Electromagnetism I: Electrostatics

The material here is covered at the right level in chapters 1–3 of Purcell. For a separate introduction to vector calculus, see the resources mentioned in the syllabus, or chapter 1 of Griffiths. Electrostatics is covered in more mathematical detail in chapter 2 of Griffiths. For interesting general discussion, see chapters II-1 through II-5 of the Feynman lectures. There is a total of **80** points.

1 Coulomb's Law and Gauss's Law

We'll begin with some basic problems which can be solved with symmetry arguments.

Idea 1

Gauss's law is written in integral form as

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0}.$$

In practice, you will only apply this form to situations with high symmetry, where

$$E = \begin{cases} Q/4\pi\epsilon_0 r^2 & \text{spherical symmetry,} \\ \lambda/2\pi\epsilon_0 r & \text{cylindrical symmetry,} \\ \sigma/2\epsilon_0 & \text{infinite plane.} \end{cases}$$

Example 1

Consider a spherical shell of uniform surface charge density σ. A small hole is cut out of the surface of the shell. What is the electric field at the center of this hole?

Solution

We use the principle of superposition. First, consider the entire spherical shell, without a hole. By Gauss's law and spherical symmetry, the radial electric field at a point P infinitesimally outside the sphere is σ/ϵ_0, while the electric field at a nearby point P' infinitesimally inside is zero.

This field is the superposition of the fields of the charges near P and P', and charges from the entire rest of the sphere. Consider the effect of a small piece of the surface, near P and P'. From the perspective of these points, this piece looks like an infinite plane, so its radial electric field is $\sigma/2\epsilon_0$ at P, and $-\sigma/2\epsilon_0$ at P'. Therefore, the entire rest of the sphere must contribute a radial electric field of $\sigma/2\epsilon_0$, at both P and P'. Therefore, when one cuts out a hole, this is the only contribution that remains, so the field is just $\sigma/2\epsilon_0$.

[2] **Problem 1** (Griffiths 2.18)**.** Some questions about uniformly charged spheres.

(a) Consider a sphere of radius R and uniform charge density ρ. Find the electric field everywhere.

(b) Now two spheres, each of radius R and carrying uniform charge densities ρ and $-\rho$, are placed so that they partially overlap. Call the vector from the positive center to the negative center $\mathbf{d}$. Find the electric field in the overlap region.

Solution. (a) The field inside a uniform sphere of density ρ and center $\mathbf{a}$ is

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(\mathbf{r} - \mathbf{a}).$$

Outside the sphere, the field falls off as an inverse square,

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}\frac{R^3}{|\mathbf{r} - \mathbf{a}|^3}(\mathbf{r} - \mathbf{a}).$$

(b) If the two centers are $\mathbf{a}_1$ and $\mathbf{a}_2$, then by superposition,

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}((\mathbf{r} - \mathbf{a}_1) - (\mathbf{r} - \mathbf{a}_2)) = \frac{\rho}{3\epsilon_0}\mathbf{d}$$

which is a constant.

[2] **Problem 2.** A charge q sits just inside a cube, next to one of the corners. What is the flux through each face of the cube? (More precisely, if a corner is at the origin, and the sides are parallel to the x, y, and z axes, let the charge be at coordinates $(\epsilon, \epsilon, \epsilon)$ for tiny ϵ.)

Solution. There are three "opposite" faces with the same flux, and three "adjacent" faces with the same flux. Now consider adding seven more cubes, so that the charge is now at the center of a $2 \times 2 \times 2$ cube. The total flux through the outer faces of the cube is q/ϵ_0, and there are 24 unit faces, so the flux out of each "opposite" face is $q/24\epsilon_0$. Now consider the original cube. By Gauss's law the total flux out must be q/ϵ_0, which means the flux out of each "adjacent" face is $7q/24\epsilon_0$.

(Note that if the charge were instead *exactly* at one of the corners, the fluxes through the opposite faces would still be $q/24\epsilon_0$, while the fluxes though the adjacent faces would technically be undefined, since the electric field blows up on the face. But roughly speaking, the flux ought to be zero. Then the total flux out of the cube is only $q/8\epsilon_0$, and that's because the corner cuts out one "octant" of the point charge's field.)

[2] **Problem 3** (BAUPC)**.** In both parts below, take the potential to be zero at infinity.

(a) Consider a solid sphere of uniform charge density. Find the ratio of the electrostatic potential at the surface to that at the center.

(b) Consider a solid cube of uniform charge density. Find the ratio of the electrostatic potential at a corner to that at the center. (Hint: use symmetry.)

Solution. (a) Let the uniform charge density be ρ and the sphere have radius R, so the total charge is $Q = \frac{4}{3}\pi\rho R^3$. We can treat the field outside the sphere to be like a point charge, so the potential at the surface relative to infinity is $U_0 = Q/4\pi\epsilon_0 R$.

To go from the surface to the center, we need to go against the field lines and change the potential by $\Delta U = -\int_R^0 E(r)\,dr$. The electric field inside the sphere can be found with Gauss's law inside the sphere: $E = \frac{4}{3}\pi\rho r^3/4\pi\epsilon_0 r^2$.

$$\Delta U = \int_0^R \frac{kQr}{R^3}dr = \frac{1}{2}U_0$$

Thus the potential at the center of the sphere is $U_0 + \frac{1}{2}U_0 = \frac{3}{2}U_0$, so the ratio of the potential at the surface to that at the center is $\frac{2}{3}$.

(b) Being at the center of the cube is like being at the corner of 8 identical cubes with half the length. From $U \sim kQ/r \sim k\rho r^2$, we see that the potential is proportional to the square of the length scale. Let the potential at the corner be U_0. For each cube with half the length, the potential from that cube is $\frac{1}{4}U_0$. With eight of those half cubes at the center, the potential at the center of the cube is $2U_0$. So the ratio of the potentials at corner to center is $\frac{1}{2}$.

Idea 2

If you follow an electric field line, the potential monotonically decreases along it.

[2] **Problem 4.** Two questions about electrostatic equilibrium.

(a) Prove that when a system of point charges is in equilibrium (i.e. the net force on *each* of the charges due to the others vanishes), the total potential energy of the system is zero.

(b) Show that for a positive point charge in the electric fields of fixed, positive point charges, there is a path along which the charge can be moved to infinity without ever needing positive external work, i.e. a path along which the potential only decreases.

Solution. (a) Fix some point O not on any of the charges, and scale the system up about O continuously, to send all the charges to infinity. At all points in time, there are no forces on any of the charges, so no work is done. The final potential energy is zero, so the initial potential energy must also have been zero.

(b) Consider the field line going through the test charge. It can't end on a negative charge, since there are none, so it must end at infinity. Moving the charge along this field line gives the desired path.

Idea 3

Gauss's law is written in differential form as

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

The divergence of a vector field $\mathbf{F} = F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}$ is

$$\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$$

in Cartesian coordinates, where ∂_x stands for $\partial/\partial x$, and so on.

Example 2

Show that the two forms of Gauss's law are equivalent.

Solution

To do this, we need to establish the geometric meaning of the divergence. For simplicity we consider two dimensions; the proof for three dimensions is similar. Consider a small rectangle prism with one corner at the origin, with axes aligned with the Cartesian coordinate axes and side lengths Δx and Δy. To apply Gauss's law in integral form, we need to compute the flux through each side. The flux going out the top side is

$$\int_0^{\Delta x} E_y(x, \Delta y)\, dx$$

while the flux going out the bottom side is

$$-\int_0^{\Delta x} E_y(x, 0)\, dx.$$

The sum of these two terms is

$$\int_0^{\Delta x} (E_y(x, \Delta y) - E_y(x, 0))\, dx \approx \Delta y \int_0^{\Delta x} (\partial_y E_y)|_{(x,0)}\, dx$$

where we applied a tangent line approximation, and the subscript indicates where the function $\partial_y E_y$ is evaluated. Higher-order terms in the Taylor series would be proportional to higher powers of Δy, which is small, so we can ignore them.

The integrand is still a function of x, but we can Taylor expand it about the origin as

$$(\partial_y E_y)|_{(x,0)} = (\partial_y E_y)|_{(0,0,0)} + \Delta x(\dots) + \dots.$$

These extra terms are again higher-order in Δx and Δy, so we ignore them. The net flux through the top and bottom faces is hence, to lowest order,

$$\Delta y \int_0^{\Delta x} (\partial_y E_y)|_{(0,0)}\, dx = \Delta x \Delta y\, (\partial_y E_y)|_{(0,0)}.$$

By similar reasoning, pairing up the left and right faces gives

$$\text{flux} = \Delta x \Delta y\, (\partial_x E_x + \partial_y E_y)|_{(0,0)} = \Delta x \Delta y\, (\nabla \cdot \mathbf{E})|_{(0,0)}.$$

Thus the divergence is the outgoing flux per unit area, or volume in three dimensions.

This shows us why the two forms of Gauss's law area equivalent. For example, starting from the differential form, the left-hand side is the flux per volume, while the right-hand side is the charge per volume, divided by ϵ_0. Integrating both sides over some volume relates the total flux to the total charge divided by ϵ_0, which is Gauss's law in integral form.

If the above derivation was a bit abstract, we can also show the idea using specific examples.

Example 3

Suppose the region $0 < x < d$ has charge density $-\rho$, and the region $-d < x < 0$ has charge density ρ. Find the electric field everywhere.

Solution

By translational symmetry, the field always points along $\hat{\mathbf{x}}$ and only depends on x, $\mathbf{E}(\mathbf{r}) = E(x)\,\hat{\mathbf{x}}$. By applying the integral form of Gauss's law to a rectangular prism, with one side at x_l and another at x_r, we have

$$E(x_r) - E(x_l) = \frac{1}{\epsilon_0}\int_{x_l}^{x_r} \rho(x)\,dx, \quad E(x) = \frac{1}{\epsilon_0}\int_0^x \rho(x)\,dx + E_0.$$

Since the divergence of $\mathbf{E}(\mathbf{r})$ is just $\partial E(x)/\partial x$, this clearly satisfies the differential form of Gauss's law. To fix the undetermined constant E_0, we could demand the field be zero on both sides of the charge distribution, motivated by symmetry. Then we have

$$E(x) = \frac{\rho}{\epsilon_0} \times \begin{cases} d - x & 0 < x < d, \\ d + x & -d < x < 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Example 4

Find the electric field of a spherically symmetric charge density $\rho(r)$.

Solution

By spherical symmetry, the field always points radially and only depends on r, $\mathbf{E}(\mathbf{r}) = E(r)\,\hat{\mathbf{r}}$. By applying the integral form of Gauss's law to a sphere of radius r,

$$4\pi r^2 E(r) = \frac{1}{\epsilon_0}\int_0^r dr'\,4\pi r'^2 \rho(r'), \quad E(r) = \frac{1}{\epsilon_0}\frac{1}{r^2}\int_0^r dr'\,r'^2\rho(r').$$

Let's check that this indeed satisfies the differential form of Gauss's law, using the divergence in spherical coordinates. For any vector field $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_\varphi\hat{\boldsymbol{\varphi}}$, the divergence is

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}.$$

This looks complicated, but things turn out simple because $\mathbf{E}$ only has a radial component, $E_r = E(r)$, which gives

$$\nabla\cdot\mathbf{E} = \frac{1}{r^2}\frac{\partial(r^2E(r))}{\partial r} = \frac{1}{r^2\epsilon_0}\frac{\partial}{\partial r}\int_0^r dr'\,r'^2\rho(r') = \frac{r^2\rho(r)}{r^2\epsilon_0} = \frac{\rho(r)}{\epsilon_0}$$

just as desired.

[3] **Problem 5.** Consider a vector field expressed in polar coordinates, $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$ where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the radial and tangential directions. Gauss's law in differential form still works in these coordinates, but the form of the divergence is different.

By considering the flux per unit area out of a small region bounded by r and $r + dr$, and θ and $\theta + d\theta$, and applying Gauss's law in integral form, find what the divergence in polar coordinates must be for Gauss's law in differential form to hold. (Optional: try generalizing to spherical coordinates.)

Solution. By summing up contributions from each of the four sides, and letting (F_r, F_θ) be the vector field at one of the corners, the flux through the region is

$$d\Phi = (F_r + dF_r)((r + dr)d\theta) - F_r(rd\theta) + (F_\theta + dF_\theta)dr - F_\theta dr.$$

In two dimensions, the divergence is the flux per area, $dA = r\,dr\,d\theta$, so

$$\nabla \cdot \mathbf{F} = \frac{d\Phi}{dA} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}.$$

[4] **Problem 6.** This problem is quite subtle, but will enhance your understanding of electromagnetism. Suppose that all of space is filled with uniform charge density ρ.

(a) Show that $\mathbf{E} = (\rho/\epsilon_0)x\hat{\mathbf{x}}$ obeys the differential form of Gauss's law.

(b) Show that $\mathbf{E} = (\rho/3\epsilon_0)r\hat{\mathbf{r}}$ also obeys Gauss's law.

(c) Argue that by symmetry, $\mathbf{E} = 0$. Show that this does not obey Gauss's law.

(d) ⋆ What's going on? Which, if any, is the actual field? If you think there's more than one possible field, how could that be consistent with Coulomb's law, which gives the answer explicitly? For that matter, what does Coulomb's law say about this setup, anyway?

Solution. (a) We see that $\nabla \cdot \mathbf{E} = \partial_x((\rho/\epsilon_0)x) = \rho/\epsilon_0$, as desired.

(b) In Cartesian coordinates, this field is

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

whose divergence is ρ/ϵ_0, as desired.

(c) This has to hold by symmetry because the electric field can't point in any particular direction, by rotational symmetry. It also can't just point radially, because that breaks translational symmetry; the center is a special point. So the only option is $\mathbf{E} = 0$, but $\nabla \cdot \mathbf{E} = 0$, so Gauss's law is not obeyed.

(d) The issue is *boundary conditions*. Just like any differential equation, the solution for the electric field is not defined without boundary conditions (or initial conditions, as we called them in mechanics). Usually, we get a unique solution by demanding the fields go to zero at infinity. (Though in some cases, this might not be the right physical answer. For example, the electric field of a capacitor in the lab doesn't have to be zero outside, because it might be inside some bigger capacitor.) However, we can't do this here because the charge density goes to infinity too. By taking different choices of boundary conditions, we can get (a), (b), or many other answers. The symmetry argument in (c) fails, because any choice of boundary conditions will break the perfect translational symmetry.

At first glance, it could seem that Coulomb's law could give us a unique answer. Coulomb's law for a point charge is itself derived by implicitly assuming that there are no "extra" fields flying around, just the spherically symmetric field of the point charge itself. This looks very reasonable, so what stops us from just saying that each charge in this problem has such a field, and then integrating over the charges? Well, if you write down the integral, you'll find that it's divergent, analogous to the integral $\int_{-\infty}^{\infty} x\,dx$. By itself, the integral is not even well-defined.

In order to get an answer, you have to "regulate" the integral (i.e. change it in a way that makes it well-defined). One possible regulator, for example, is to just chop off the limits of integration at finite values, like $\int_{-L}^{L} x\,dx$. But that particular regulator is equivalent to just replacing the charge distribution with a finite one centered at the origin! In other words, Coulomb's law also fails to give a unique answer, because it requires a regulator to give a well-defined answer, and there are many possible regulators. If you treat the charge distribution as a giant ball with center at the origin, you get the result of part (b). If you treat it as a thick, huge slab along the yz plane centered at the origin, you get the result of part (a). The symmetry argument fails once again, because all the regulators break translational symmetry. This is a simple example of an "anomalous symmetry", an important idea in theoretical physics.

The exact same problem applies to Newtonian cosmology, where charge density is replaced with mass density, and this problem confused Newton himself, who incorrectly thought that $\mathbf{g} = 0$ by symmetry. In this context, *all* regulators/boundary conditions are unsatisfactory. Of course, we want a rotationally symmetric universe to match experiment, so we have to put that in by hand. But then every solution has a center towards which everything collapses, so to keep the solar system an inertial frame, we'd have to put it at the center of the universe! Surely, this would make Copernicus roll in his grave.

Some of these problems are fixed in general relativity. You still have to postulate rotational symmetry (again, on the basis of experimental data), but once you do that, there are no further problems. In general relativity, acceleration is not absolute in the way it is in Newtonian mechanics. Instead, there is no center; everything just gets closer to everything else. For further discussion and references, see this paper.

Idea 4

A tricky, occasionally useful idea is to use Newton's third law: it may be easier to calculate the force of A on B than the force of B on A.

Example 5: Purcell 1.28

Consider a point charge q. Draw any imaginary sphere of radius R around the charge. Show that the average of the electric field over the surface of the sphere is zero.

Solution

Imagine placing a uniform surface charge σ on the sphere. Then the average of the point charge's electric field over the sphere times $4\pi R^2\sigma$ is the total force of the point charge on the charged sphere. But this is equal in magnitude to the force of the charged sphere on the point charge, which must be zero by the shell theorem. Thus the average field over the sphere has to vanish.

[3] **Problem 7** (Purcell 1.28)**.** Some extensions of the previous example.

(a) Show that if the charge q is instead outside the sphere, a distance $r > R$ from its center, the average electric field over the surface of the sphere is the same as the electric field at the center of the sphere.

(b) Show that for any overall neutral charge distribution contained within a sphere of radius R, the average electric field over the interior of the sphere is $-\mathbf{p}/4\pi\epsilon_0 R^3$ where $\mathbf{p}$ is the total dipole moment.

Solution. The same Newton's third law trick will work for both parts.

(a) Let the desired answer be $\mathbf{E}_{\text{avg}}$ and let the charge q be at $\mathbf{r}$. Now imagine a charge Q is uniformly distributed over the surface of the sphere. The force of the charge q on the distributed charge Q is precisely $\mathbf{F}_{qQ} = Q\mathbf{E}_{\text{avg}}$. But we also know that

$$\mathbf{F}_{qQ} = -\mathbf{F}_{Qq} = -\frac{kQq}{r^2}\hat{\mathbf{r}}$$

by Newton's third law and the shell theorem. Therefore we have

$$\mathbf{E}_{\text{avg}} = -\frac{kq}{r^2}\hat{\mathbf{r}}$$

which is precisely the electric field at the center of the sphere due to q. (Note that $\hat{\mathbf{r}}$ points from the center of the sphere to the charge q.)

(b) Let the desired answer be $\mathbf{E}_{\text{avg}}$. Now imagine a charge Q is uniformly distributed over the volume of the sphere. The force of the charge distribution (with charge density $\rho(\mathbf{x})$) on the distributed charge Q is precisely $\mathbf{F}_{qQ} = Q\mathbf{E}_{\text{avg}}$. But we also know that

$$\mathbf{F}_{qQ} = -\mathbf{F}_{Qq} = -\int \rho(\mathbf{r})\mathbf{E}_Q(\mathbf{r})\, d\mathbf{r}$$

where $\mathbf{E}_Q$ is the field due to Q. Now, this field is easy to find, as it is just the field of a uniformly charged sphere, so

$$\mathbf{E}_Q = \frac{kQ}{R^3}\mathbf{r}$$

as shown in problem 1. Putting this in the integral, we have

$$Q\mathbf{E}_{\text{avg}} = -\frac{kQ}{R^3}\int \rho(\mathbf{r})\mathbf{r}\, d\mathbf{r}$$

so by the definition of the dipole moment,

$$\mathbf{E}_{\text{avg}} = -\frac{k}{R^3}\int \rho(\mathbf{r})\mathbf{r}\, d\mathbf{r} = -\frac{k\mathbf{p}}{R^3}$$

as desired.

[3] **Problem 8.** There are two point charges, $q_1 > 0$ and $q_2 < 0$, in empty space. An electric field line leaves q_1 at an angle α from the line connecting the two charges. Determine whether this field line hits q_2, and if so, at what angle β from the line connecting the two charges. (Hint: this can be done without solving any differential equations.)

Solution. Suppose the field line does hit q_2. Rotate the field line about the line connecting the two charges, to form a Gaussian surface. Because no electric field lines go across this surface, the total charge inside must be zero. Now, this surface envelopes "slices" of each point charge. (If you're not happy with "slicing a point charge", just replace the point charges with tiny uniformly charged spheres; everything stays the same.) The solid angle of the first point charge enveloped is

$$\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\alpha} \sin\theta \, d\theta = 2\pi(1 - \cos\alpha)$$

so the amount of charge enclosed is

$$\frac{\Omega}{4\pi} q_1 = \frac{1 - \cos\alpha}{2} q_1 = q_1 \sin^2 \frac{\alpha}{2}.$$

Reasoning similarly for the other surface, we have

$$q_1 \sin^2 \frac{\alpha}{2} = |q_2| \sin^2 \frac{\beta}{2}$$

and the field line hits q_2 if there is a solution for β, i.e. when $|q_1/q_2| \sin^2(\alpha/2) < 1$.

Idea 5

The integral $\int d\mathbf{S}$ over a surface with a fixed boundary is independent of the surface.

We proved this in a mechanical way in **M2**. If you want to see a proof using vector calculus, see problem 1.62 of Griffiths.

[3] **Problem 9.** A hemispherical shell of radius R has uniform charge density σ and is centered at the origin. Find the electric field at the origin. (Hint: combine the previous two ideas.)

Solution. Place a point charge q at the origin. To find the magnitude of the field, we will compute the force on the hemisphere divided by q. The force on the hemisphere is

$$\int \frac{q}{4\pi\epsilon_0 R^2} \sigma \, d\mathbf{S} = \frac{q\sigma}{4\pi\epsilon_0 R^2} \int d\mathbf{S}.$$

By idea 5, we can replace the surface of integration with a flat disk, so $|\int d\mathbf{S}| = \pi R^2$. Thus, the force is $F = q\sigma/4\epsilon_0$, so the field is

$$E = \frac{\sigma}{4\epsilon_0}.$$

[3] **Problem 10.** A point charge q is placed a distance $a/2$ above the center of a square of charge density σ and side length a. Find the force of the square on the point charge.

Solution. This is a tricky problem, whose solution uses a one-time trick. It's equivalent to find the force of the point charge on the square. Set up coordinates so that the square is in the xy plane, and its center is the origin. Then we have

$$\mathbf{F} = \sigma \int \mathbf{E} \, dS$$

where the surface integral is over the square. On the other hand, we know that $\mathbf{F}$ is along the $\hat{\mathbf{z}}$ direction by symmetry, so

$$F = \mathbf{F} \cdot \hat{\mathbf{z}} = \sigma \int E_z \, dS.$$

Now, since $d\mathbf{S}$ is parallel to $\hat{\mathbf{z}}$, this is in fact the same thing as

$$F = \sigma \int \mathbf{E} \cdot d\mathbf{S}$$

where the integral is just the electric flux through the square! By symmetry, this flux is $q/6\epsilon_0$, so

$$F = \frac{\sigma q}{6\epsilon_0}.$$

[4] **Problem 11** (Griffiths 2.47, PPP 113, MPPP 140)**.** Consider a uniformly charged spherical shell of radius R and total charge Q.

(a) Find the net electrostatic force that the southern hemisphere exerts on the northern hemisphere.

(b) Generalize part (a) to the case where the sphere is split into two parts by a plane whose minimum distance to the sphere's center is h.

(c) Generalize part (a) to the case where the two hemispherical shells have uniform charge density, opposite orientation, and the same center, but have different total charges q and Q, and different radii r and R, where $r < R$.

Hint: see example 9, and use superposition and symmetry when applicable.

Solution. (a) The net force that the northern hemisphere exerts on itself is 0, so it is equivalent to find the force on the north due to the entire sphere. The surface charge density is $\sigma = Q/(4\pi R^2)$. By the result of example 9, the outward pressure on the northern hemisphere is $\sigma^2/2\epsilon_0$. Therefore, the total force is

$$F = \frac{\sigma^2}{2\epsilon_0} \int_N d\mathbf{S} = \frac{\sigma^2}{2\epsilon_0}(\pi R^2) = \frac{Q^2}{32\pi\epsilon_0 R^2}$$

where N refers to the northern hemisphere, and the surface integral was done as in problem 9.

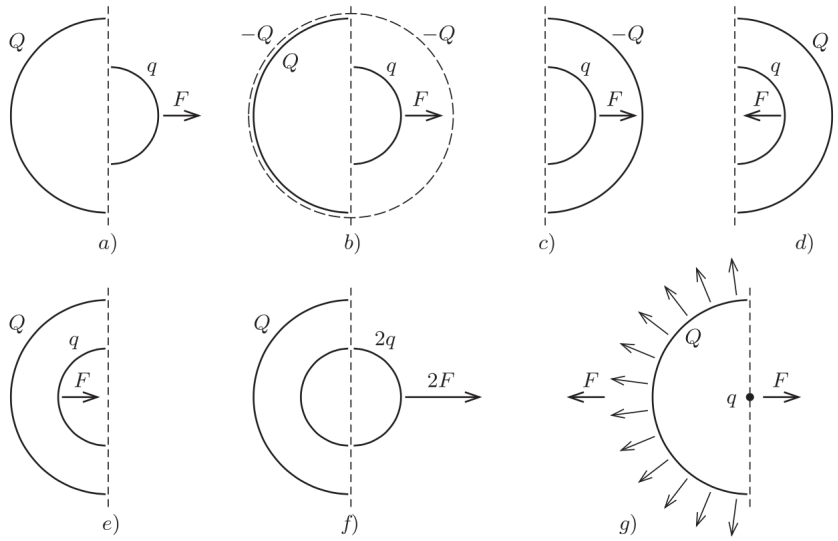
(b) This is exactly the same as in part (a), except that now the integral over the piece is

$$\int d\mathbf{S} = \pi(R^2 - h^2)$$

which gives the result

$$F = (\sigma^2/2\epsilon_0)\pi(R^2 - h^2) = \frac{Q^2}{32\pi\epsilon_0 R^2}\,(1 - h^2/R^2).$$

(c) This can be solved using an ingenious superposition and symmetry argument.



The force we want to compute is shown in (a). Now consider superposing a uniformly negatively charged sphere with radius just larger than R, as shown in (b). By the shell theorem, this doesn't change the force on the hemisphere of radius r. The result of the superposition is (c). Flipping the charge of one of the hemispheres in (c) flips the force, leading to (d). Finally, reflecting (d) gives (e).

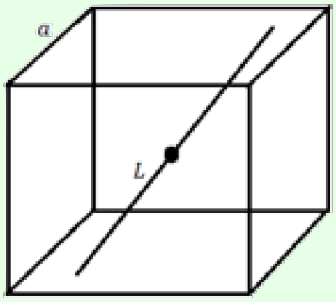
This has all been preamble to the ingenious step: superpose (a) and (e) to get (f), which involves the force on a *complete* sphere of radius r. Using Newton's third law, $2F$ can now be computed by finding the force on the hemisphere. But that is easy because of the shell theorem, which tells us that F is the net force on the hemisphere shown in (g). Using the method of problem 9 again, we conclude

$$F = \frac{q}{4\pi\epsilon_0 R^2}(\pi R^2)\frac{Q}{2\pi R^2} = \frac{Qq}{8\pi\epsilon_0 R^2}$$

which is independent of r! (Setting $r = 0$ and $r = R$ recovers the answers to two previous problems.)

Example 6: IdPhO 2020.1A

A point charge of mass m and charge $-q$ is placed at the center of a cube with side length a, whose volume has uniform charge density ρ. The point charge is allowed to slide along a straight line, which has an arbitrary orientation, so that the distance along the line from the center to one of the cube's faces is L.



Find the frequency of small oscillations.

Solution

The official solution goes as follows: consider displacing the point charge by some small amount $\Delta\mathbf{r}$. The cube of charge can then be decomposed into (1) a slightly smaller cube of charge centered around the point charge's new position, and (2) three thin plates of charge on the faces opposite to the charge's motion. By symmetry, (1) contributes nothing, and we know what (2) contributes from the answer to problem 10. The result is a restoring force proportional to $-\Delta\mathbf{r}$, whose magnitude has no dependence on the orientation of $\Delta\mathbf{r}$, so the oscillation frequency doesn't depend on L. Once you know this, you can orient the line any way you want, so the problem is simple to finish.

Personally, I don't like this problem because the intended solution requires knowing the answer to problem 10, which itself is pretty tricky. That is, the difficulty of the problem depends mostly on whether you've seen that tough, but standard problem elsewhere. However, I'm including it as an example because there's another way to solve it, which is more advanced, but quite illustrative.

Since this is a question about small oscillations, it suffices to expand the potential energy to second order about the center of the cube. The most general possible exprssion is

$$V(x,y,z) = a + b_1x + b_2y + b_3z + c_1x^2 + c_2y^2 + c_3z^2 + c_4xy + c_5yz + c_6xz + O(r^3).$$

The constant a doesn't matter, so we can just ignore it. And since $\mathbf{E}$ vanishes at the center, the linear terms b_i are all zero as well. Because the x, y, and z axes are all equivalent by cubical symmetry (e.g. we can rotate them into each other, while keeping the cube the same),

$$c = c_1 = c_2 = c_3, \quad c' = c_4 = c_5 = c_6.$$

Thus, our complicated Taylor series boils all the way down to

$$V(x,y,z) = c(x^2 + y^2 + z^2) + c'(xy + yz + xz) + O(r^3)$$

without even having to do any work! Finally, notice that the cube is symmetric under reflections $x \to -x$, $y \to -y$, or $z \to -z$. These reflections keep the c term the same, but flip the c' term. Therefore, we must have $c' = 0$, so

$$V(r) = cr^2 + O(r^3)$$

which is remarkably simple. The potential near the origin is spherically symmetric, even though the setup as a whole isn't! It's not automatic: it wouldn't have been this simple if we had had a slightly more complex shape. This "accidental" spherical symmetry is a consequence of the combination of cubical symmetry and the simplicity of Taylor series.

Therefore, to finish the problem we only need to find the coefficient c. While there are simpler ways to do this, I'll do it in a way that introduces some useful facts. Combining the definition of V and Gauss's law, we have

$$\nabla \cdot (\nabla V) = -\nabla \cdot \mathbf{E} = -\frac{\rho}{\epsilon_0}.$$

This is a standard and fundamental result in electrostatics, called Poisson's equation, which we will see again later. The divergence of a gradient is also called a Laplacian, and written as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon_0}.$$

Using this, we can easily compute the value of c, giving

$$V(\mathbf{r}) = -\frac{\rho r^2}{6\epsilon_0} + O(r^3).$$

Therefore, for a displacement $\Delta\mathbf{r}$ in *any* direction, the restoring force is $\rho q r/3\epsilon_0$ in the opposite direction, which means

$$\omega = \sqrt{\frac{\rho q}{3\epsilon_0 m}}$$

independent of the orientation of the line.

Remark

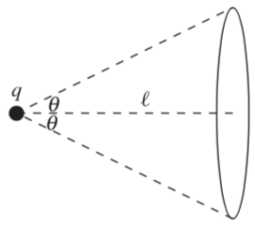
Accidental symmetry is important in modern physics. For example, protons are stable because of an accidental symmetry in the Standard Model, which ensures that baryon number is conserved. That explains why we often expect proton decay to occur in extensions of the Standard Model, such as grand unified theories, as explained in this nice article.

2 Continuous Charge Distributions

Idea 6

In almost all cases in Olympiad physics, there will be sufficient symmetry to reduce any multiple integral to a single integral. Remember that when using Gauss's law, the Gaussian surface may be freely deformed as long as it doesn't pass through any charges.

[2] **Problem 12** (Purcell 1.15)**.** A point charge q is located at the origin. Compute the electric flux that passes through a circle a distance ℓ from q, subtending an angle 2θ as shown below.



Solution. Let $\ell = R\cos\theta$, and deform the disk into a spherical cap with radius R. Then the answer is then just kq/ϵ_0, where k is the ratio of the area of the cap to the total area of the sphere. In spherical coordinates,

$$k = \frac{1}{4\pi}\int_0^\theta 2\pi \sin\theta\, d\theta = \frac{1-\cos\theta}{2}$$

so the answer is

$$\frac{1-\cos\theta}{2}\frac{q}{\epsilon_0}.$$

You can also show this using the original flat Gaussian surface, though that takes more work.

[3] **Problem 13** (Purcell 1.8)**.** A ring with radius R has uniform positive charge density λ. A particle with positive charge q and mass m is initially located in the center of the ring and given a tiny kick. If the particle is constrained to move in the plane of the ring, show that it exhibits simple harmonic motion and find the frequency.

Solution. Suppose it is moved by $r \ll R$ in the x direction. Set up polar coordinates with $\theta = 0$ being the positive x axis. By the law of cosines, we have

$$\begin{aligned} U(r) &= 2\int_0^\pi \frac{1}{4\pi\epsilon_0}\frac{q(\lambda R d\theta)}{\sqrt{R^2+r^2-2Rr\cos\theta}} \\ &= \frac{q\lambda}{2\pi\epsilon_0}\int_0^\pi \frac{d\theta}{\sqrt{1+(r^2/R^2)-2(r/R)\cos\theta}}. \end{aligned}$$

Next, we can expand the square root using a Taylor series. If we expand to first order in r/R, then the result will be proportional to the integral of $\cos\theta$, which vanishes. Thus, to get the leading contribution we must expand to second order, giving

$$\begin{aligned} U(r) &= \frac{q\lambda}{2\pi\epsilon_0}\int_0^\pi \left[1 - \frac{1}{2}\frac{r^2}{R^2} + \frac{3}{8}\left(-\frac{2r}{R}\cos\theta\right)^2\right] d\theta \\ &= \frac{q\lambda}{2\pi\epsilon_0}\int_0^\pi \frac{r^2}{2R^2}(3\cos^2\theta - 1)\, d\theta + \text{const} \\ &= \frac{q\lambda r^2}{8\epsilon_0 R^2} + \text{const}. \end{aligned}$$

Therefore, the effective spring constant is $k = q\lambda/4\epsilon_0 R^2$, so

$$\omega = \sqrt{\frac{q\lambda}{4m\epsilon_0 R^2}}.$$

[3] **Problem 14** (Purcell 1.12)**.** Consider the setup of problem 9. If the hemisphere is centered at the origin and lies entirely above the xy plane, find the electric field at an arbitrary point on the z-axis. (This is a bit complicated, and is representative of the most difficult kinds of integrals you might have to set up in an Olympiad. For a useful table of integrals, see Appendix K of Purcell.)

Solution. Set up spherical coordinates with the hemisphere being the equation of $r = R$ and $\theta \in [0, \pi/2]$. Suppose our location is $(0, 0, z)$. The hemisphere has surface charge σ. We see that the field points in the z-direction by symmetry, so we'll only worry about that piece. The ring at angle θ with width $d\theta$ provides fields at an angle, and some geometry shows that we have to correct by a factor of $\frac{R\cos\theta - z}{r}$ where $r \equiv \sqrt{R^2 + r^2 - 2Rz\cos\theta}$. We then have

$$dE_z = -\frac{\sigma(2\pi R^2 \sin\theta\, d\theta)}{4\pi\epsilon_0 r^2} \cdot \frac{R\cos\theta - z}{r},$$

so

$$E(z) = -\frac{\sigma R^2}{2\epsilon_0} \int_0^{\pi/2} \frac{(R\cos\theta - z)\sin\theta\, d\theta}{(R^2 + r^2 - 2Rz\cos\theta)^{3/2}}.$$

Consulting Appendix K tells us that

$$E(z) = \frac{\sigma R^2}{2\epsilon_0 z^2}\left(\frac{R}{\sqrt{R^2 + z^2}} - \frac{R - z}{\sqrt{(R - z)^2}}\right).$$

Taking some care with the square root, we conclude

$$E(z) = \frac{\sigma R^2}{2\epsilon_0 z^2} \times \begin{cases} \frac{1}{\sqrt{1 + z^2/R^2}} - 1 & z < R \\ \frac{1}{\sqrt{1 + z^2/R^2}} + 1 & z > R \end{cases}.$$

[3] **Problem 15.** USAPhO 2018, problem B1.

Idea 7: Electric Dipoles

The dipole moment of two charges q and $-q$ separated by $\mathbf{d}$ is $\mathbf{p} = q\mathbf{d}$. More generally, the dipole moment of a charge configuration is defined as

$$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}\, d\mathbf{r}.$$

For an overall neutral charge configuration, the leading contribution to its electric potential far away is the dipole potential,

$$\phi(r, \theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

where θ is the angle of $\mathbf{r}$ to $\mathbf{p}$.

Remark

Here's a trick to remember the dipole potential. Let $\phi_0(\mathbf{r}) = k/r$ be the potential for a unit charge at the origin. An ideal point dipole of dipole moment p consists of charges $\pm p/d$ separated by d, in the limit $d \to 0$. So the potential is

$$p \lim_{\mathbf{d}\to 0} \frac{\phi_0(\mathbf{r}) - \phi_0(\mathbf{r} + \mathbf{d})}{d}.$$

But this is precisely the (negative) derivative, so you can get the dipole potential by differen-

tiating the ordinary potential! Indeed, for a dipole aligned along the $\hat{\mathbf{z}}$ axis,

$$-\frac{d}{dz}\frac{kp}{r} = \frac{kp}{r^2}\frac{dr}{dz} = \frac{kp}{r^2}\frac{z}{r} = \frac{kp\cos\theta}{r^2}$$

which matches the above result. You can use the same trick for quadrupoles and higher multipoles, which we'll see in **E8**.

[3] **Problem 16.** In this problem we'll derive essential results about dipoles, which will be used later.

(a) Using the binomial theorem, derive the dipole potential given above, for a dipole made of a pair of point charges $\pm q$ separated by distance d, oriented along the z-axis.

(b) Differentiate this result to find the dipole field,

$$\mathbf{E}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

where the expression above is in spherical coordinates. (Hint: feel free to use the expression for the gradient in spherical coordinates.)

(c) Show that this may also be written as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}(3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}).$$

You don't need to memorize these expressions, but it's useful to remember what a dipole field looks like, the fact that its magnitude is roughly $p/4\pi\epsilon_0 r^3$, and the fact that the numeric prefactor is 2 along the dipole's axis and 1 perpendicular to it.

Solution. (a) Let the charges be at $(0,0,0)$ and $(0,0,d)$. Then

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0 r}\left(-1 + \frac{1}{\sqrt{1-2(d/r)\cos\theta + (d/r)^2}}\right) \approx \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}.$$

(b) We use the definition $\mathbf{E} = -\nabla V$, along with the gradient in spherical coordinates. Then

$$E_r = -\frac{\partial V}{\partial r} = \frac{p}{4\pi\epsilon_0 r^3}\cdot 2\cos\theta$$

and

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p}{4\pi\epsilon_0 r^3}\cdot\sin\theta,$$

as desired.

(c) We see that $\mathbf{p}\cdot\hat{\mathbf{r}} = p\cos\theta$ and $\mathbf{p} = p\hat{\mathbf{z}} = p(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)$. Thus,

$$3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p} = 3p\cos\theta\,\hat{\mathbf{r}} - p(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\,\sin\theta) = p(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}),$$

as desired.

[3] **Problem 17.** USAPhO 2002, problem B2.

[3] **Problem 18.** USAPhO 2009, problem B2. This essential problem introduces useful facts about dipole-dipole interactions.

Idea 8

The potential energy of a set of point charges is

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{2} \sum_i q_i V(\mathbf{r}_i).$$

We sum over $i \neq j$ to avoid computing the energy of a single point charge due to its interaction with itself, which would be infinite. For a continuous distribution of charge, we don't have this problem, and instead find

$$U = \frac{1}{2} \int \rho(\mathbf{r}) V(\mathbf{r}) \, d\mathbf{r} = \frac{\epsilon_0}{2} \int |\mathbf{E}(\mathbf{r})|^2 \, d\mathbf{r}.$$

Unlike the other quantities we've considered, energy doesn't obey the superposition principle.

[3] **Problem 19.** In this problem we'll apply the above results to balls of charge.

(a) Compute the potential energy of a uniformly charged ball of total charge Q and radius R.

(b) Show that the potential energy of two point charges of charge $Q/2$ separated by radius R is lower than the result of part (a).

(c) Hence it appears that it is energetically favorable to compress a ball of charge into two point charges. Is this correct?

Solution. (a) We can find the potential by building up the ball by placing charges from infinity. Consider a shell of charge at radius r, and let the charge density by $\rho = Q/(\frac{4}{3}\pi R^3)$. The energy needed to put the shell there is $dU = kQ_{enc}\, dQ/r$, where $Q_{enc} = \frac{4}{3}\rho\pi r^3$ is the charge inside and $dQ = 4\rho\pi r^2\, dr$ is the charge in the shell added to the sphere. Then the energy needed to build the ball, which is the potential energy of the ball, is

$$U_a = \int_0^R kQ \frac{r^3}{R^3} (3Qr^2\, dr/R^3)/r = \frac{3kQ^2}{R^6} \int_0^R r^4 dr = \frac{3kQ^2}{5R} = \frac{3Q^2}{20\pi\epsilon_0 R}.$$

(b) From $U = kq_1q_2/r$, we find that for two point charges the potential energy is

$$U_b = \frac{kQ^2}{4R} = \frac{Q^2}{16\pi\epsilon_0 R}$$

which is less than U_a.

(c) It's wrong because in part (b), the energy needed to create the point charges, by squeezing the two halves of the ball down, is not included. Plugging in a radius of zero into part (a), we see that this energy is actually infinite. (Of course, in reality it doesn't take infinite energy to produce electrons, which are point charges. Classical electrodynamics breaks down when describing such a process, which can only be properly understood within relativistic quantum field theory.)

[3] **Problem 20.** An insulating circular disc of radius R has uniform surface charge density σ.

(a) Find the electric potential on the rim of the disc.

(b) Find the total electric potential energy stored in the disc.

Solution. (a) Place the origin at a point on the rim and use polar coordinates. Because the polar equation of a circle is $r = 2R\cos\theta$, we have

$$V = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2R\cos\theta} \frac{\sigma}{4\pi\epsilon_0}\, dr = \int_{-\pi/2}^{\pi/2} \frac{\sigma R}{2\pi\epsilon_0} \cos\theta\, d\theta = \frac{\sigma R}{\pi\epsilon_0}.$$

(b) Consider building up the ring outward in radius. When we add charges to bring the radius from r to $r + dr$, we do work

$$dW = V\, dq = \frac{\sigma r}{\pi\epsilon_0} (2\pi r\sigma\, dr) = \frac{2\sigma^2 r^2}{\epsilon_0} dr$$

which means

$$W = \int_0^R \frac{2\sigma^2 r^2}{\epsilon_0}\, dr = \frac{2\sigma^2 R^3}{3\epsilon_0}.$$

[3] **Problem 21.** Consider a uniformly charged ball of total charge Q and radius R. Decompose this ball into two parts, A and B, where B is a ball of radius $R/2$ whose center is a distance $R/2$ of the ball's center, and A is everything else. Find the potential energy due to the interaction of A and B, i.e. the work necessary to move B to infinity.

Solution. By the shell theorem, everywhere within B, the electric field due to A is constant, with magnitude

$$E = \frac{k(Q/8)}{(R/2)^2} = \frac{kQ}{2R^2}.$$

This is just the same reasoning as in problem 1. Now, if we take B and squish it down to its center, this takes no net work against A's electric field. If we then move that point a distance $R/2$ out to the edge of A, then this takes work

$$W = E\frac{Q}{8}\frac{R}{2} = \frac{kQ^2}{32R}.$$

At this point, we still need to move B out to infinity. The field that B experiences is the superposition of a uniform ball (the original one) plus a uniform ball of negative density where B was. So again by the shell theorem, the potential energy of B is

$$U = \frac{Q}{8}\left(-\frac{kQ}{R} + \frac{k(Q/8)}{R/2}\right).$$

The total work needed to move B out to infinity is the work we already did, plus what we need to move it out of this potential well,

$$W - U = \frac{kQ^2}{8R}.$$

Note that we haven't counted the work harvested by squishing B, because the question is asking about the interaction between A and B, so this contribution is irrelevant.

[2] **Problem 22** (PPP 149)**.** A distant planet is at a very high electric potential compared with Earth, say 10^6 V higher. A metal space ship is sent from Earth for the purpose of making a landing on the planet. Is the mission dangerous? What happens when the astronauts open the door on the space ship and step onto the surface of the planet?

Solution. As the space ship approaches the planet, its potential gradually increases from that of the Earth, to that of the distant planet. Meanwhile, all the astronauts inside are doing just fine since the ship acts like a Faraday cage. Once the ship lands, it's already at the same potential as the planet, and when the astronauts step out, nothing happens. In other words, it's electric field that's dangerous, not potential, and the electric fields in this problem are always small.

Another way to see that there's no danger is to replace electric fields with gravitational fields, and thus electric potential with gravitational potential. An elevator in a skyscraper takes you from a low to a very high gravitational potential. But nothing violent happens when you get off!

Example 7

Since Newton's law of gravity is so similar to Coulomb's law, the results we've seen so far should have analogues in Newtonian gravity. What are they? For example, what's the gravitational Gauss's law?

Solution

The fundamental results to compare are

$$F = -\frac{Gm_1m_2}{r^2}, \quad F = \frac{q_1q_2}{4\pi\epsilon_0 r^2}$$

where the minus sign indicates that the gravitational force is attractive, while the electrostatic force between like charges is repulsive. Then we can transform a question involving (only positive) electric charges to one involving masses if we map

$$q \to m, \quad \frac{1}{4\pi\epsilon_0} \to -G, \quad \mathbf{E} \to \mathbf{g}.$$

Thus, while electrostatics is described by

$$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0},$$

the gravitational field is described by

$$\nabla \times \mathbf{g} = 0, \quad \nabla \cdot \mathbf{g} = -4\pi G\rho_m, \quad \oint \mathbf{g} \cdot d\mathbf{S} = -4\pi GM$$

where ρ_m is the mass density. Similarly, the potential energy can be written in two ways,

$$U = \frac{1}{2}\int \rho_m(\mathbf{r})\phi(\mathbf{r})\, d\mathbf{r} = -\frac{1}{8\pi G}\int |\mathbf{g}(\mathbf{r})|^2\, d\mathbf{r}$$

where $\phi(\mathbf{x})$ is the gravitational potential. This result was first written down by Maxwell.

Remark

Here's a philosophical question: is potential energy "real"? It's not as obvious as you might think! In the 1700s, there was a lively debate over whether the ideas of kinetic energy and momentum, which at the time were given various other names, were worthwhile. Which one

of the two was the *true* measure of motion? In our modern language, proponents of energy pointed out that the momentum always vanished in the center of mass frame, which made it trivial, while supporters of momentum replied that kinetic energy was clearly not conserved in even the simplest of cases, like inelastic collisions.

In the 1800s, the theory of thermodynamics was developed, allowing the energy seemingly lost in inelastic collisions to be accounted for as internal energy. But there still remained the problem that kinetic energy was lost in simple situations, such as when balls are thrown upward. By the mid-1800s, the modern language that "kinetic energy is converted to potential energy" was finally standardized, but it was still common to read in textbooks that potential energy was fake, a mathematical trick used to patch up energy conservation. After all, potential energy has some suspicious qualities. If a ball has lots of potential energy, you can't see or feel it, or even know it's there by considering the ball alone. It doesn't seem to be located anywhere in space, and its amount is arbitrary, as a constant can always be added. In the late 1800s, a revolution on physics answered some of these questions. Maxwell and his successors recast electromagnetism as a theory of fields, and showed that the dynamics of charges and currents were best understood by allowing the fields themselves to carry energy and momentum. We'll cover this in detail in **E7**, but for now, it implies that electrostatic potential energy is fundamentally stored in the field, with a density of $\epsilon_0 E^2/2$. This implies that its location and total amount are directly measurable.

Maxwell believed that the dynamics of fields emerged from the microscopic motions and elastic deformations of an all-pervading ether, in the same way that, say, a fluid's velocity field emerges from the average motion of fluid molecules. This makes it manifestly positive, so he was disturbed to find that the energy density of a gravitational field is *negative*!

A few decades later, the arrival of special relativity answered some questions and reopened others. On one hand, it demolished Maxwell's vision of the ether. On the other hand, it finally answered the question of whether all kinds of potential energy are "real", and it got rid of the freedom to add arbitrary constants. That's because in special relativity, the total energy of a system at rest is related to its mass by $E = mc^2$, and the mass is directly measurable. This finally puts thermal energy, elastic potential energy, and field energy on an equal footing.

Here's the most modern view of energy conservation. All particles and their interactions are fundamentally described by relativistic quantum fields. A famous result called Noether's theorem implies that whenever such a theory is time-translationally symmetric, there is a conserved quantity which we call the energy. (The distinction between kinetic and potential energy becomes irrelevant; it's all just energy.) The density of energy in space can be computed from the state of the fields, but it doesn't need to be explained, as Maxwell imagined, by the internal motion of whatever the fields are made of. The fields are fundamental: they aren't made of anything; instead, they make up everything!

What happens when we throw gravity into the mix? As we'll discuss further in **R3**, it turns out that at nonrelativistic velocities, the dynamics of gravitating particles can be described by "gravitoelectromagnetism", a theory closely analogous to electromagnetism, where moving

masses also source "gravitomagnetic" fields $\mathbf{B}_g$, which result in $m\mathbf{v} \times \mathbf{B}_g$ forces. But the situation gets much more subtle when we upgrade to full general relativity. Here, the notion of a gravitational field disappears completely, and is replaced by the curvature of spacetime, making it hard to define an energy density for it at all. For an accessible overview of the debate, see this paper. Ultimately, though, it doesn't matter that much, since it doesn't impair our ability to use either Newtonian gravity or general relativity.

Example 8

For an infinite line of linear charge density λ, find the potential $V(r)$ by dimensional analysis.

Solution

This example illustrates a famous subtlety of dimensional analysis. The only quantities in the problem with dimensions are λ, ϵ_0, and r. To get the electrical units to balance, we have

$$V(r) = \frac{\lambda}{2\pi\epsilon_0} f(r)$$

where $f(r)$ is a dimensionless function. But there are *no* nontrivial dimensionless functions of a dimensionful quantity r. The only possibilities are that $f(r)$ is a dimensionless constant, or that $f(r)$ is infinite. In the first case, the electric field would vanish, which can't be right. In the second case, it is unclear how to calculate the electric field at all.

In fact, the electric potential *is* infinite, if you insist on the usual convention of setting $V(\infty) = 0$. In that case, we have

$$V(r) = \int_r^\infty \frac{\lambda}{2\pi\epsilon_0} \frac{dr}{r} = \infty$$

independent of r. But this is useless; to get a finite result we can actually work with, we need to subtract off an infinite constant from the potential. Equivalently, we need to set the potential to be zero at some finite distance $r = r_0$. This process is known as renormalization, and it is extremely important in modern physics. After renormalization, we have

$$V(r) = \frac{\lambda}{2\pi\epsilon_0} \log \frac{r_0}{r}$$

which is perfectly consistent with dimensional analysis.

Notice that in the process of renormalization, a new dimensionful quantity r_0 appeared out of nowhere. This phenomenon is known as dimensional transmutation. Of course, physical predictions don't depend on this new scale (e.g. the electric field is independent of r_0), but you can't write down quantities like the potential without it.

3 Conductors

Idea 9

In electrostatic conditions, $\mathbf{E} = 0$ inside a conductor, which implies the conductor has constant electric potential V. This further implies that $\mathbf{E}$ is always perpendicular to a conductor's surface. By Gauss's law, the conductor has $\rho = 0$ everywhere inside, so all charge resides on the surface.

Example 9

Consider a point on the surface of a conductor with surface charge density σ. Show that the outward pressure on the charges at this point is $\sigma^2/2\epsilon_0$.

Solution

Gauss's law tells us that the difference of the electric fields right inside and outside the conductor at this point is

$$E_{\text{out}} - E_{\text{in}} = \frac{\sigma}{\epsilon_0}$$

by drawing a pillbox-shaped Gaussian surface. But we also know that $E_{\text{in}} = 0$ since we're dealing with a conductor, so $E_{\text{out}} = \sigma/\epsilon_0$.

Let's think about how this electric field is made. If there were no charges around except for the ones at this surface, then the interior and exterior fields would have been $\pm\sigma/2\epsilon_0$. This means that all of the other charges, that lie elsewhere on the surface of the conductor, must provide a field $\sigma/2\epsilon_0$ here, so that E_{in} cancels out.

The pressure on the charges at this point on the surface is equal to the product of the surface charge density with the field due to the *rest* of the charges, since the charges at this point can't exert an overall force on themselves, so

$$P = \sigma\left(\frac{\sigma}{2\epsilon_0}\right) = \frac{\sigma^2}{2\epsilon_0}$$

as required. Equivalently, we can conclude that $P = \epsilon_0 E_{\text{out}}^2/2$.

Example 10

Is the charge density at the surface of a charged conductor usually greater at regions of higher or lower curvature?

Solution

Of course, we can't answer this question directly, because it is essentially impossible to find the charge distribution of an irregularly shaped conductor. However, we can get some insight by considering the limiting case of a conductor made of two spheres of radii R_1 and

R_2, connected by a very long rod.

For the potential to be the same at both spheres, we must have $Q_1/R_1 = Q_2/R_2$, so the charge is proportional to the radius, and the charge density is inversely proportional to the radius. Thus, there's generally higher charge density at sharper points of the conductor.

[1] **Problem 23.** Show that any surface of charge density σ with electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ immediately on its two sides experiences a force $\sigma(\mathbf{E}_1 + \mathbf{E}_2)/2$ per unit area. (This is a generalization of the example above, where one side was inside a conductor.)

Solution. Let $\mathbf{E}$ be the field due to all the other charges. Again, we have $\mathbf{E}_1 = \mathbf{E} + \sigma/2\epsilon_0\hat{\mathbf{n}}$ and $\mathbf{E}_2 = \mathbf{E} - \sigma/2\epsilon_0\hat{\mathbf{n}}$. Thus, $\mathbf{E} = \frac{1}{2}(\mathbf{E}_1 + \mathbf{E}_2)$, and the force per area is $\sigma\mathbf{E}$.

[2] **Problem 24.** Is it possible for a connected, completely isolated conductor with a positive charge to have a negative surface charge density at any point? If not, prove it. If so, sketch an example.

Solution. This can't happen. Note that the surface of the conductor has a constant, positive potential. Now suppose there was a region with negative charge on the conductor, and consider a field line that ends on such a charge. It can't have come from infinity, because the potential at infinity is lower than that of the conductor. And it can't have come from elsewhere on the conductor, because the conductor is an equipotential. This yields a contradiction.

Idea 10: Existence and Uniqueness

In a system of conductors where the total charge or potential of each conductor is specified, there exists a unique charge configuration that satisfies those boundary conditions.

This is very useful because in many cases, it is difficult to directly derive the charge distributions or fields. Instead, sometimes one can simply insightfully guess an answer; then it must be the correct answer by uniqueness. For further discussion, see section 2.5 of Griffiths.

Example 11

Consider a conductor with nonzero net charge, and an empty cavity inside. Show that the electric field is zero in the cavity.

Solution

Let's consider a second conductor with the same net charge and the same shape, but without the cavity. By the existence and uniqueness theorem, we know there exists some charge configuration on the second conductor's surface which satisfies the boundary conditions, namely that the electric field vanishes everywhere inside the conductor. In particular, that means the field is zero where the cavity of the original conductor would have been.

Now consider the original conductor again. If we give this conductor precisely the same surface charge distribution, then this will again solve the boundary conditions, and it'll have

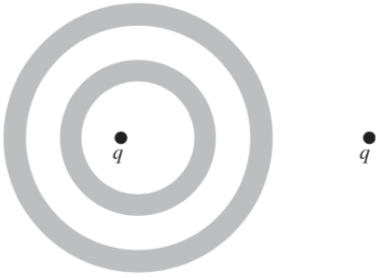
no field in the cavity. But by the existence and uniqueness theorem, the charge distribution is unique, so this is the only possible answer: the field *must* be zero in the cavity.

If this is your first time seeing this, it can sound like a fast-talking swindle (which is why I made it an example rather than a problem!). It looks like we used no effort and got a strong conclusion out. Of course, that's because all the work is done by the uniqueness theorem.

[1] **Problem 25.** Consider a spherical conducting shell with an arbitrary charge distribution inside, with net charge Q. Find the electric field outside the shell.

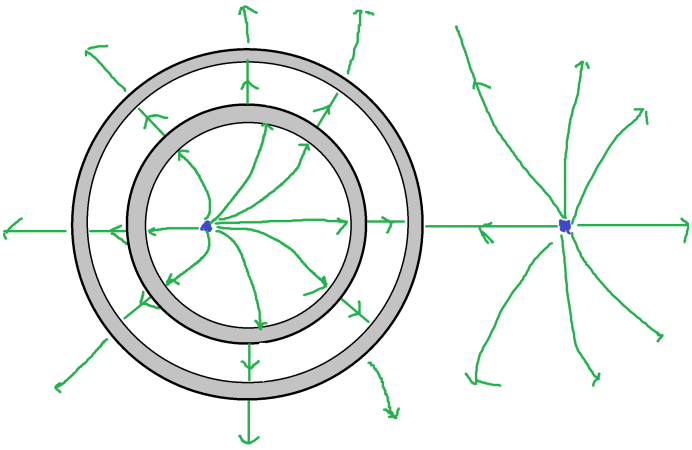
Solution. The shell is an equipotential. The field of a point charge Q at the center of the shell hence satisfies the boundary conditions. By the uniqueness theorem, this is the only solution.

[2] **Problem 26** (Purcell 3.33)**.** The shaded regions represent two neutral conducting spherical shells.



Carefully sketch the electric field. What changes if the two shells are connected by a wire?

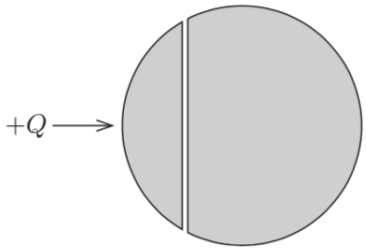
Solution. The results are shown below.



Whether or not there are any field lines coming from the charge outside the shell depends on how close that charge is to the shell. (The entire field configuration in this problem can be found exactly using the method of images, as shown in **E2**.) In the case that the two spheres are connected with a wire, the field in between the two spheres disappears, but nothing else changes.

[3] **Problem 27.** USAPhO 2014, problem A4.

[4] **Problem 28** (MPPP 150)**.** A solid metal sphere of radius R is divided into two parts by a planar cut, so that the surface area of the curved part of the smaller piece is πR^2. The cut surfaces are coated with a negligibly thin insulating layer, and the two parts are put together again, so that the original shape of the sphere is restored. Initially the sphere is electrically neutral.



The smaller part of the sphere is now given a small positive electric charge Q, while the larger part of the sphere remains neutral. Find the charge distribution throughout the sphere, and the electrostatic interaction force between the two pieces of the sphere.

Solution. We know that distributing charge uniformly on the outer surface of the entire sphere will give a valid configuration, in the sense that the field is everywhere perpendicular to the conductors. Similarly, distributing equal and opposite charges uniformly on the two flat faces will give a valid configuration, since it acts like a parallel plate capacitor, making the field vanish everywhere outside.

Neither of these solutions have the right total charge on each piece, but we can fix this by superposing the two. By solving a system of two equations, we find the charge distribution is

- total charge Q distributed uniformly on the sphere,
- charges $\pm(3/4)Q$ distributed uniformly on the flat surfaces.

The two flat faces attract each other and the two curved faces repel each other; there are no other forces by the shell theorem. The pressure on the flat faces is $\sigma^2/2\epsilon_0$. With a little trigonometry, we find the area of the flat faces is $(3/4)\pi R^2$, giving a force

$$F_1 = \frac{3Q^2}{8\pi\epsilon_0 R^2}.$$

As for the repulsive force, using the result of problem 11 we get

$$F_2 = -\frac{3Q^2}{128\pi\epsilon_0 R^2}, \quad F_{\text{tot}} = F_1 + F_2 = \frac{45}{128}\frac{Q^2}{\pi\epsilon_0 R^2}.$$

[3] **Problem 29.** In this problem we'll work through a heuristic proof of a version of the uniqueness theorem. In particular, we will show that for a system of conductors in empty space, specifying the total charge on each conductor alone specifies the entire surface charge distribution.

(a) Suppose for the sake of contradiction that two different charge distributions can exist, and consider their difference, which has zero total charge on each conductor. Argue that at least one conductor must have electric field lines both originating from and terminating on it.

(b) Show that at least one of these field lines must originate from or terminate on another one of the conductors.

(c) By generalizing this reasoning, prove the desired result. (Hint: consider the conductors with the highest and lowest potentials.)

Solution. (a) Since the overall charge distributions are difference, at least one conductor C must have different charge distributions in the two cases. So when we consider the difference, C has areas of both positive and negative surface charge. Field lines come out of the latter, and go into the former.

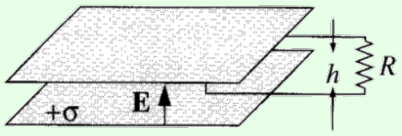
(b) The field lines can't go back to C, because by following the field line, one would prove that C has a higher potential than itself, which is impossible. They also can't all go off to infinity, because we can consider "infinity" to just be a big, far away neutral conductor at zero potential. If lines both came from infinity to C and from C to infinity, then C would again have a higher potential than itself, which is impossible. So some field line must go between C and another conductor C'.

(c) Consider the conductor with the highest potential $\phi_{\max}$ which has nontrivial surface charges. Field lines have to go into this conductor, but they can't come from anywhere except from infinity. Since infinity is at zero potential, we have $\phi_{\max} \le 0$.

Now consider the conductor with the lowest potential $\phi_{\min}$ which has nontrivial surface charges. Field lines have to leave this conductor, but they can't go anywhere except for infinity. Since infinity is at zero potential, $\phi_{\min} \ge 0$. This forces $\phi_{\min} = \phi_{\max} = 0$, so everything must be at zero potential, which means there aren't any electric field lines at all.

Example 12: Griffiths 7.6

A wire loop of height h and resistance R has one end placed inside a parallel plate capacitor with electric field $\mathbf{E}$, as shown.

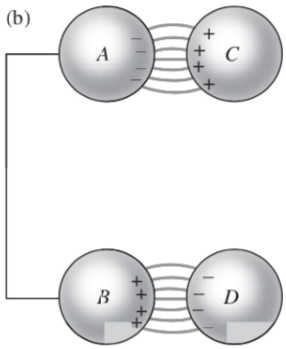


The other end of the loop is far away, where the field is negligible. Find the emf in the loop.

Solution

Of course, this is a trick question; if the answer were nonzero, the current would run forever, yielding a perpetual motion machine. Electrostatic fields always produce zero total emf along any loop. The $\sigma h/\epsilon_0$ voltage drop inside the capacitor is canceled out by the voltage drop due to the fringe fields, which are small, but accumulate over a long distance. The point of this example is that, while we can ignore fringe fields for some calculations, they are often essential to get a consistent overall picture. We'll consider the subtleties of fringe fields in much more detail in **E2**.

[2] **Problem 30** (Purcell 3.2)**.** Spheres A and B are connected by a wire; the total charge is zero. Two oppositely charged spheres C and D are brought nearby, as shown.



The spheres C and D induce charges of opposite sign on A and B. Now suppose C and D are connected by a wire. Then the charge distribution should not change, because the charges on C and D are being held in place by the attraction of the opposite charge density. Is this correct?

Solution. This isn't correct. To see this rigorously, we can use the uniqueness theorem. After connecting the wires, we have two conductors (A/B, and C/D), each with zero net charge. One possible solution is to have zero charge everywhere. By uniqueness, this is the only possible solution, so anything else cannot have been in equilibrium.

That is rigorous, but it might not be intuitive; after all, it sure looks like the charges on C are stuck where they are. However, though the charges on C are attracted towards A, they also strongly repel each other. It's this repulsion that causes charge on C to start flowing to D when the wire is connected.