Fourier Optics

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1 Introduction

In this handout, we will give a brief introduction to Fourier Transforms (without the rigorous proofs), Convolution Theorem, and how they can be applied to determine the slit interference pattern of various apertures, and vice versa.

In a nutshell, the Fourier Transform takes a function expressed in one variable, and expresses it in another variable, without losing information about the original function.

For example, if a music note produces a perfectly sinusoidal sound wave with amplitude $A = \cos(3t)$. This is currently in the "time domain." If we instead plotted this in the "angular frequency domain" (i.e. after taking the Fourier transform), we should expect the function to have a peak at $\omega = 3$, and zero elsewhere. This is exactly what we get, except there is also a peak at $\omega = -3$. This negative sign comes from the fact that the Fourier Transform doesn't break the function up into sinusoidal but rather complex exponentials, but this fact isn't too important.

The magic is that Fourier Transforms allow us to break up a function into sinusoidals, and the reverse: generating the original function from the sinusoidals.

2 Properties of Fourier Transform

In this section, we will show how we can apply the Fourier Transform on simple functions. Specifically, we will apply the fourier transform on $f(x)$ to get $\hat{F}[f(x)] = \hat{f}(\xi)$. Note that x and ξ will have inverse dimensions.

Notation: There are various ways of writing notation, but the convention we will be using is that $f(x)$ denotes the original function and $\hat{f}(\xi)$ denotes the function after taking the Fourier transform.

2.1 The Delta Function

The delta function is given by $\delta(x)$ and comes up in many places in physics. Without going into the mathematical rigour, we can define it as follows[1]:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{elsewhere} \end{cases}$$

with the special property that

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1.$$

It is essentially a normal distribution with an infinitely high peak. This is an important function as it is the *identity* of the Fourier transform, i.e.:

$$\hat{F}[1] = \delta(\xi)$$
$$\hat{F}[\delta(x)] = 1$$

Note that this makes sense intuitively. If we have a constant function, then the frequency is 0, so we should expect the Fourier transform to be zero everywhere except at the origin.

2.2 Basic Properties

The Fourier Transform satisfies the following properties:

- Linearity:
$$\hat{F}[af(x) + bg(x)] = a\hat{f}(\xi) + b\hat{g}(\xi)$$
- Shift in Time Domain:
$$\hat{F}[f(x-a)] = e^{-2\pi i a\xi}\hat{f}(\xi)$$
- Shift in Frequency Domain:
$$\hat{F}[f(x)e^{iax}] = \hat{f}\left(\xi - \frac{a}{2\pi}\right)]$$
- Scaling in Time Domain:
$$\hat{F}[f(ax)] = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right)$$

[1] which I'm sure a cranky mathematician will come knocking at my door

Note that since Fourier Transforms are usually between time and frequency, x is often referred to as time and ξ is referred to as frequency.

2.3 Fourier Transforms of Common Functions

Here are the Fourier transforms of common functions:

- Complex Exponentials
$$\hat{F}[e^{iax}] = \delta\left(\xi - \frac{a}{2\pi}\right)$$
- Cosine
$$\hat{F}[\cos(ax)] = \frac{\delta\left(\xi - \frac{a}{2\pi}\right) + \delta\left(\xi + \frac{a}{2\pi}\right)}{2}$$
- Sine
$$\hat{F}[\sin(ax)] = \frac{\delta\left(\xi - \frac{a}{2\pi}\right) - \delta\left(\xi + \frac{a}{2\pi}\right)}{2}$$
- Rectangular Pulse
$$\hat{F}[\mathrm{rect}(ax)] = \frac{1}{|a|} \cdot \mathrm{sinc}\left(\frac{\xi}{a}\right)$$
  Here, we define
$$\mathrm{rect}(x) = \begin{cases} 0 & |t| > 1/2 \\ 1/2 & |t| = 1/2 \\ 1 & |t| < 1/2 \end{cases}$$

2.4 Inverse Fourier Transform

It turns out that the Inverse Fourier Transform is very similar to the Fourier Transform. All the same basic properties apply. The only change is a flip in the sign:
$$\hat{F}[\hat{f}(x)] = f(-\xi)$$

This means for even functions, the Fourier Transform and the Inverse Fourier Transform are inverses of each other.

Further Note: These Fourier Transforms are all in one-direction. This handout will purely be focusing on the one-dimensional case, though the concepts can be extended and applied to higher dimensions.

3 Applying the Fourier Transform

3.1 Fundamental Principles

The fundamental principle is that applying the Fourier Transform on the aperture (slit) pattern, we will get the amplitude function of the interference pattern on the wall. It might be confusing as both of these are functions of a length, but this is not the case. Let $f(x)$ be the aperture pattern and let $\hat{f}(\xi)$ be the interference pattern, where $\xi = \frac{y}{\lambda D}$. Here, y describes the physical distance on the screen, λ is the wavelength of light, and D is the distance to the screen. We can represent slits via:

- Thin slit centered at $x = a$ is represented by
$$\delta(x-a)$$
.
- Slit with width w centered at $x = a$ is represented by
$$\operatorname{rect}\left(\frac{x-a}{w}\right)$$

Then, to get the interference pattern we just need to construct the aperture pattern and apply the Fourier Transform on it.

3.2 Examples

3.2.1 Double Slit

If we have two slits at $x = -d/2$ and $x = d/2$, then the aperture pattern is

$$f(x) = \delta(x - d/2) + \delta(x + d/2).$$

Taking the Fourier Transform gives:

$$\begin{aligned}\hat{f}(\xi) &= e^{-2\pi i(d/2)\xi}\hat{\delta}(x) + e^{2\pi i(d/2)\xi}\hat{\delta}(x)\\ &= e^{-\pi i d\xi} + e^{\pi i d\xi}\\ &= 2\cos(\pi d\xi)\\ &= 2\cos\left(\frac{\pi d}{D\lambda}y\right)\end{aligned}$$

where in the first line we applied linearity, and in the second line we applied the property of $\delta(x)$.

3.2.2 Single Wide Slit

If we have a single slit at $x = 0$ with width w, then the aperture pattern is

$$f(x) = \operatorname{rect}\left(\frac{x}{w}\right).$$

The Fourier Transform then gives the amplitude of the interference pattern to be

$$\hat{f}(\xi) = \frac{1}{w}\operatorname{sinc}(w\xi) = \frac{1}{w}\operatorname{sinc}\left(\frac{w}{\lambda D}y\right)$$

which we get for free!

3.2.3 Dealing with Complex Amplitude

Suppose we have two slits at $x = 0$ and $x = d$, then the aperture pattern is

$$f(x) = \delta(x) + \delta(x - d)$$

and the Fourier transform gives

$$\hat{f}(\xi) = 1 + e^{-2\pi i d\xi}.$$

Unlike the previous times, we get a complex valued function! We can make more sense of this by taking the modulus squared to get

$$\begin{aligned}
|\hat{f}(\xi)|^2 &= (1+\cos(-2\pi d\xi))^2 + \sin(-2\pi d\xi)^2 \\
&= 1 + 2\cos(-2\pi d\xi) + \cos^2(-2\pi d\xi) + \sin^2(-2\pi d\xi) \\
&= 2\,(1+\cos(2\cdot \pi d\xi)) \\
&= 4\cos^2(\pi d\xi) \\
&= 4\cos^2\left(\frac{\pi d}{\lambda D} y\right)
\end{aligned}$$

which gives the same intensity pattern as the double slit. Notice that this isn't offset, as one might expect. This is because we assumed the screen was far away, such that the actual positions of the slits don't really matter: only their relative positions do. As a result, it's often easier to exploit symmetry.

3.2.4 Dealing with Phase Change

Suppose we have a typical double slit experiment, with two thin slits at $x = -d/2$ and one at $x = d/2$. However, the left slit causes a ϕ phase change. To account for this, we can represent that slit as $e^{i\phi}\delta(x)$. Similarly, if we wish to reduce the amplitude by half, we can represent it as $\frac{1}{2}\delta(x)$. As a result, the aperture pattern is

$$f(x) = e^{i\phi}\delta(x-d/2) + \delta(x+d/2)$$

Applying linearity, we can conclude that

$$\hat{f}(\delta) = e^{i(\phi - \pi d\xi)} + e^{\pi i d\xi}$$

so the intensity is

$$\begin{aligned}
|\hat{f}(\delta)|^2 &= (\cos(\phi-\pi d\xi) + \cos(\pi d\xi))^2 + (\sin(\phi - \pi d\xi) + \sin(\pi d\xi))^2 \\
&= 4\cos^2(\pi d\xi - \phi/2) \\
&= 4\cos^2\left(\frac{\pi d}{\lambda D} y - \phi/2\right)
\end{aligned}$$

3.2.5 Triple Slit

Suppose we have three slits centered at $x = -d/2$, $x = 0$, and $x = d/2$, then applying linearity and the result from earlier, we can immediately conclude that the amplitude function is

$$\hat{f}(\xi) = 1 + 2\cos\left(\frac{\pi d}{D\lambda} y\right) +$$

4 Convolution Theorem

The previous examples show how powerful Fourier Transforms can be when determining the interference pattern caused by a combination of simple slit patterns. However, there is a deeper mathematical result related to the Fourier Transform that can further assist in solving problems.

4.1 Definition of a Convolution

The convolution of two variables of two functions f and g can be written as

$$(f * g)(t) := \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \, \mathrm{d}\tau \,.$$

This doesn't make any sense, but it has a nice visual representation. We can construct the convolution $(f * g)(t)$ via the following (from Wikipedia):

1. Express each function in terms of a dummy variable τ.
2. Fix $f(\tau)$ in place.
3. Reflect g such that: $g(\tau) \to g(-\tau)$.
4. Add a time-offset (translation), which allows $g(t-\tau)$ to slide across the τ-axis
5. Starting from $t = -\infty$ and ending at $t = +\infty$, slide $g(t-\tau)$ across every point, and at each point t, determine the integral of their product. That integral at $t = t_0$ is equal to $(f * g)(t_0)$.

Here are two helpful animations from Wikipedia:

- Animation 1
- Animation 2

4.2 Properties of Convolutions

Here are some important properties:

- Commutative:
$$f * g = g * f$$
- Associative:
$$f * (g * h) = (f * g) * h$$
- Associative with Scalar Multiplication:
$$c(f * g) = (cf) * g$$
- Distributive
$$f * (g + h) = f * g + f * h$$

And δ is the multiplicative identity:

$$f * \delta = f.$$

Since the delta function $\delta(x)$ is so important, here are more properties related to it:

- $\delta(t-a) * \delta(t-b) = \delta(t-(a-b))$
- $f(x)\delta(t-a) = f(x-a)$

4.3 Convolution Theorem

The Convolution Theorem states that

$$\hat{F}[f * g] = \hat{F}(f) \cdot \hat{F}(g)$$

and

$$\hat{F}[f(x)g(x)] = (\hat{f} * \hat{g})(\xi)$$

Therefore, multiplication in the ξ domain is equivalent to convolution in the x domain, and vice versa. This is important because if you have two slit patterns $f(x)$ and $g(x)$ and you know the interference patterns $\hat{f}(\xi)$ and $\hat{g}(\xi)$ that they produce, then you know that if you take the convolution of the two slit patterns, you will get the interference pattern for free.

5 Applying Convolutions

In the solution to Problem 28 of OPhO Open 2021, the convolution theorem is mentioned and a paarticular exercise is given (but not proven):

> "The diffraction pattern of two slits is the same as the product of the amplitudes for one slit and two light sources."

We will show why this is true. The two light sources (which correspond to two thin slits) can be represented by the aperture function $\delta(x-a)+\delta(x+a)$ and the slit can be represented by $\mathrm{rect}(x/w)$. The convolution is then:

$$\begin{aligned}(\delta(x-a)+\delta(x+a)) * \mathrm{rect}(x/w) &= \delta(x-a) * \mathrm{rect}(x/w) + \delta(x+a) * \mathrm{rect}(x/w) \\ &= \mathrm{rect}\left(\frac{x-a}{w}\right) + \mathrm{rect}\left(\frac{x+a}{w}\right)\end{aligned}$$

Which is the slit pattern of two wide slits! Notice how powerful this result is: If we know the interference pattern created by one wide slit and two thin slits, then the amplitude function will then be:

$$\hat{F}\left[\mathrm{rect}\left(\frac{x-a}{w}\right) + \mathrm{rect}\left(\frac{x+a}{w}\right)\right] = \hat{F}[(\delta(x-a)+\delta(x+a)) * \mathrm{rect}(x/w)] = \hat{F}[\delta(x-a)+\delta(x+a)] \cdot \hat{F}[\mathrm{rect}(x/w)]$$

and both of these Fourier Transforms (or for someone who doesn't know Fourier Transforms, the interference patterns) are both well known! This gives a concrete example of how the Convolution Theorem is so powerful: It allows us to decompose complicated slit patterns into simple ones which we know the interference pattern for. Then the interference pattern of the complicated aperture is just the product of the interference patterns of the simple slits!

5.1 Examples

5.1.1 What Slit Has an Amplitude Pattern of $\cos^2(x)$?

Suppose we want to construct a series of slits such that the amplitude pattern behaves like $\cos^2(x)$. Specifically, we want the amplitude to be proportional to $4\cos^2\left(\frac{\pi d}{D\lambda}y\right)$. We want to take the Inverse Fourier Transform. However for even functions, the Inverse Fourier Transform is the same as the regular Fourier Transform, so we want to compute:

$$\hat{F}\left[4\cos^2\left(\frac{\pi d}{D\lambda}y\right)\right]$$

However, applying the Convolution Theorem, this is equivalent to

$$\begin{aligned}\hat{F}\left[\cos^2\left(\frac{\pi d}{D\lambda}y\right)\right] &= \hat{F}\left[2\cos\left(\frac{\pi d}{D\lambda}y\right)\right] * \hat{F}\left[2\cos\left(\frac{\pi d}{D\lambda}y\right)\right] \\ &= (\delta(x-d/2)+\delta(x+d/2)) * (\delta(x-d/2)+\delta(x+d/2)) \\ &= \delta^2(x-d/2) + 2\delta(x+d/2)\delta(x-d/2) + \delta^2(x+d/2) \\ &= \delta(x-d) + 2\delta(x) + \delta(x+d)\end{aligned}$$

Note that the factor of 2 before the $\delta(x)$ means that we have a thin slit centered at $x=0$, which doubles the amplitude of light coming through it (i.e. increases the intensity by a factor of $\sqrt{2}$).