresponds to less kinetic energy in the men after jumping, which would ultimately have ended up as heat after they slid to a stop. Accounting properly for the kinetic energy of everything in the system solves a lot of paradoxes involving energy, as we'll see below.

In the case of many men, by similar reasoning we have

$$v = \frac{m_{\text{tot}}}{M + m_{\text{tot}}}\, u$$

in the first case, while in the second case the answer is the sum

$$v = \sum_{i=1}^{N} \frac{m_{\text{tot}} u}{N} \frac{1}{M + (i/N) m_{\text{tot}}}.$$

This can be converted into an integral, by letting $x = i/N$, in which case $\Delta x = 1/N$ and

$$v = \sum_i \Delta x\, \frac{m_{\text{tot}} u}{M + x m_{\text{tot}}} \approx \int_0^1 dx\, \frac{m_{\text{tot}} u}{M + x m_{\text{tot}}} = \log\left(\frac{M + m_{\text{tot}}}{M}\right) u.$$

Note that this is essentially the rocket equation, which we'll derive in a different way in **M6**.

3 Energy

Idea 3

The work done on a point particle is

$$W = \int \mathbf{F} \cdot d\mathbf{x}$$

and is equal to the change in kinetic energy, as you showed in **P1**.

Remark: Dot Products

The dot product of two vectors is defined in components as

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$$

and is equal to $|\mathbf{v}|\,|\mathbf{w}|\cos\theta$ where θ is the angle between them. For example, if $\mathbf{A}$ and $\mathbf{B}$ are the sides of a triangle,

$$|\mathbf{A} + \mathbf{B}|^2 = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) = A^2 + B^2 + 2AB\cos\theta.$$

Since the left-hand side is the length squared of the third side of the triangle, we've proven the law of cosines. (Or, if you accept the law of cosines, you could regard this as a proof that the dot product depends on $\cos\theta$ as claimed.)

Like the ordinary product, the dot product obeys the product rule. For example,

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{w}) = \dot{\mathbf{v}} \cdot \mathbf{w} + \mathbf{v} \cdot \dot{\mathbf{w}}.$$

Using this, it's easy to generalize the derivation of the work-kinetic energy theorem in **P1** to three dimensions; we have

$$\frac{1}{2}d(v^2) = \frac{1}{2}d(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot d\mathbf{v} = \frac{d\mathbf{x}}{dt} \cdot d\mathbf{v} = \frac{d\mathbf{v}}{dt} \cdot d\mathbf{x} = \mathbf{a} \cdot d\mathbf{x}$$

and this is equivalent to the desired theorem. As you can see, it's all basically the same, since the product and chain rule manipulations work the same way for vectors and scalars.

Example 4: IPhO 1996 1(b)

A skier starts from rest at point A and slowly slides down a hill with coefficient of friction μ, without turning or braking, and stops at point B. At this point, his horizontal displacement is s. What is the height difference h between points A and B?

Solution

Since the skier begins and ends at rest, the change in height is the total energy lost to friction,

$$mgh = \int f_{\text{fric}}\, ds$$

where the integral over ds goes over the skier's path. Since the skier is always moving slowly, the normal force is approximately $mg\cos\theta$. (More generally, there would be another contribution to provide the centripetal acceleration.) But then

$$\int f_{\text{fric}}\, ds = \int \mu mg\cos\theta\, ds = \int \mu mg\, dx = \mu mgs$$

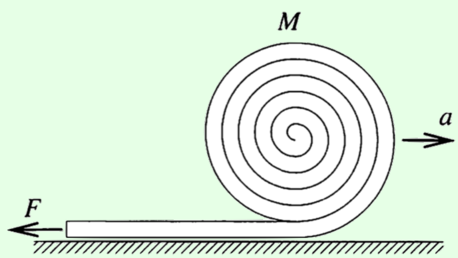
which gives an answer of $h = \mu s$. (If the skier's path turned around, then this would still hold as long as s denotes the total horizontal distance traveled.)

Idea 4

If a problem can be solved using either momentum conservation or energy conservation alone, it usually means one of the two isn't actually conserved. In particular, many processes are inherently inelastic and inevitably dissipate energy. For more about inherently inelastic processes, see section 5.8 of Morin.

Example 5: PPP 108

A fire hose of mass M and length L is coiled into a roll of radius R. The hose is sent rolling along level ground, with its center of mass given initial speed $v_0 \gg \sqrt{gR}$. The free end of the hose is held fixed.



The hose unrolls and becomes straight. How long does this process take to complete?

Solution

First, we need to find what is conserved. The horizontal momentum is not conserved, because there is an external horizontal force needed to keep the end of the hose in place. On the other hand, the energy *is* conserved, even though this process looks inelastic. The hose "sticks" to the floor as it unrolls, but this process dissipates no energy because the circular part of the hose rolls without slipping, so the bottom of this part always has zero velocity.

Once we figure out energy is conserved, the problem is straightforward. The assumption $v_0 \gg \sqrt{gR}$ means we can neglect the change in gravitational potential energy as the hose unrolls. After the hose travels a distance x,

$$\frac{1}{2}\left(1+\frac{1}{2}\right) M v_0^2 = \frac{1}{2}\left(1+\frac{1}{2}\right) m v^2$$

where the 1/2 terms are from rotational kinetic energy. Since $m(x) = M(1 - x/L)$, we have

$$v(x) = \frac{v_0}{\sqrt{1 - x/L}}$$

which gives a total time

$$T = \int_0^L \frac{dx}{v(x)} = \frac{L}{v_0} \int_0^1 \sqrt{1-u}\, du = \frac{2L}{3v_0}.$$

Evidently, the hose accelerates as it unrolls.

Idea 5

Any temporary interaction between two objects that conserves energy and momentum is a perfectly elastic collision.

Example 6

Two masses are constrained to a line. The mass m_1 moves with velocity v_1, and the mass m_2 moves with velocity v_2. The masses collide perfectly elastically. Find their speeds afterward.

Solution

The usual method is to directly invoke conservation of energy and momentum, which leads to a quadratic equation. A slicker method is to work in the center of mass frame instead. (This is useful for collision problems in general, and it'll become even more useful for the relativistic collisions covered in **R2**.)

The center of mass of the system has speed

$$v_{\text{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

Moreover, by momentum conservation, the center of mass never accelerates. Now we boost into the frame moving with the center of mass. Since the total momentum is by definition zero in the center of mass frame, the momenta of the particles cancel out. The only way for this to remain true after the collision is if we multiply their velocities by the same number. Energy is only conserved if this number is ± 1, with the latter representing no collision at all.

Therefore, during an elastic collision, the velocities in the center of mass frame simply reverse. The initial velocities in that frame are

$$v_{1,\text{CM}} = v_1 - v_{\text{CM}}, \quad v_{2,\text{CM}} = v_2 - v_{\text{CM}}.$$

The final velocities in that frame are

$$v'_{1,\text{CM}} = -v_1 + v_{\text{CM}}, \quad v'_{2,\text{CM}} = -v_2 + v_{\text{CM}}.$$

Finally, going back to the original frame gives the final velocities

$$v'_1 = -v_1 + 2v_{\text{CM}}, \quad v'_2 = -v_2 + 2v_{\text{CM}}.$$

There are many special cases we can check. For example, if $m_1 = m_2$, then the two masses simply swap their velocities, as if they just passed through each other. As another check, consider the case where the second mass is initially at rest, $v_2 = 0$. Then

$$v'_1 = v_1 \frac{m_1 - m_2}{m_1 + m_2}, \quad v'_2 = v_1 \frac{2m_1}{m_1 + m_2}.$$

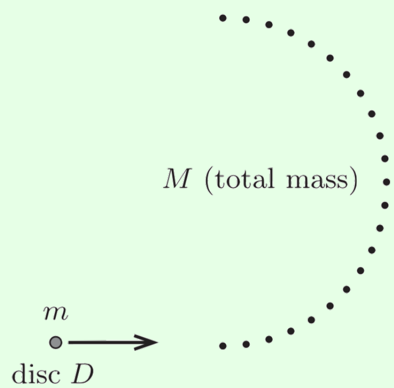
When $m_1 = m_2$, the first mass gives all its velocity to the second. When m_2 is large, the first mass just rebounds off with velocity $-v_1$. When m_1 is large, the first mass keeps on going and the second mass picks up velocity $2v_1$. Finally, when $m_1 = m_2/3$, then the final speeds are $v'_1 = -v_1/2$ and $v'_2 = v_1/2$, a nice result which is worth committing to memory.

Idea 6

For problems involving many collisions, a nice way to keep track of everything is to make a 2D plot of $x(t)$ for all the masses.

Example 7: MPPP 42

There are N identical tiny discs lying on a table, equally spaced along a semicircle, with total mass M. Another disc D of mass m is very precisely aimed to bounce off all of the discs in turn, then exit opposite the direction it came.



In the limit $N \to \infty$, what is the minimal value of M/m for this to be possible? Given this value, what is the ratio of the final and initial speeds of the disc?

Solution

The reason that there is a lower bound on M is that, by the result of part (c) of problem 20, there is a maximal angle that each tiny disc can deflect the disc D. For large N, the deflection is π/N for each disc, so

$$\frac{\pi}{N} = \sin^{-1}\frac{M/N}{m} \approx \frac{M}{Nm}$$

which implies that $M/m \geq \pi$.

To see how much energy is lost in each collision, work in the center of mass frame and consider the first collision. In this frame, the disc D is initially approximately still, and the tiny disc comes in horizontally with speed v. To maximize the deflection angle in the table's frame, the tiny disc should rebound vertically, as this provides the maximal vertical impulse to the disc D.

Thus, going back to the table's frame, where the disc D has speed v, the tiny disc scatters with speed $\sqrt{v^2+v^2} = \sqrt{2}v$. By conservation of energy,

$$\Delta\left(\frac{1}{2}mv^2\right) = -\frac{1}{2}\frac{M}{N}(\sqrt{2}v)^2.$$

This simplifies to

$$\frac{\Delta v}{v} = -\frac{\pi}{N}$$

which means that after N collisions, we have

$$\frac{v_f}{v_i} = \left(1 - \frac{\pi}{N}\right)^N \approx e^{-\pi}$$

where in the last step we used a result from **P1**.

Example 8: EFPhO 2003.1

In this question, we consider a simplified model of how an elastic collision actually happens. Consider a spherical volleyball inflated with excess pressure ΔP, radius r, and mass m. If it hits the ground with a large, but not huge, speed v, estimate how long the subsequent elastic collision takes.

Solution

When the volleyball hits the ground, it will keep going, deforming the part that touches the ground into a flat circular face. Specifically, when the ball has moved a distance y into the ground, the flat face has area

$$A = \pi \left(\sqrt{r^2 - (r-y)^2} \right)^2 = \pi y(2r - y) \approx 2\pi r y$$

where we assumed that $y \ll r$ at all times, which is reasonable as long as the initial speed is not huge. As a result, the pressure of the volleyball exerts a force

$$F = 2\pi r \, \Delta P \, y$$

on the ground. This assumes the pressure inside the volleyball remains uniform, and that the rest of the volleyball stays approximately spherical, which is again reasonable as long as the initial speed is not huge.

Assuming the initial velocity is not too small, gravity is negligible during the collision, so during the collision the force on the volleyball is effectively that of an ideal spring. The collision lasts for half a period, giving

$$\tau = \pi \sqrt{\frac{m}{k_{\text{eff}}}} = \sqrt{\frac{\pi m}{2r \, \Delta P}}.$$

If we plug in realistic numbers, the result is of order 10 ms, which is plausible.

5 Continuous Systems

Example 9

As shown in **M2**, a hanging chain takes the form of a catenary. Suppose you pull the chain down in the middle. How does the center of mass of the chain move? Does the answer depend on how hard you pull?

Solution

No matter how hard you pull, or in what direction, the height of the center of mass always goes up! This is because this quantity measures the total gravitational potential energy of the chain. If you pull a chain in equilibrium, in any direction whatsoever, you will do work on it. So this raises its potential energy, and hence the center of mass.

Another way of saying this is that the equilibrium position, without the extra pull you supply, is already in the lowest energy state, and hence already has the lowest possible center of mass. Changing this shape in any way raises the center of mass.

Idea 7: Center of Mass Energy

The work done on a part of a system is

$$dW = F\,dx$$

where F is the force on that specific part of the system, and dx is its displacement. Then $dW = dE$ where E is the total energy of the system.

Similarly, the "center of mass work" done on a system is

$$dW_{\text{cm}} = F\,dx_{\text{cm}}$$

where F is the total force on the system and dx_{cm} is the displacement of the center of mass. Then $dW_{\text{cm}} = dE_{\text{cm}}$ where the "center of mass energy" is defined as $E_{\text{cm}} = Mv_{\text{cm}}^2/2$.

It should be noted that, like regular energy and work, center of mass energy and work depend on the reference frame you're using.

Example 10

Consider a cyclist who pedals their bike to accelerate. The wheels roll without slipping on the ground. The cyclist moves a distance d, with the bike experiencing a constant friction force f from the ground. Analyze the situation using both energy and center of mass energy.

Solution

Since the wheels roll without slipping, their contact point with the ground is always zero, so the friction force does exactly zero work. Thus the net energy of the cyclist/bike system is conserved. The additional kinetic energy of the cyclist/bike comes from the chemical energy of the cyclist, which ultimately came from what they ate. So conservation of energy is correct, but it doesn't tell us anything useful at all.

Now consider center of mass energy. Considering the cyclist/bike system, the center of mass work is fd, which is the change in $Mv_{\text{cm}}^2/2$. This allows us to compute the change in velocity of the cyclist/bike.

Example 11

Consider the same setup as in the previous example, but now the cyclist brakes hard. The wheels slip on the ground, and experience a friction force $-f$ while the cyclist moves a distance d. Analyze the situation using both energy and center of mass energy.

Solution

The center of mass work equation tells us about the overall deceleration of the cyclist/bike, just as in the previous example.

On the other hand, the work done by the friction force is indeterminate! It can be any quantity between zero and $-fd$. When it is 0, the total energy of the cyclist/bike system is again conserved, which means all the kinetic energy lost is dissipated as heat inside the bike itself. When it is $-fd$, all the kinetic energy lost is dissipated as heat in the *ground*, and hence energy is removed from the cyclist/bike system. In general, the work will be an intermediate value, meaning that both the ground and the bike heat up, but we can't calculate what it is without a microscopic model of how the friction works. It depends on, e.g. how easily the ground and bike tire surface deform.