Mechanics II: Statics

For review, read chapter 2 of Morin or chapter 2 of Kleppner and Kolenkow. Statics is covered in more detail in chapter 7 of Wang and Ricardo, volume 1. Surface tension is covered in detail in chapter 5 of *Physics of Continuous Matter* by Lautrup, which is an upper-division level introduction to fluids in general. There is a total of **82** points.

1 Balancing Forces

Idea 1

In principle, you can always solve every statics problem by balancing forces on every individual particle in the setup, but often you can save on effort by considering appropriate systems.

Idea 2

Any problem where everything has a uniform velocity is equivalent to a statics problem, by going to the reference frame moving with that velocity. Any problem where everything has a uniform acceleration $\mathbf{a}$ is also about statics, by going to the noninertial frame with acceleration $\mathbf{a}$, where there is an extra effective gravitational acceleration $-\mathbf{a}$. The same principle applies to uniform rotation, where a centrifugal force appears in the rotating frame, acting like an effective gravitational acceleration $\omega^2\mathbf{r}$.

Example 1

Six blocks are attached in a horizontal line with rigid rods, and placed on a table with coefficient of friction μ. The blocks have mass m and the leftmost block is pulled with a force F so the blocks slide to the left. Find the tension force in the rod in the middle.

Solution

There are six objects here and five rods, each with a different tension, so a direct analysis would involve solving a system of six equations. Instead, first consider the entire set of six blocks as one object; we can do this because the rigid rods force them to move as one. The total mass is $6m$, and applying Newton's second law gives

$$F - 6mg\mu = 6ma, \quad a = \frac{F}{6m} - \mu g.$$

Next, consider the rightmost three blocks as one object. Their total mass is $3m$, and their acceleration is the same acceleration a we computed above. This system experiences two horizontal force: tension and friction. Newton's second law gives

$$T - 3mg\mu = 3ma$$

and solving for T gives

$$T = \frac{F}{2}.$$

This is intuitive, because the differences of any two adjacent tension forces are the same; that's the amount of tension that needs to be spent to accelerate each block. So the middle rod, which has to accelerate only half the blocks, has half the tension.

The reason we could ignore the tension forces in the other four rods is that the only thing they do is ensure the blocks move with the same acceleration. Once we assume this is the case, the specific values of the tensions don't matter; we can just zoom out and forget them. It's just like how *within* each block there is also an internal tension which keeps it together, but we rarely need to worry about its details.

Idea 3

To handle a problem where something is just about to slip on something else, set the frictional force to the maximal value μN and assume slipping is not yet occurring, so the two objects move as one. The same idea holds for problems which ask for the minimal force needed to make something move, or the minimal force needed to keep something from moving.

2 Balancing Torques

Idea 4

A static rigid body will remain static as long as the total force on it vanishes, and the total torque vanishes, where the torque about the origin is

$$\boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

where $\mathbf{r}_i$ is the point of application of force $\mathbf{F}_i$. If the total force vanishes, the total torque doesn't depend on where the origin is, because shifting the origin by $\mathbf{a}$ changes the torque by

$$\Delta\boldsymbol{\tau} = \sum_i \mathbf{a} \times \mathbf{F}_i = \mathbf{a} \times \left(\sum_i \mathbf{F}_i \right) = 0.$$

The origin should usually be chosen to set as many torques as possible to zero.

Idea 5

The center of mass $\mathbf{r}_{\text{CM}}$ of a set of masses m_i at locations $\mathbf{r}_i$ with total mass M satisfies

$$M\mathbf{r}_{\text{CM}} = \sum_i m_i \mathbf{r}_i.$$

If a system experiences no external forces, its center of mass moves at constant velocity.

Idea 6

A uniform gravitational field exerts no torque about the center of mass. Thus, for the purposes of applying torque balance on an *entire* object, the gravitational force $M\mathbf{g}$ can be taken to act entirely at its center of mass. (This is a formal substitution; of course, the actual gravitational force remains distributed throughout the object.)

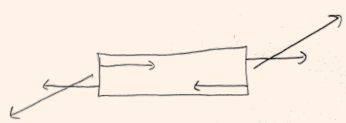
Torque balance works in noninertial frames, as long as one accounts for the torques due to fictitious forces. Thus, for an accelerating frame, the $-M\mathbf{a}$ fictitious force can be taken to act at the center of mass. In a uniformly rotating frame, the total centrifugal force is $M\omega^2\mathbf{r}_{\text{cm}}$, and for the purposes of balancing torques, can be taken to act entirely at the center of mass.

Example 2

Show that the tension in a completely flexible static rope, massive or massless, points along the rope everywhere in the rope.

Solution

Consider a tiny segment $d\ell$ of the rope. Since the rope is static, the tension forces on both ends balance, so they are opposite. Let them both be at an angle θ to the rope direction. Then the net torque on the segment is $(T\,d\ell)\sin\theta$. Since this must vanish for static equilibrium, we must have $\theta = 0$ and hence the tension is along the rope. In other words, flexible ropes can transmit force, but they can't transmit torque.



It's important to note that the argument above doesn't work for a rigid rod, because the internal forces in a rigid object can look like the picture above. In other words, there can be extra shear forces from the adjacent pieces of the rod that provide the compensating torque. If one tried to set up forces like this in a rope, it would flex instead.

In general, the force distribution within a massless rigid rod can be quite complicated, but if we zoom out, we can replace it with a single tension which does not necessarily point along the rod. This transmits both a force and a torque through the rod, in the sense that a torque is eventually exerted by whatever holds the end of the rod in place. Note that if the rod's supports are free to rotate, then they can't absorb torque, so the rod acts just like a rope, with tension always along it.

Remark

Sometimes, problem writers will intentionally not introduce any variables that are irrelevant to the answer. This can occur in two ways. First, the variables might just cancel out, as one can often see by dimensional analysis. Second, the specific values of the variables might not matter in the limit when they are very large or small. For instance, if a problem simply states a mass is "very heavy" but doesn't give it a name like m, it is asking for the answer in the limit $m \to \infty$.

Idea 7

To handle problems where an object is just about to tip over, note that at this moment, the entire normal force will often be concentrated at a point. (For example, when you're about to fall forward, all your weight goes on your toes.) That often means it's a good idea to take torques about this point.

Example 3: Povey 5.6

Suppose that on level ground, a car has a distance d between its left and right tires, and its center of mass is a height h above the ground. Now suppose the car turns as in problem 2, but in the extreme case $\theta = 90^\circ$, with speed v. For what v is this motion possible?

Solution

Again working in the noninertial frame of the car, force balance gives

$$f_{\text{fric}} = mg, \quad N = \frac{mv^2}{R}$$

where f_{fric} and N are the total friction and normal forces on the four tires. Since $f_{\text{fric}}/N \leq \mu$,

$$v \geq \sqrt{gR/\mu}$$

which matches the general solution to problem 2. But in that problem, we only considered force balance. In this extreme situation, we also have to consider torque balance, i.e. the possibility that the car might topple over. When the car is about to topple over, all the normal and friction force is on the bottom tires. About this point, we have only torques from gravity and the centrifugal force, giving

$$mgh = \frac{mv^2}{R}\frac{d}{2}$$

and solving for v gives $v = \sqrt{2gRh/d}$. Toppling is less likely the higher v is, so the answer is

$$v \geq \sqrt{gR}\,\max(1/\sqrt{\mu}, \sqrt{2h/d}).$$

Idea 8

An extended object supported at a point may be static if its center of mass lies directly above or below that point. More generally, if the object is supported at a set of points, it can be static if its center of mass lies above the convex hull of the points.

3 Trickier Torques

Idea 9

Sometimes, a clever use of torque balance can be used to remove any need to have explicit force equations at all. Rarely, the same situation can occur in reverse.

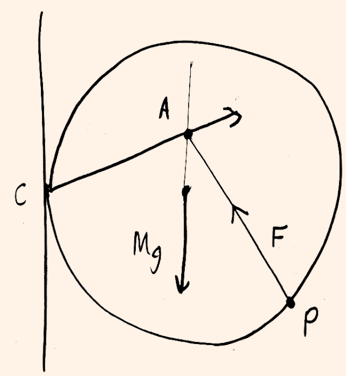
Example 4: EFPhO 2010.4

A spherical ball of mass M is rolled up along a vertical wall, by exerting a force F to some point P on the ball. The coefficient of friction is μ. What is the minimum possible force F, and in this case, where is the point P?

Solution

Following the logic of idea 3, when the minimum possible force is used, the frictional force with the wall must be maximal, $f = \mu N$, and directed upward. (If friction weren't pushing the ball up as hard as possible, we could get by using a smaller force F.) So even though we don't know the magnitude of the normal or the frictional force, we know the direction of the sum of these two forces, so we'll consider them as one combined force.

This reduces the number of independent forces in the problem to three: gravity (acting at the center of mass), the force F (acting at P), and the combined normal and friction forces (acting at the point of contact C with the wall). Therefore, by the result of problem 5, the lines of these forces must all intersect at some point A, as shown.



This ensures that the torques will balance, when taken about point A.

Next, we need to incorporate the information from force balance. Doing this directly will lead us to some nasty trigonometry, but there's a better way. There are in principle two force balance equations, for horizontal and vertical forces. However, one of these equations is just going to tell us the magnitude of the normal/frictional force, which we don't care about. So in reality, we just need one equation, which preferably doesn't involve that force.

The trick is to use torque balance *again*, about the point C, which says that the torques due to gravity and F must cancel. Now you might ask, didn't we already use torque balance? We did, but recall from idea 4 that taking the torque about a different point can give you a different equation if the forces don't balance. So by demanding the torque vanish about two different points, we actually are using force balance! (Specifically, we are using the linear combination of the horizontal and vertical force balance equations that *doesn't* involve the normal/friction force, which we don't need to find anyway.)

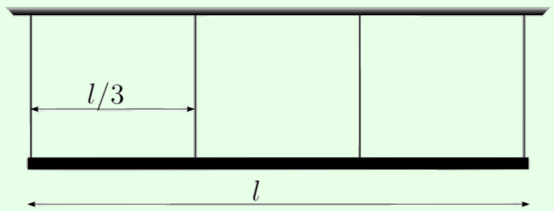
When taking the torque about C, we see that F is minimized if P is chosen to maximize the lever arm of the force. This occurs when $CA \perp PA$, in which case the lever arm is $R\sqrt{1+\mu^2}$, where R is the radius of the ball. So we have

$$MgR = FR\sqrt{1+\mu^2}, \quad F = \frac{Mg}{\sqrt{1+\mu^2}}$$

and P is determined as described above.

Example 5

A uniform bar with mass m and length ℓ hangs on four equally spaced identical light wires. Initially, all four wires have tension $mg/4$.



Find the tensions after the leftmost wire is cut.

Solution

This illustrates a common issue with setups involving rigid supports: there are often more normal forces than independent equations, so there is not a unique solution. In the real world, the result is determined by imperfect characteristics of the wires. For example, if one of the wires was slightly longer than the others, it would go slack, reducing the number of normal forces by one and yielding a solution.

A reasonable assumption, if you aren't given any further information, is to assume that the supports are identical, very stiff springs. In equilibrium, the bar will tilt a tiny bit, so that the length of the middle wire will be the average of the lengths of the other two. By Hooke's law, the force in that wire will than be the average of the other two, so the tensions are $mg/3 - x$, $mg/3$, and $mg/3 + x$. Applying torque balance yields $7mg/12$, $mg/3$, and $mg/12$.

The general point here is that concepts like rigid bodies or strings characterized by a single tension force are abstractions, made for the idealized problems we study in mechanics classes. A real civil engineer designing a structure would instead use a sophisticated computer program which simulates the complex internal forces, torques, and strains throughout the material.

Remark: Subtleties of Friction

Statics problems involving friction can also get quite elegant, but it's important to remember that at the end of the day, they're just a decent approximation for the real world. In reality, laws like $F_{\text{fric}} = -\mu_k N$ are only approximately true for some regimes of behavior of some materials. In fact, the particular law $F_{\text{fric}} = -\mu_k N$ can sometimes yield equations with no solutions, a phenomenon called the Painleve paradox!

The source of the paradox is the discontinuous dependence of the direction of the friction force on the direction of the velocity. If you assume an object slips left, it is possible that after you do all the calculations (assuming a rightward friction force) you instead find the object slips right. And the reverse happens if you assume the object slips right, which makes the actual slipping direction completely indeterminate.

You won't have to worry about this paradox for Olympiads, which never contain setups that trigger it. But it's worth keeping in mind that friction is not just a simple law, but the subject of an entire field of study called tribology, which is essential for engineering. In these paradoxical cases, you would need to use a more refined model of friction to figure out what actually happens.

4 Extended Bodies

Idea 10

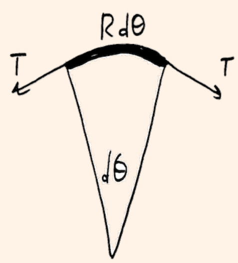
To deal with extended bodies, one can consider an infinitesimal piece of the body, or consider a well-chosen symmetrical piece of it, or consider the whole body as a system. Often, multiple approaches are needed.

Example 6

Find the tension in a circular rope of radius R spinning with angular velocity ω and mass per length λ.

Solution

Consider an infinitesimal segment of the rope, spanning an angle $d\theta$.



The mass of this segment is $dm = R\lambda\, d\theta$. The total force is downward, with magnitude

$$dF = 2T \sin \frac{d\theta}{2} \approx T\, d\theta$$

where we used the small angle approximation. This is the centripetal force, so

$$dF = (dm)\, \omega^2 R.$$

Combining these results yields $T = R^2\omega^2\lambda$.

Example 7

Find the distance d of the center of mass of a uniform semicircle of radius R to its center. (Note that a semicircle is half of a circle, not half of a disc.)

Solution

This can be done by taking the setup of the previous problem, and taking a subsystem comprising exactly half of the rope. In this case the net tension force is simply

$$F = 2T.$$

The total mass is $m = \pi R \lambda$, and the force must provide the centripetal force, so

$$F = (\pi R \lambda)(\omega^2 d)$$

But we also know that $T = R^2 \omega^2 \lambda$ as before, so plugging this in gives

$$d = \frac{2}{\pi} R.$$

Alternatively, we could have worked in the frame rotating with the rope. The equations would be the same, but instead we would say the tension balances the centrifugal force.

Example 8

A chain is suspended from two points on the ceiling a distance d apart. The chain has a uniform mass density λ, and cannot stretch. Find the shape of the chain.

Solution

First, we note that the horizontal component of the tension T_x is constant throughout the chain; this just follows from balancing horizontal forces on any piece of it. Moreover, by similar triangles, we have $T_y = T_x y'$ everywhere.

Now consider a small segment of chain with horizontal projection Δx. The length of the piece is $\Delta x\sqrt{1+y'^2}$ which determines its weight, and this be balanced by the difference in vertical tensions. Thus

$$\Delta T_y = \lambda g \sqrt{1+y'^2}\,\Delta x.$$

For infinitesimal Δx, we have $\Delta T_y = T_x d(y') = T_x y''\,dx$, so we get the differential equation

$$y'' = \frac{\lambda g}{T_x}\sqrt{1+y'^2}.$$

Usually nonlinear differential equations with second derivatives are very hard to solve, but this one isn't because there is no direct dependence on y, just its derivatives. That means we can treat y' as the independent variable first, and the equation is effectively first order in y'.

Writing $y'' = d(y')/dx$ and separating, we have

$$\int \frac{dy'}{\sqrt{1+y'^2}} = \int \frac{\lambda g}{T_x}\,dx.$$

Integrating both sides gives

$$\sinh^{-1}(y') = \frac{\lambda g x}{T_x} + C.$$

Choosing $x = 0$ to be the lowest point of the chain, the constant C is zero, and

$$y' = \sinh\left(\frac{\lambda g x}{T_x}\right).$$

Integrating both sides again gives the solution for y,

$$y = \frac{T_x}{\lambda g}\cosh\left(\frac{\lambda g x}{T_x}\right)$$

where we suppressed another constant of integration. This curve is called a catenary.

Example 9

A uniform spring of spring constant k, mass m, and relaxed length L is hung from the ceiling. Find its length in equilibrium, as well as its center of mass.

Solution

Problems like this contain subtleties in notation. For example, if you talk about "the piece of the slinky at z", this could either mean the piece that's actually at this position in equilibrium, or the piece that was originally at this place in the absence of gravity. Talking about it the first way automatically tells you where the piece is now, but talking about it the second way makes it easier to keep track of, because then the z of a specific piece of the spring stays the same no matter what happens.

In fluid dynamics, these are known as the Eulerian and Lagrangian approaches, respectively. If you don't use one consistently, you'll get nonsensical results, and it's easy to mix them up.

There are many ways to solve this problem, but I'll give one that reliably works for me. We're going to use the Lagrangian approach, and avoid confusion with the Eulerian approach by breaking the spring into discrete pieces. Let the spring consist of $N \gg 1$ pieces, of masses m/N, spring constants Nk, and relaxed lengths L/N.

The i^{th} spring from the bottom has tension $(i/N)mg$, and thus is stretched by

$$\Delta L_i = \frac{1}{kN} \frac{i}{N} mg = \frac{mg}{kN^2}\, i.$$

The total stretch is

$$\sum_{i=1}^{N} \Delta L_i = \frac{mg}{kN^2} \int_0^N i\, di = \frac{mg}{2k}.$$

This makes sense, since the average tension is $mg/2$. To find the center of mass, note that the j^{th} spring is displaced downward by a distance

$$\Delta y_j = \sum_{i=j}^{N} \Delta L_i = \frac{mg}{2k} \left(1 - \frac{j^2}{N^2}\right)$$

downward from its position in the absence of gravity. The center of mass displacement is

$$\Delta y_{\text{CM}} = \frac{1}{N} \sum_{j=1}^{N} \Delta y_j \propto \frac{1}{N} \sum_{j=1}^{N} \left(1 - \frac{j^2}{N^2}\right) = \frac{1}{N^3} \int_0^N N^2 - j^2\, dj = \frac{2}{3}$$

so restoring the proportionality constant gives

$$\Delta y_{\text{CM}} = \frac{mg}{3k}.$$

If you want to test your understanding of slinkies, you can also try doing this problem with the Eulerian approach. The first steps would be finding a relation between the density $\rho(z)$ and tension $T(z)$ from Hooke's law, and finding out how to write down local force balance.

Remark

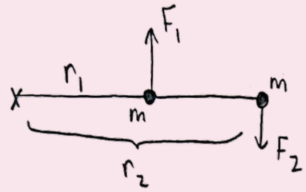
In this problem set, we've given some examples involving static, continuous, one-dimensional objects such as strings and ropes. The general three-dimensional theory of elasticity is mathematically quite complicated, but extremely important in engineering. For more about this subject, which requires comfort with tensors, see chapters 6 through 11 of Lautrup. It is also covered in chapters II-31, II-38, and II-39 of the Feynman lectures.

Remark: Why Use Torque?

Here's a seemingly naive question. Why is the idea of torque so incredibly useful in physics problems, even though in principle, everything can be derived from $F = ma$ alone? Why is it almost impossible to solve any nontrivial problem without referring to torques, and how would a student who's never heard of torque come up with it in the first place?

We don't need torque to analyze the statics of a single, featureless point particle. Torque only became useful in this problem set when we started analyzing rigid bodies with spatial extent. The reason we couldn't reduce torque balance to force balance easily is because the internal forces in these bodies, which maintain their rigidity, are generally very complicated.

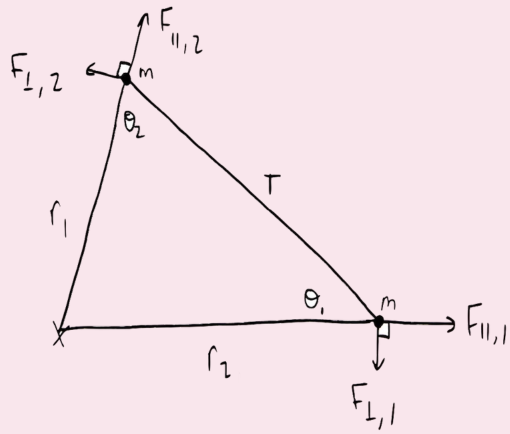
It's possible to derive torque balance from force balance in simple cases, but it's subtle. For example, suppose we try to make a rigid body by attaching two ideal springs of infinite spring constant to masses m and a pivot, as shown.



If forces F_1 and F_2 are applied, we would like to show that the setup if static only if the torques are balanced, $r_1 F_1 = r_2 F_2$, using only force balance on the masses. But our derivation fails at the very first step: the forces of ideal springs only point along the direction of the spring. Therefore, they can't balance the forces at all!

The problem is that this setup isn't actually rigid; it will bend at the first mass. We could make the setup rigid by replacing the spring with a rod, because, as we saw in example 2, a rigid rod can exert downward shear forces on the first mass and upward shear forces on the second. But in that case, it becomes nontrivial to decide when the *rod* can remain in equilibrium. You would need to decompose it microscopically, to see what is responsible for the shear forces in the first place, and balance forces on all those microscopic pieces. We tried to make a simple rigid body out of two masses, but now we need to keep track of the rod's infinitely many degrees of freedom!

Luckily, there's a simple modification of our original setup that works. Consider a triangle whose sides are made of ideal massless springs, pivoted at one vertex, as shown.



This actually *is* a rigid body, and the internal forces are simple: each spring just carries some tension. Let T be the tension in the spring between the two masses, opposite the pivot. The forces on each mass can be decomposed into a component parallel to the sides r_1 and r_2, and a component perpendicular to those sides. The former components can be balanced by adjusting the tensions in the other two springs. The perpendicular components can only be balanced by the tension T, from which we conclude

$$F_{\perp,1} = T \sin\theta_1, \quad F_{\perp,2} = T \sin\theta_2.$$

Upon eliminating T and using the law of sines, we conclude that force balance on the masses is only possible if $r_2 F_{\perp,1} = r_1 F_{\perp,2}$, which is precisely the statement of torque balance.

Tricky, right? This is the simplest example of a rigid body I know of, and in general, the internal forces can be much more complicated. The miracle of torque is that it automatically takes care of all those details for us, no matter what they are.

5 Pressure and Surface Tension

Example 10

A sphere of radius R contains a gas with a uniform pressure P. Find the total force exerted by the gas on one hemisphere.

Solution

The pressure provides a force per unit area orthogonal to the sphere's surface, so the straightforward way to do this is to integrate the vertical component of the pressure force over a hemisphere. However, there's a neat shortcut in this case.

Momentarily forget about the sphere and just imagine we have a sealed hemisphere of gas at pressure P. The net force of the gas on the hemisphere must be zero, or else it would just begin shooting off in some direction, violating conservation of momentum. So the force on the curved face must balance the force on the flat face, which is $\pi R^2 P$. The same logic must hold for the sphere, since the forces on the curved face are the same, so the answer is $\pi R^2 P$.

This trick works whenever one has a uniform outward pressure on a surface, and it'll come in handy for several future problems. For example, it's the quick way to do $F = ma$ 2018 B24.

Idea 11

The surface of a fluid carries a surface tension γ. If one imagines dividing the surface into two halves, then γ is the tension force of one half on the other per length of the cut. Specifically,

$$d\mathbf{F} = \gamma \, d\mathbf{s} \times \hat{\mathbf{n}}$$

which means the tension acts along the surface and perpendicular to the cut.

Example 11

A soap bubble of radius R and surface tension γ is in air with pressure P, and contains air with pressure $P + \Delta P$. Compute ΔP.

Solution

We use the result of the previous problem to conclude that the force of one hemisphere on another is $\pi R^2 \Delta P$. This must be balanced by the surface tension force. By imagining cutting the surface of the bubble in half, the surface tension force is γL where L is the total length of the surface connecting the hemispheres.

At this point, we can write $L = 2\pi R$, giving

$$\Delta P = \frac{2\gamma}{R}.$$

This is called the Young–Laplace equation. However, in this particular case, this is not the right answer. The reason is that we should actually take $L = 4\pi R$ because the surface tension is exerted at both the inside and outside surfaces of the bubble wall, and thus the answer is

$$\Delta P = \frac{4\gamma}{R}.$$

The increased pressure inside balances the surface tension, which wants to collapse the bubble.

If you're confused about why $L = 4\pi R$, you can also think about it in terms of energy. Surface tension arises from the fact that it costs energy to take soapy water and stretch it out into a surface, because this breaks some of the attractive intermolecular bonds. The Young–Laplace equation would give the correct answer for a *ball* of soapy water. But for a *bubble* of soapy water, twice as much soapy water/air surface is created. So the energy cost is double, and the force is double.

[2] **Problem 28.** One can also derive the Young–Laplace equation by just considering energy. Suppose the bubble radius changes by dr. The energy of the bubble changes for two reasons: first, there is net $\Delta P\, dV$ work from the two pressure forces, and there is the $\gamma\, dA$ surface tension energy cost. At equilibrium, the energy must be at a minimum, so it should not change at all under an infinitesimal displacement. Using this idea, rederive the Young–Laplace equation.

Solution. The work done by the surface tension should be balanced by the work done by the pressure difference. Noting that the total surface area is $8\pi R^2$, we have

$$\Delta P\, dV = \Delta P\, d\left(\frac{4}{3}\pi R^3\right) = \Delta P(4\pi R^2)\, dR = d(8\pi R^2\gamma) = 16\pi\gamma R\, dR$$

from which we conclude

$$\Delta P = \frac{4\gamma}{R}.$$

Remark

The general idea used in the above problem, of thinking about how energy would change in order to find an unknown force, is known as the principle of virtual work. The principle works for all kinds of forces. For example, if the bubble was charged, it would grow due to electrostatic repulsion, and the new equilibrium radius could be found using virtual work.

Remark

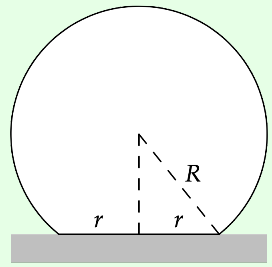
Note that the Young–Laplace equation we gave above only holds for spherical surfaces. More generally, a surface has two principle radii of curvature R_1 and R_2 at each point. These are both equal to R for a sphere of radius R, but for, e.g. a cylinder of radius R, one is equal to R and the other is infinity. For general surfaces, the Young–Laplace equation is

$$\Delta P = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where the R_i can each be positive or negative, depending on the direction of curvature.

Example 12

A solid ball of radius R, density ρ, and Young's modulus Y rests on a hard table. Because of its weight, it deforms slightly, so that the area in contact with the table is a circle of radius r.



Estimate r, assuming that it is much smaller than R.

Solution

Recall from **P1** that the Young's modulus is defined by

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{restoring force/cross-sectional area}}{\text{change in length/length}}$$

and has dimensions of pressure. By dimensional analysis, you can show that

$$r = R\, f(\rho g R/Y)$$

but dimensional analysis alone can't tell us anything more about f. Moreover, an exact analysis using forces would be very difficult, because different parts of the ball are compressed in different amounts, and in different directions; there's little symmetry here. Instead, we resort to an order of magnitude approach. To balance gravity, we need a typical stress of

$$\text{stress} \sim Mg/r^2 \sim \rho R^3 g/r^2$$

in the deformed region. This deformed region has typical length r. To estimate the strain, note that if the ball were not deformed at all, it would be standing a height $\delta \sim r^2/R$ taller, as you can show by the Pythagorean theorem. Thus,

$$\text{strain} \sim \delta/r \sim r/R.$$

Using the definition of the Young's modulus, we conclude

$$r \propto R \left(\frac{\rho g R}{Y}\right)^{1/3}.$$

By the way, there's a whole field of study devoted to figuring out how the normal and other forces behave for realistic, deformable solids, known as contact mechanics, which is essential in engineering. For an authoritative reference, see *Contact Mechanics and Friction* by Popov.