Mechanics I: Kinematics

See chapters 3 and 4 of Morin for material on solving differential equations. For general review on kinematics, see chapter 1 of Kleppner and Kolenkow. For fun, see chapters I-1 through I-8 of the Feynman lectures. There is a total of **87** points.

1 Motion in One Dimension

Example 1

When a projectile moves slowly through air, the drag is linear in the velocity, $F = -\alpha m v$. Find the velocity $v(t)$ of a projectile thrown upward at time $t = 0$ with speed v_0.

Solution

We write Newton's second law as

$$\frac{dv}{dt} = -g - \alpha v$$

and multiply through by dt. Integrating both sides from the initial condition to time t_f gives

$$\int_{v_0}^{v(t_f)} \frac{dv}{g + \alpha v} = -\int_0^{t_f} dt.$$

Performing the integrals gives

$$\frac{1}{\alpha} \log(g + \alpha v)\Big|_{v_0}^{v(t_f)} = -t_f.$$

Renaming t_f to t and solving for v yields

$$v(t) = e^{-\alpha t} v_0 + \frac{g}{\alpha}(e^{-\alpha t} - 1).$$

This renaming is necessary because we don't want to confuse t, the dummy variable that we integrating over, with t_f, the time at which we want to evaluate the velocity; t ranges from zero to t_f. Unfortunately, often people just call both of these t, so you need to watch out.

Example 2

Find how the speed of a rowing boat depends on the number of rowers N.

Solution

A fast-moving boat experiences quadratic friction, so a drag force

$$F \propto v^2 A$$

where A is the submerged cross-sectional area of the boat. Since the submerged volume scales as $V \propto N$ in hydrostatic equilibrium, we have $A \propto N^{2/3}$. (This is the sketchy step of the analysis, since the scaling of A depends on how we adjust the shape of the boat as N increases.) Thus, the power the rowers need to provide scales as $P = Fv \propto v^3 N^{2/3}$, but we also have $P \propto N$. Combining gives the exceptionally weak dependence $v \propto N^{1/9}$, which agrees decently with Olympic rowing times.

Idea 1

An ordinary differential equation is any equation involving a quantity $x(t)$ and its derivatives. In introductory physics, we are usually concerned with a few very simple differential equations, with the following nice properties.

- The differential equation is at most second order, meaning that it can contain x, $\dot{x} = v$, and $\ddot{x} = a$, but no higher derivatives. This implies that the solution can be determined by an initial position and initial velocity. (We'll focus on second order differential equations for the rest of this section; most first order differential equations can simply be solved by separation and integration, as we've already seen above.)
- The differential equation is linear, meaning that terms don't contain products of x, $\dot{x}$, and $\ddot{x}$. For example, a damped driven harmonic oscillator with time-dependent drag,

  $$m\ddot{x} = -b(t)\dot{x} - kx + f(t)$$

  is a second order linear differential equation. Solutions to such differential equations obey the superposition principle: if $x_1(t)$ and $x_2(t)$ are both solutions, so is $c_1 x_1(t) + c_2 x_2(t)$.
- The differential equation is homogeneous, meaning that each term is proportional to exactly one power of x or its derivatives. The above differential equation is not homogeneous, but it would be if we removed the driving $f(t)$.
- The differential equation is time-translation invariant, meaning that no functions of time appear except for x and its derivatives. The above equation isn't, but it would be if we set $f(t)$ and $b(t)$ to constants.

Idea 2

Linear, homogeneous, time-translation invariant differential equations are very special, and they can all be solved by the exact same method. First, note that we can promote $x(t)$ to a complex variable $\tilde{x}(t)$ and solve the differential equation over the complex numbers. As long as we have a complex solution, we can recover a real solution by taking the real part. Now, the method of solution, which works for *almost* all equations of this form, is to guess a complex exponential solution

$$\tilde{x}(t) = e^{i\omega t}.$$

Plugging this into the differential equation will yield the allowed values of ω, and the general solution can be found by superposing the complex exponentials.

Example 3

Solve the simple harmonic oscillator, $m\ddot{x} + kx = 0$, using the above principles.

Solution

First, we pass to a complex differential equation,

$$m\ddot{\tilde{x}} + k\tilde{x} = 0.$$

We guess $\tilde{x}(t) = e^{i\omega t}$. Plugging this in and using the chain rule gives

$$m(i\omega)^2 e^{i\omega t} + ke^{i\omega t} = 0$$

and canceling $e^{i\omega t}$ and solving gives two solutions,

$$\omega = \pm\omega_0, \quad \omega_0 = \sqrt{k/m}.$$

Since this a second-order linear differential equation, the general solution is given by the superposition of these two complex exponentials,

$$\tilde{x}(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

where A and B are general complex numbers. The real part of $\tilde{x}(t)$ satisfies the original real differential equation $ma + kx = 0$, and is

$$\operatorname{Re} x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

where C and D are real numbers.

2 Tricks

In this section we'll consider some kinematics problems that require cleverness, not computation.

Idea 3

Many problems can be solved by a clever choice of reference frame. It is often useful to go to the frame moving with one of the objects in the problem, or to go into a frame that makes the motion in the problem more symmetric. For the purposes of kinematics it can even be useful to use noninertial reference frames, such as a falling frame where projectiles don't accelerate, or a rotating frame, though this will introduce fictitious forces into the dynamics. It is also useful to tilt the coordinate axes to be parallel to various objects.

Example 4: $F = ma$ 2022 B4

A firework explodes, sending shells in all directions. Suppose the shells are all launched with the same speed, and ignore air resistance, but not gravity. What shape do the shells make?

Solution

In the absence of gravity, the shells would always form a sphere. Adding gravity simply shifts all of their locations downward by $gt^2/2$, so the shape is still always a sphere.

Idea 4

To find the minimum value of some quantity, it's often useful to think about all possible values of that quantity. This can reveal a solution using geometry or symmetry.

Idea 5

In problems with friction, the best reference frame to use is almost always the frame of whatever is causing the friction.

Idea 6

For a variety of kinematics problems, it can be useful to think about the motion from a different perspective. For example, if your problem involves complicated accelerations, it can be useful to think in "velocity space", i.e. directly think about how the velocity vector evolves over time, and deal with the position later. Or, if your problem involves complicated processes occurring in time, it can be useful to think in "spacetime", meaning to visualize the process on a space where time is one of the axes. It can also be useful to parametrize motion in terms of quantities other than the usual Cartesian coordinates.

Idea 7

Often, motion in two dimensions can be treated as two independent one-dimensional problems. A change of reference frame may be necessary first.

Idea 8

In problems involving an inclined plane, always set the angle θ to be much closer to either 0° or 90° than to 45°. This reduces mistakes, because almost every angle will be either θ or $90^\circ - \theta$, and you can identify which by sight.

Example 5

Consider projectile motion where wind provides a constant horizontal force F. At what angle should a projectile of mass m be launched in order to return to the thrower?

Solution

The key idea is to use tilted coordinate systems. Clearly, when the only force is downward, the projectile must be launched straight upward. Now, the horizontal force acts like an effective horizontal gravitational acceleration of F/m, so that gravity is effectively tilted an angle $\tan^{-1}(F/mg)$ away from the vertical. One must launch the projectile directly "upward" with respect to this effective gravitational field, so the launch angle is an angle $\tan^{-1}(F/mg)$ from the vertical. (For a related problem, see the infamous $F = ma$ 2014 problem 19.)

Example 6: $F = ma$ 2022 A23

For projectiles, the force of air resistance can be modeled as proportional to the speed ("linear drag") or proportional to the square of the speed ("quadratic drag"), depending on the circumstances. Two identical objects, A and B, are dropped from the same height h simultaneously, but object A is given an initial horizontal velocity v. The objects hit the ground at times t_A and t_B. How do these times compare, assuming linear or quadratic drag?

Solution

For linear drag, the horizontal and vertical components of the motion are independent,

$$a_x = -bv_x, \quad a_y = -g - bv_y$$

for some coefficient b. That means the time to hit the ground, which depends on the vertical motion, is independent of the initial horizontal velocity, so $t_A = t_B$. But for quadratic drag,

$$a_y = -g - bv_y|v|$$

which means the upward drag force is larger when the horizontal velocity is larger, so $t_A > t_B$.

Since the components are independent for linear drag, it's not too hard to write down an expression for the trajectory, by recycling the results of example 1. But for quadratic drag, the results of problem 2 won't help much; the two-dimensional problem is much harder.

Example 7

A bug flies towards a light with constant speed v, always making an angle α with the radial direction. If the initial distance to the lamp is L and the radius of the lamp is R, through what total angle does it turn before hitting the lamp?

Solution

In this case we can't avoid solving differential equations, but they're not too hard. It's easiest to work in polar coordinates, with the center of the lamp at the origin. By decomposing the velocity into radial and tangential components, we have

$$\frac{dr}{dt} = -v\cos\alpha, \quad r\frac{d\theta}{dt} = v\sin\alpha.$$

We only care about the path, not the time-dependence, so we divide these equations to get

$$\frac{dr}{d\theta} = -\frac{r}{\tan\alpha}$$

where we manipulated differentials as in **P1**. Separating and integrating,

$$-\int_L^R \frac{dr}{r} = \frac{\Delta\theta}{\tan\alpha}$$

which tells us that

$$\Delta\theta = (\tan\alpha)\log\frac{L}{R}.$$

The shape traced out is a logarithmic spiral.

4 Optimal Launching

Finally, we'll consider projectile motion questions that involve optimization. These are rare on the USAPhO, but they are quite fun problems, with occasionally very slick solutions.

Example 8

A bug wishes to jump over a cylindrical log of radius R lying on the ground, so that it just grazes the top of the log horizontally as it passes by. What is the minimum launch speed v required to do this?

Solution

Let P be the point at the top of the log. For the bug to be moving horizontally at P, energy conservation applied to the vertical motion gives an initial v_y obeying

$$\frac{1}{2}mv_y^2 = 2mgR, \quad v_y = 2\sqrt{gR}.$$

Thus, we need to find the minimum v_x for the motion to be possible. If v_x is too low, the hypothetical trajectory of the bug will instead pass through the log. At the lowest possible v_x, the bug's trajectory is not just tangent to the log at point P, but also has the same radius of curvature (i.e. the trajectory and the log's shape have the same first and second derivatives).

For uniform motion in a circle of radius r, the acceleration is $a = v^2/r$. Conversely, when an object follows a trajectory of instantaneous radius of curvature r, its acceleration component normal to the path must be $a = v^2/r$. So applying this to the bug at P gives

$$g = \frac{v_x^2}{R}, \quad v_x = \sqrt{gR}.$$

Thus, the minimum initial speed is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5}\, gR.$$

This radius of curvature trick doesn't come up often, but it's cool when it does.

Idea 9

Since mechanics is time-reversible, and the speed of a projectile only depends on its height and not the path taken, finding the way to reach point B from point A with the lowest possible initial speed is the same as finding the way to reach point A from point B with the lowest possible initial speed.