4th Singapore Astronomy Olympiad Suggested Answers

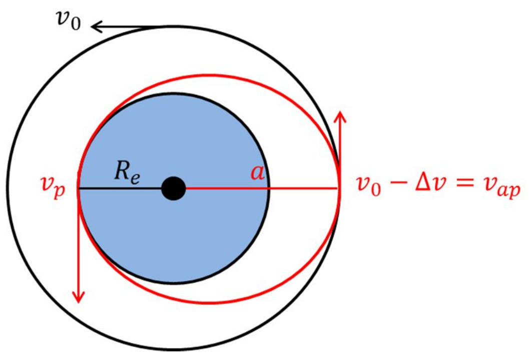
Last updated 27/11/2016
Provided by 4th SAO Committee

Table of Contents

Q1 Celestial Mechanics (14m)

a)	**(First principles)** The gravitational force provides the centripetal force for the HST's circular orbit, thus $\frac{GM_e m}{a^2} = \frac{mv^2}{a}$ $\because v = a\omega, \;\; \omega = \frac{2\pi}{T}$ $\therefore \; GM_e = a\left(\frac{2\pi a}{T}\right)^2$ **(Solution)** Rearranging, Kepler's Third Law: $T^2 = \frac{4\pi^2}{GM_e}a^3$ Hence, rearranging to find a, $a = \sqrt[3]{\frac{GM_e T^2}{4\pi^2}}$ $= \sqrt[3]{\frac{(6.674 \times 10^{-11}\ m^3\ kg^{-1}\ s^2)(5.972 \times 10^{24}\ kg)(5736\ s)^2}{4\pi^2}}$ $= \mathbf{6.93 \times 10^6\ m}$	A1 B1
	Note the full equation is in fact, $T^2 = \frac{4\pi^2}{G\mu}a^3$ where μ is the reduced mass; however since $M_e \gg m$ then $\mu \simeq M_e$. Common Mistakes: • There is no need to use the vis-viva equation for circular orbits – remembering it incorrectly is self-penalisation • $T^2 = a^3$ is incorrect here! This form of Kepler's Third Law is only applicable to satellites of the Sun, i.e. planets, comets, where T and a are in units of *yr* and *AU* respectively.	
b)	For a circular orbit, $v_0 = \frac{2\pi a}{T}$ $= \frac{2\pi(6.93 \times 10^6\ m)}{5736\ s}$ $= \mathbf{7.59 \times 10^3\ m\ s^{-1}}$	M1 A1

c) The problem can in fact be visualised as a **Hohmann Transfer Orbit**, with corresponding diagram (not to scale),



Conserving orbital energy U,

$$U = -\frac{GM_e m}{2\left(\frac{a+R_e}{2}\right)} = \frac{1}{2}mv_{ap}^2 - \frac{GM_e m}{a}$$ **A1**

Solving for v_{ap},

$$2GM_e\left(\frac{1}{a} - \frac{1}{a+R_e}\right) = v_{ap}^2$$

$$v_{ap} = \sqrt{\frac{2GM_eR_e}{a(a+R_e)}}$$ **M1**

$$= \sqrt{\frac{2(6.674\times10^{-11}\ m^3\ kg^{-1}\ s^2)(5.972\times10^{24}\ kg)(6.371\times10^6\ m)}{(6.926\times10^6\ m)\big((6.926\times10^6\ m)+(6.371\times10^6\ m)\big)}}$$

$$= 7.426\times10^3\ m\ s^{-1}$$ **A1**

$$\Delta v = v_0 - v_{ap}$$
$$= (7.591\times10^3\ m\ s^{-1}) - (7.426\times10^3\ m\ s^{-1})$$ **B1**
$$= \mathbf{165\ m\ s^{-1}}$$

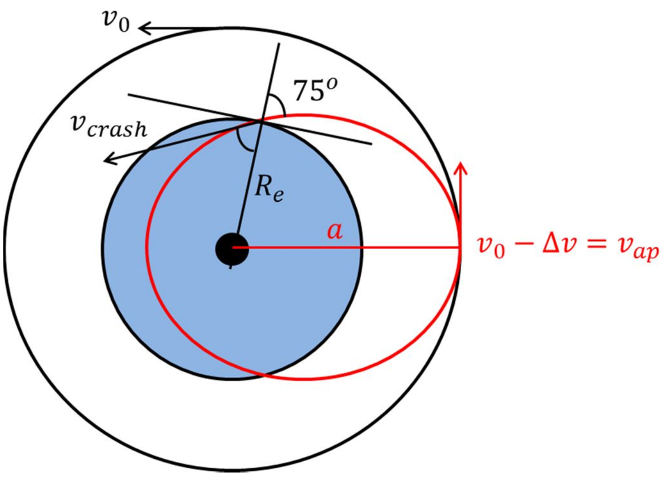
Such a trajectory would skim the Earth's surface. As such it would be hard to control where the HST impacts the surface. **A1**

There is no need to consider orbital angular momentum here as there is only one unknown, also $v_{ap}r_{ap} = v_p r_p$ gives no useful information.

Common Mistakes:

- Δv is a scalar – it has no direction!
- Quoting Kepler's first law in polar form or vis-viva equation without a supporting diagram or working
- Question does not involve escape velocity!
- Illogical conclusion unrelated to the question (e.g. damaging buildings)

d) The diagram for this scenario is shown below.



Conserving orbital energy U,

$$\frac{1}{2}mv_{crash}^2 - \frac{GM_em}{R_e} = \frac{1}{2}mv_{ap}^2 - \frac{GM_em}{a}$$ **A1**

Conserving orbital angular momentum L,

$$\vartheta = 180° - 75° = 105°$$
$$L = mv_{ap}a = mv_{crash}(\sin 105°)R_e$$ **A1**
$$\Rightarrow v_{crash} = \frac{a}{R_e(\sin 75°)}v_{ap}$$

Rearranging and substituting,

$$v_{ap}^2\left(1 - \left(\frac{a}{R_e(\sin 75°)}\right)^2\right) = 2GM_e\left(\frac{1}{a} - \frac{1}{R_e}\right)$$ **M1**

$$v_{ap} = \sqrt{\frac{2GM_e\left(\frac{1}{a} - \frac{1}{R_e}\right)}{1 - \left(\frac{a}{R_e(\sin 75°)}\right)^2}}$$

$$= \sqrt{\frac{2GM_e(R_e - a)}{aR_e\left(1 - \frac{a^2}{R_e^2(\sin^2 75°)}\right)}}$$

$$= \sqrt{\frac{2(6.674\times 10^{-11}\ m^3\ kg^{-1}\ s^2)(5.972\times 10^{24}\ kg)((6.371\times 10^6 - 6.926\times 10^6)\ m)}{(6.926\times 10^6\ m)(6.371\times 10^6\ m)\left(1 - \frac{(6.926\times 10^6\ m)^2}{(6.371\times 10^6\ m)^2(\sin^2 75°)}\right)}}$$

$$= 6.132\times 10^3\ m\ s^{-1}$$ **A1**

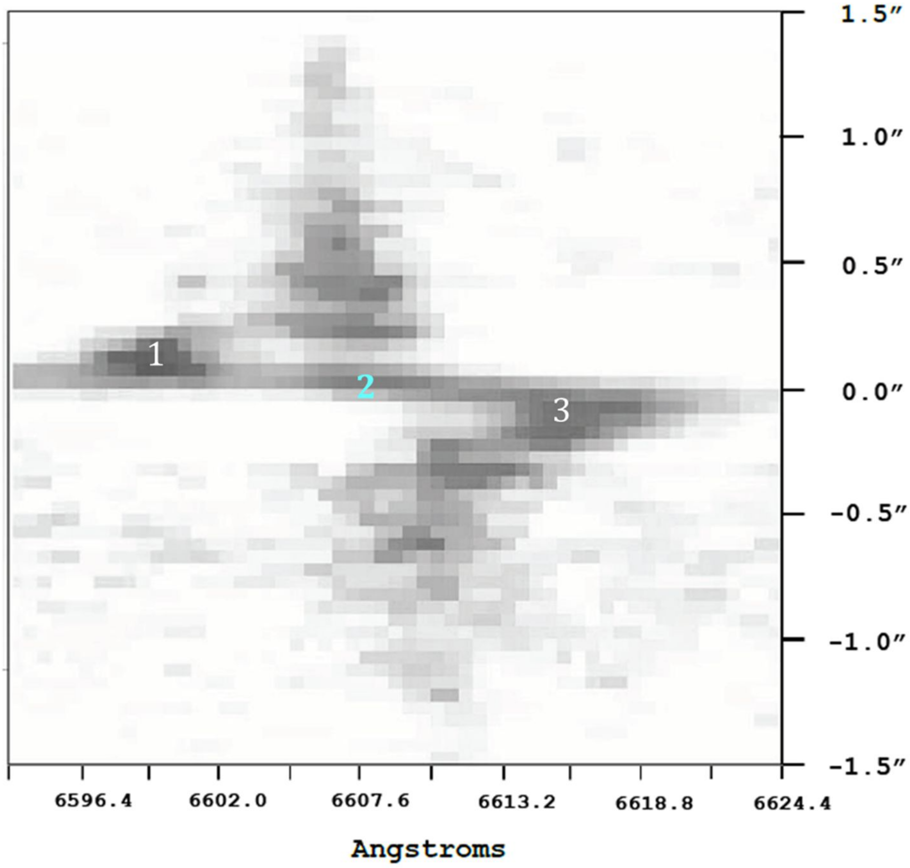
	$\Delta v = v_0 - v_{ap}$ $= (7.591 \times 10^3\ m\ s^{-1}) - (6.132 \times 10^3\ m\ s^{-1})$ $= \mathbf{1.46 \times 10^3}\ \boldsymbol{m\ s^{-1}}$	**B1**
	Recall **L** = **r** × **p**, hence $\lvert\mathbf{L}\rvert = L = mvr\sin\vartheta$, where ϑ is the angle between the vectors **r** and **p**. Recall *sin* 105° = *sin* 75°. Also observe that as the impact angle from the vertical α decreases from 90°, the periapsis of the new orbit moves closer to the focus (Earth's centre). Common Mistakes: • Note the actual angle ϑ between the **r** and **v** vectors is 105°, **not** 75° • Some candidates misinterpreted the question, calculating for the scenario where the HST impacts at 15° to the vertical (i.e. 75° to the horizontal).	

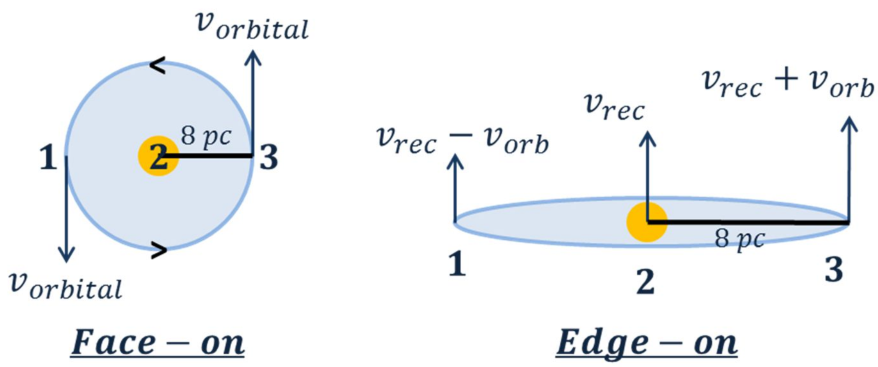
Q2 Data response: observational astronomy (8m)

a)	First, estimate the redshift of the nucleus of M84, by reading from the center, $z = \frac{6608Å}{6583Å} - 1 = \mathbf{0.00380 \pm 0.0003}$ Accept all answers to the above error corresponding to maximum error of 2 pixels from the center. Then, by Hubble's law, for small z, $v = H_0 D \approx cz$ $\therefore D = \frac{cz}{H_0}$ $= \frac{(3.00 \times 10^8\ m\ s^{-1})(0.00380)}{67.8\ km\ s^{-1}\ Mpc^{-1}}$ $= \mathbf{17\ Mpc}$	**B1** **B1**
	Many participants did not know how to interpret the data, even though it's technically the easiest question. Common Mistakes: • Many used parallax to obtain the distance to the galaxy, even though parallax is impossible to use at such distances due to instrumental limitations (parallax is often used only out to a few kpc). • Parallax method also does not make use of the spectrograph data • Some left their answers to other units of distance • Lack of understanding that the recession velocity of the galaxy is causing the entire spectrum to be redshifted from the emitted wavelength. • Some participants brought in unnecessary special relativistic formula	
b)	Convert the angle from degrees to radians, $\theta = 1'' = \frac{1}{3600}\ degrees = \frac{1/3600}{360}(2\pi)$ $= 4.84 \times 10^{-6}\ radians$ $Length\ of\ arc = \theta D$ $= (4.84 \times 10^{-6})(16.8 \times 10^6\ pc)$ $= \mathbf{81\ pc}$ Accept answers from 72 to 86pc.	**A1** **B1**
	Common Mistakes: • Equations are not written out in full with variables – marks lost as markers have to guess your meaning • Equation for arc length s = rθ requires θ to be in **radians.**	

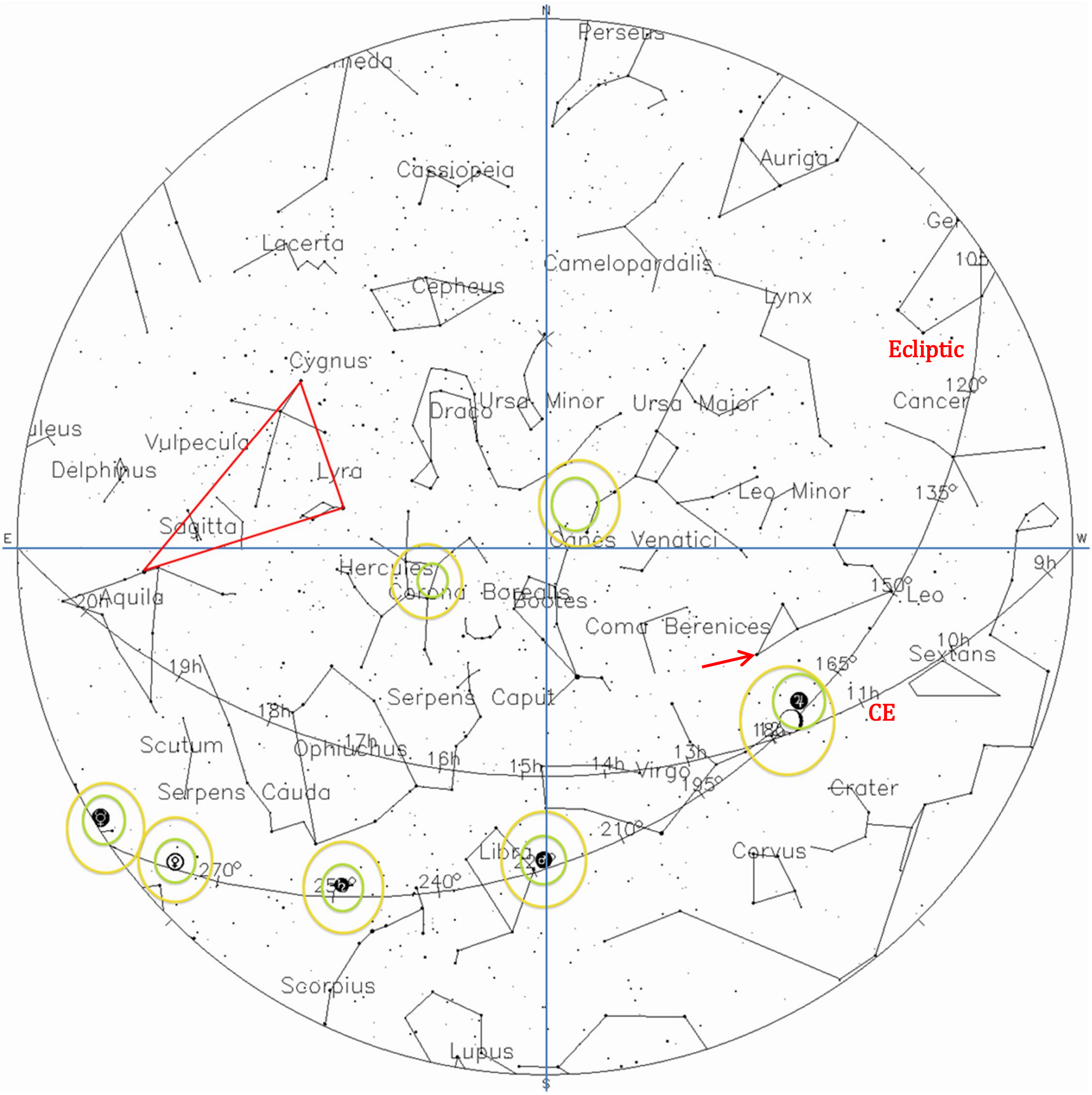
c)	From part (b), $8pc$ corresponds to about 0.1". Thus read the wavelengths of the spectrum at $\pm$0.1" along the vertical axis, at the darkest parts of the spectrum (corresponding to localised concentrations of gas). Reading from the spectrum, $\lambda_1 = 6599\text{Å}$ **and** $\lambda_2 = 6616\text{Å}$ Award this 1m for <u>either</u> of the above wavelengths read off. Approximating the orbital speed of the gas at this $8pc$ radius, using both λ_2, $v_{orb} \approx c\overline{\Delta z} = c\left(\frac{\lambda_2 - \lambda_1}{2\lambda_0}\right)$ $= (3.00 \times 10^8\ m\ s^{-1})\left(\frac{6616\text{Å} - 6599\text{Å}}{2(6608\text{Å})}\right)$ $= 386\ km\ s^{-1}$ Award the M1 mark only for calculating the redshift due to orbital motion of the gas $\overline{\Delta z}$ using both λ_1 and λ_2. Allow rounding off of v_{orb} to 1s.f. or 2s.f. Thus, within an $8pc$ radius, $M = \frac{v_{orb}^2 R}{G}$ $= \frac{(8\ pc)\left(386\frac{km}{s}\right)^2}{(4.302 \times 10^{-3}\ pc\ km^2\ s^2\ M_{solar}^{-1})}$ $\approx \mathbf{2.8 \times 10^8} \boldsymbol{solar\ masses}$ Allow answers from 2.0×10^8 to 3.0×10^8 solar masses. Accept answers given to 1 s.f.	B1 M1 A1 B1
	This subpart was badly done; almost no one scored full credit. The orbital speed of gas at $8pc$ radius out is due to the gravitational force exerted by the mass contained within. Common Mistakes: • Assuming the galaxy is seen face-on despite being told otherwise in the question, leading to nonsensical working • Some candidates did not take the hint from part (b) about the angular distance $\Delta\theta$ along the vertical axis you should be reading the wavelength from; they incorrectly used parallax (again) • Candidates did not understand the significance of the two dark patches in the spectrum (local concentrations of gas at $8pc$ orbital distance)	

The problem can be visualised as below.





Q3 Practical astronomy (20m)



a)	Marking Scheme: **2 marks** – All three stars (Vega in Lyra, Deneb in Cygnus, Altair in Aquila) in the Summer Triangle connected correctly **1 mark** – Any two out of three stars in the Summer Triangle connected correctly **0 marks** – One or none of the three stars identified correctly	

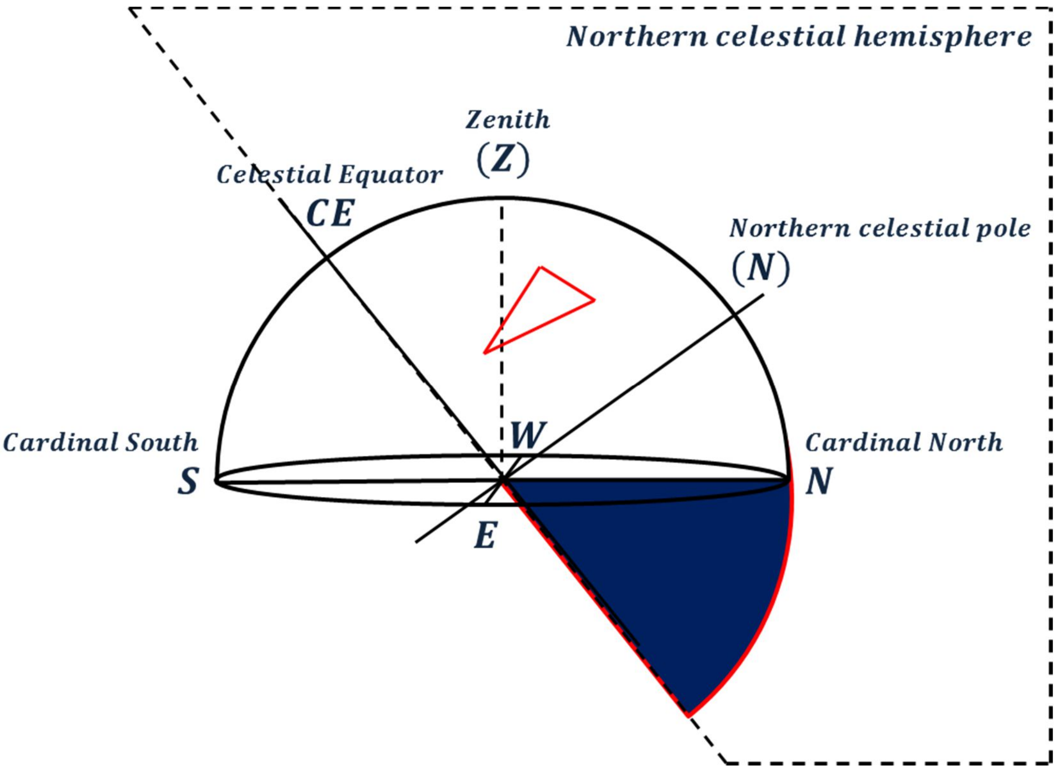
Poorly attempted overall despite the Summer Triangle being a bright and easy visible asterism in Singapore and the Northern hemisphere.

In December at **local midnight**, the Winter constellations reach their culmination (Winter Solstice), and Spring constellations are rising from the East.

Since the planetary display event occurred at 5-7am in the morning, that means the Winter constellations would be setting in the West and Summer constellations will be rising in the East **in December**.

Since the planetary display event occurred in **January-February**, thus the Summer constellations would have risen 30-45^{o} above the horizon.

From part (b), the observer is at a moderately northern latitude (~45^{o}), thus all three constellations (Lyra, Aquilla and Cygnus), which are north of the celestial equator, will appear close to the great circle connecting the zenith, cardinal East and West (drawn as a straight horizontal line above). See diagram below.



Thus, the Summer Triangle is **expected to be located in the centre-left of the starchart.**

Common Mistakes:

- The angle at the vertex with Vega is not approximately a right angle.
- Selecting other dimmer stars in the three constellations for the vertices
- Summer Triangle drawn with stars in other constellations for vertices
- Not using a ruler to connect the stars (no penalisation)
- Deneb is Arabic for *tail*, hence the tail of the swan (Cygnus) is at the short end, i.e. the eagle and swan are flying in **opposite directions**

<table>
<tr><td rowspan="2">b)</td><td>Marking Scheme:
Connecting all seven key stars of Ursa Minor
Marking out Polaris, the axle of the Little Dipper, with a cross
Reasonable approximation of the observer's latitude (40° – 50°)</td><td rowspan="2">A1
A1
A1</td></tr>
<tr><td>Many participants left this part completely blank. Amongst those that didn't, it was poorly attempted, showing poor understanding of the Celestial Sphere.

First, draw a vertical line connecting Cardinal North, South and the zenith on the starchart, representing the local meridian (a great circle in 3D). The Northern celestial pole (N), will always be found on the local meridian, and by extension, the pole star Polaris is too.

Scanning along the vertical line, we find Ursa Major and Polaris in the northern top-centre of the starchart, about 1/3 of the line down.

Note however that a stereographic projection distorts the shape and area of constellations, with the effect becoming more severe away from the zenith towards the horizon. Hence, whilst by linear proportion,

$$\theta_{lat} = \frac{3.2\ cm}{7.9\ cm} \times 90° \approx 36°$$
A correction should be made to account for this stereographic effect, by adding 5-10°, giving ~45° (which is the actual value).

Common Mistakes:
Polaris is not found along the local meridian
Ursa Major (Big Dipper) is traced out instead of Ursa Minor (Little Dipper)
Tracing of the Little Dipper is not closed (no penalisation)
Estimates of the latitude using linear proportion without correcting for error of stereographic projections (generally falling around 30-40°)
Estimates of latitude widely outside reasonable bounds, e.g. $\theta_{lat} > 60°$ or $\theta_{lat} < 0°$ indicating zero understanding of the Celestial Sphere.</td></tr>
<tr><td rowspan="2">c)</td><td>Marking Scheme:
Correctly identifying Denebola in Leo (tip of arrow has to point unambiguously at it)
Within green circle – full credit
Within yellow circle – half-credit
Correctly identifying M101, the Pinwheel Galaxy in Ursa Major
Correctly identifying M13, the Great Hercules Cluster in Hercules</td><td rowspan="2">A1

A1
A1</td></tr>
<tr><td>Virtually no marks awarded, showing that almost all participants lack general knowledge of prominent stars and deep-sky objects in Singapore's sky</td></tr>
</table>

	Common Mistakes: • Selecting another star in *Leo* as *Denebola*, f.y.i. *Denebola* contains *"Deneb"*, which refers to the *tail of the Lion* • Randomly marking out stars in *Hercules* or *Ursa Major* hoping to luckily locate *M13* or *M101*	
d)	Marking Scheme: • Celestial Equator passes through East and West, regardless of shape • CE is a **smooth arc** with curvature such that it intersects the local meridian (vertical line) approximately the same distance from the zenith as the North Celestial pole (N), since the CE intersects the local meridian at 45° altitude as well.	**M1** **A1**
	Poorly attempted, demonstrating little understanding of the celestial coordinate system. Benefit of doubt was given to ambiguous tracings of the CE. Common Mistakes: • CE is not a smooth arc (e.g. horizontal line, vertical line, abstract curves) or is a closed trace. • CE does not pass through cardinal East and West • The ecliptic is traced but labelled as CE.	

e)	Marking Scheme:
	• **Up to 2 marks for accuracy of final answer**
	  ○ **2 marks** – for answers between 14^h and 15^h inclusive
	  ○ **1 marks** – for answers from 13^h to 13^h59^m or 15^h01^m to 16^h inclusive
	  ○ **0 marks** – for answers outside the range of 13^h to 16^h
	• **Award 1 mark for correct logic and working**
	• **For incorrect logic or unsupported answers, award 0 marks.**
	
	(Method 1: Using RA of known object and estimating LHA to obtain LST)
	From the definitions of the following variables, the following equation relates *LST*, *RA* at meridian and local hour angle (LHA),
	
	$LHA_{obj} = LST - RA_{obj}$
	
	Then, find an object with known RA, and estimate its LHA. For an object on the meridian, *LHA* = 0. For rising and setting objects, *LHA < 0* or *LHA > 0* respectively.
	
	e.g. Arcturus has a RA of 14^h15^m; it is estimated to have crossed the meridian (LHA) about 30 mins ago, i.e. $LHA_{Arcturus} = 0^h30^m$
	
	Hence, $LST = LHA_{obj} + RA_{obj} = 14^h15^m + 0^h30^m \approx \mathbf{14^h45^m}$
	
	(Method 2: By first principles, definition of LST)
	By definition, the $LST = LHA_{Vernal\ Equinox}$, and $RA_{Vernal\ Equinox} = 0°$.
		
		M1
		
		A2

	In a tropical (calendar) year, there are **365.25 solar days**, but **366.25 sidereal days**. By definition, **at local noon** on the day of Vernal Equinox (usually around March 20-21), the vernal equinox is at the local meridian everywhere, thus LST = 0^h. Thus, the *LST* at local noon every day after that increases by a small amount equal to the time required for the Earth to rotate an extra small angle due to its orbital motion, i.e. for each complete solar day the Earth completes a small fraction of the extra sidereal day. This amount is, $\Delta LST = \frac{1\ day}{366.25\ days} \times 24^h 00^m = 3.93^m = 3^m 56^s$ Hence, in January 28, approximately 10 months and 7 days after Vernal Equinox (there is inaccuracy introduced here as each month is not of equal length), $LST_{noon\ of\ 28\ Jan} = \left(\frac{10}{12} \times 365.25 + 7\right) \times 3^h 56^m = 1197^m = 20.40^h$ If we do have a calendar, we can get a more accurate calculation, with 313 days between March 21 and January 28 $LST_{noon\ of\ 28\ Jan} = (313) \times 3^h 56^m = 1231^m = 20.51^h$ Since the starchart was taken at 06:15 UTC, i.e. 6.15am at the Greenwich Meridian (0^o latitude, which coincidentally is the latitude of this observer), $LST_{6.15am\ on\ 28\ Jan} \approx 20.51^h - (12^h - 6.25^h) = 14.76^h \approx \mathbf{14^h 45^m}$ **The actual answer according to a calculator is 14.65^h. This is correct since Kochab, the brightest star in the bowl of the Little Dipper, has RA of 14.83^h and has LHA = -0.2^h (i.e. about 10 mins from crossing the local meridian).**	 **M1** **A2**
	Only one candidate got any credit for this question (he got full credit). Disappointing as it shows most other candidates are lacking understanding in geometric astronomy and time systems. Common Mistakes: • Guessing the LST without any working • Calculating the LST based on an incorrectly remembered RA of a visible star in the starchart (no A2 marks)	
f)	Marking Scheme: • Ecliptic is inclined and intersects the CE at approximately 23.5^o • Ecliptic is a **smooth arc** of approximate curvature that intersects the CE around $11^h 30^m$ to $12^h 30^m$ RA, running through many of the zodiac constellations	 **A1** **A1**

<table>
<tr><td></td><td>Common Mistakes:
The CE is traced but labelled as the ecliptic.
The ecliptic is not a smooth arc (e.g. horizontal line, vertical line, abstract curves) or is a closed trace.
The ecliptic, when traced does not pass any of the through the zodiac constellations
The ecliptic ascends above the celestial equator at the intersections instead (the intersection at 12^h RA is a descending node).</td><td></td></tr>
<tr><td rowspan="2">g)</td><td>Within green circle – full credit
Within yellow circle – half-credit
Correctly locating Mercury near the eastern horizon and along the ecliptic
Correctly locating Venus at a higher altitude above Kaus Borealis, the top star of the teapot in Sagittarius
Correctly locating Mars to the west of Zubenelgenubi, Alpha Librae; Mars is also on the local meridian
Correctly locating Jupiter to the west of the point of Autumnal Equinox, i.e. west of RA 12^h along the ecliptic, near the descending node
Correctly locating Saturn at around 17^h RA, and around -20^o declination, i.e. close to the point of Winter Solstice but 1^h earlier along the ecliptic
Any planets not within $\pm 5°$ of the ecliptic are awarded no marks.</td><td rowspan="2">A1
A1
A1
A1
A1</td></tr>
<tr><td>Poorly attempted overall, only a handful of marks were awarded in total. This was exacerbated by the poor tracing of the ecliptic in part (f), which rendered the hints useless.

Common Mistakes:
Planets were identified too far above or below the ecliptic (the ecliptic is also the approximate orbital plane of most planets in the Solar System!)
Planets were marked out but not clearly named.</td></tr>
</table>

Q4 Physics of stars and planets (12m)

<table>
<tr><td>a)</td><td>First, derive the Schwarzchild radius by setting $v = c$ and $R = R_s$ into the equation for escape velocity.

$\because v = \sqrt{\frac{2GM}{R}}$
$\therefore R_s = \frac{2GM}{c^2}$

Therefore, the surface area of a black hole (assumed spherical) is,

$A = 4\pi R_s^2$
$= \boldsymbol{\frac{16\pi G^2 M^2}{c^4}}$</td><td>

B1

B1</td></tr>
<tr><td></td><td>Common Mistakes:
Failing to show clear derivation of the Schwarzchild radius.
Calculating area of the black hole instead of surface area, which is physically meaningless
Incorrect expression for the surface area A of a sphere</td><td></td></tr>
<tr><td>b)</td><td>Since the emissivity $\varepsilon = 1$, i.e. the black hole is a perfect blackbody radiator,

$L = A\varepsilon\sigma T^4$
$= 4\pi R_s^2 \sigma T_H^4$
$= \left(\frac{16\pi G^2 M^2}{c^4}\right)\sigma T_H^4$
$= \left(\frac{16\pi G^2 M^2}{c^4}\right)\left(\frac{\hbar c^3}{8\pi G M k_B}\right)^4$
$= \boldsymbol{\frac{\sigma \hbar^4 c^8}{256\pi^3 G^2 M^2 k_B^4}}$</td><td>

M1

B1</td></tr>
<tr><td></td><td>Common Mistakes:
Some participants did not understand emissivity – the ratio of the electromagnetic radiation from a body to the radiation from an ideal blackbody at a given temperature
Not simplifying the expression to obtain the final answer
Incorrect expression for luminosity L</td><td></td></tr>
</table>

c)	Assume the luminosities of the stars stacks linearly, i.e. none of the stars obstruct our line of sight to other stars, and that the surface temperature of the stars remain unchanged after splitting. Let the luminosity of the parent star be L_0, and its radius and surface temperature be R and T respectively. $L_0 = 4\pi R^2 \sigma T^4$ Let the radius of each daughter star be r, thus by conservation of mass, assuming density remains unchanged, $\frac{4}{3}\pi R^3 = N(\frac{4}{3}\pi r^3)$ $\Rightarrow r^2 = N^{-\frac{2}{3}} R^2$ The combined luminosity of the N daughter stars is $L = N(4\pi r^2 \sigma T^4)$ $= N(4\pi N^{-\frac{2}{3}} R^2 \sigma T^4)$ $= 4\pi N^{\frac{1}{3}} R^2 \sigma T^4$ $= N^{\frac{1}{3}} L_0$ Hence, the final apparent magnitude is $m = m_0 - 2.5 \log\left(\frac{L}{L_0}\right)$ $= \boldsymbol{m_0 - \frac{2.5}{3}\log(N)}$	A1 B1 M1 B1
	Common Mistakes: • Incorrect expression for comparing apparent magnitudes (*Pogson's law*) • Incorrect coefficient of the *lg* term in *Pogson's law*; it is 2.5, not 2.512 or $100^{1/5}$ • Forgetting to account for the N daughter stars of smaller radius in combined luminosity • Missing or incomplete assumption(s)	
d)	Use the same variable notation as part (c) above. Define $\xi = \frac{\hbar c^3}{8\pi G k_B}$ such that $T_H = \frac{\xi}{M}$	

	The surface area of each daughter black hole is $$A = \frac{16\pi G^2 \left(\frac{M}{N}\right)^2}{c^4}$$ $$= \frac{A_0}{N^2}$$	**M1**
	The luminosity of the parent black hole is $$L_0 = A_0 \sigma T_{H,0}^4$$ $$= A_0 \sigma \left(\frac{\xi}{M}\right)^4$$	**B1**
	Thus the combined luminosity of the N daughter black holes is $$L = N(A\sigma T_H^4)$$ $$= N\left(\frac{A_0}{N^2}\sigma\left(\frac{\xi}{\frac{M}{N}}\right)^4\right)$$ $$= N\left(N^2 A_0 \sigma \left(\frac{\xi}{M}\right)^4\right)$$ $$= N^3\left(A_0 \sigma \left(\frac{\xi}{M}\right)^4\right)$$ $$= N^3 L_0$$	**M1**
	Hence, the final apparent magnitude is $$m = m_0 - 2.5\log\left(\frac{L}{L_0}\right)$$ $$= m_0 - 2.5\log(N^3)$$ $$= \boldsymbol{m_0 - 7.5\log(N)}$$	**B1**
	Common Mistakes: • Refer to part (c) comments.	

Q5 Galactic and extragalactic astrophysics (24m)

a)	Given: 1μm = 10^4 Å, i.e. 0.1μm = 1000Å. Since 0.2μm corresponds to L_{2000A} = 2.3cm, then in Angstroms, $\lambda_{obs} = 8000 + \frac{L_{measured}}{L_{2000A}}(2000)$ $= 8000 + \frac{2.10 \pm 0.15cm}{2.30cm}(2000)$ $= 9826 \pm 131\text{Å}$ The wavelength of the spectral lines is redshifted by a factor of (1 + z), i.e. $\frac{\lambda_{obs}}{\lambda_{emit}} = 1 + z$ $z = \frac{9826}{1216} - 1$ $= \mathbf{7.081 \pm 0.08}$	A1 B1
	Common Mistakes: • Incorrect conversion of Angstroms into meters • Some participants gave answers where z < 7! • The redshift z is simply the ratio of the shift in wavelength $\Delta\lambda$ to the emitted wavelength λ.	
b)	Setting $v = c$ in Hubble's law to find the *Hubble radius/distance*, $d_H = \frac{c}{H_0}$ $= \frac{3.00 \times 10^8\ m\ s^{-1}}{67.80\ km\ s^{-1}\ Mpc^{-1}}$ $= 4425\ Mpc$ $d_C = d_H \chi$ $= 8850\ Mpc$ Due to cosmological expansion, the wavelength of light received is increased and the photon flux is reduced, each which introduces a (1 + z) factor. Hence, $F = \frac{L_{bol}}{4\pi d_C^2(1+z)^2} = \frac{6.3 \times 10^{13} L_{Sun}}{4\pi d_L^2}$ Award this bonus 1m only if participant has considered *luminosity distance* d_L (using *Etherington's reprocity theorem*) or $(1 + z)^2$ factor. Hence, comparing apparent magnitudes of the Sun and quasar (*Pogson's law*),	B1 M1

	$$m_{quasar} - m_{Sun} = -2.5lg\frac{F_{quasar}}{F_{Sun}}$$ $$m_{quasar} - m_{Sun} = -2.5lg\frac{\frac{6.3\times10^{13}L_{Sun}}{4\pi d_L^2}}{\frac{L_{Sun}}{4\pi d_{Sun}^2}}$$ Eliminating L_{Sun}, $$m_{quasar} = m_{Sun} - 2.5lg\frac{(6.3\times10^{13})d_{Sun}^2}{d_L^2}$$ $$= -26.74 - 2.5g\frac{(6.3\times10^{13})(1\ AU)^2}{(71510\times10^6\ pc)^2}$$ $= 19.6$ OR **15.1 if using d_C (transverse comoving distance)** Award this 1m only if participant has carried through the calculation correctly and obtained the mark for the Hubble distance.	**M1** **B1**
	This sub-part has four marking points for three marks. Common Mistakes: • Bringing in special relativity when it was not needed – the velocity we observe is not the peculiar velocity of the quasar • No contradiction between special relativity and general relativity – *Hubble radius* d_H is **not infinite** • Low wavelength approximation for redshift z does not hold for large z! • Not reading the question thus not substituting $v = c$ as directed • Apparent magnitude was poorly calculated: incorrect coefficient of *log* term, incorrect conversion between parsecs and AU, comparing only ratio of distances • Ignoring the dimensionless factor χ • $\chi(z)$ means χ is a function of redshift z; do not multiply χ with z!	
c)	The Hubble time, for a linear expansion model of the Universe, is $$t_H \approx \frac{1}{H_0}$$ $$= \frac{1}{67.80\ km\ s^{-1}\ Mpc^{-1}}$$ $$= \frac{3.086\times10^{16}\times10^6\ m}{67.80\times10^3\ m\ s^{-1}}$$ $$= 4.552\times10^{17}s$$ $$= 14.4\ billion\ years$$ **OR** quote an estimate: $t_0 = 13.8 \pm 0.7\ billion\ years$ (5% error allowed) Award this 1m only if participant has provided has provided a reasonable estimate of the age of the Universe.	**B1**

	Hence, the age of the Universe at the end of the matter-dominated era $t_{m\text{-}\Lambda}$, $t_{m-\Lambda} = 14.4 - 4.05$ $= 10.35\ billion\ years$ Since our Universe is flat, and $\Omega_{r,0}$ is negligible at the current epoch, $\Omega_{m,0} + \Omega_{\Lambda,0} = \Omega_{total,0} = 1$ $\therefore \Omega_{\Lambda,0} \simeq 0.69$ At the matter-dark energy equality, $\Omega_\Lambda = \Omega_m$, $\Rightarrow \Omega_{\Lambda,0} = \Omega_{m,0} a^{-3}$ $a_{m-\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}$ $= \left(\frac{0.31}{0.69}\right)^{1/3}$ $= 0.766$	B1
	Do not award this 1m if $\Omega_{m,0} + \Omega_{\Lambda,0} \neq 1$. At the end of the radiation-dominated era (start of the matter-dominated era), $\Omega_r = \Omega_m$, $\Omega_{r,0} a^{-4} = \Omega_{m,0} a^{-3}$ $a_{r-m} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$ $= \frac{9.1 \times 10^{-5}}{0.31}$ $= 2.94 \times 10^{-4}$ which corresponds to a redshift z of around 3400.	B1
	Between the radiation-matter equality and the matter-dark energy equality, the Universe is matter-dominated, hence use $a(t) \propto t^{2/3}$. $\frac{a_{r-m}}{a_{m-\Lambda}} = \left(\frac{t_{r-m}}{t_{m-\Lambda}}\right)^{2/3}$ Rearranging to find $t_{r\text{-}m}$, $t_{r-m} = \left(\frac{a_{r-m}}{a_{m-\Lambda}}\right)^{3/2} t_{m-\Lambda}$ $= \left(\frac{2.94 \times 10^{-4}}{0.766}\right)^{\frac{3}{2}} (10.35 \times 10^9)$ $= \boldsymbol{78,000\ years}$	B1

Award this 1m if participant has used the correct proportionality relation $a(t) \propto t^{2/3}$ and has carried through the calculation.

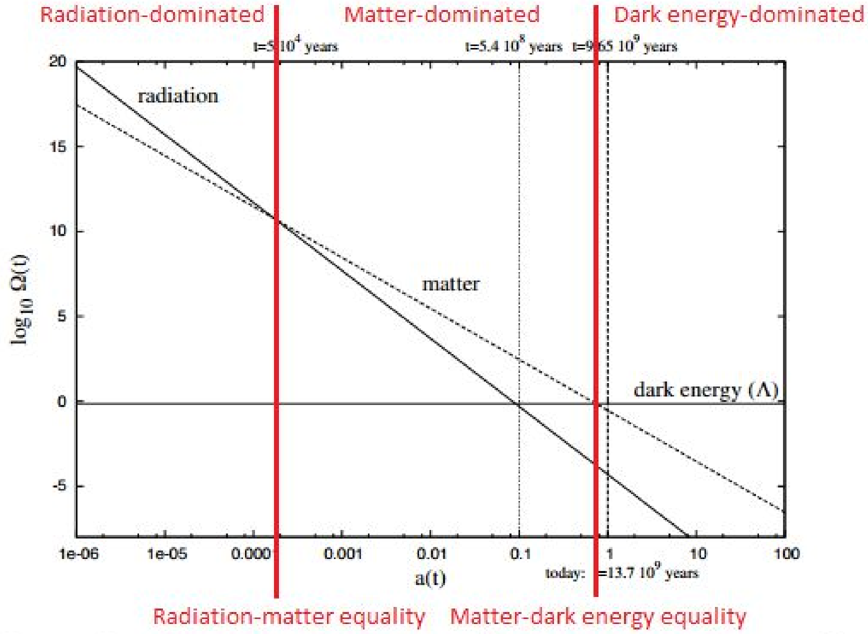


Figure 49: Three epochs in the evolution of the Universe: (1) radiation-dominated $a < a_{eq}$, (2) matter-dominated $a_{eq} < a < a_{eq2}$, (3) dark energy-dominated $a > a_{eq2}$.

Common Mistakes:

- Some candidates treated the *proportionality relations* as equations
- Little understanding of how the density of each component (radiation, matter, dark energy) change with evolution of the Universe
- Many treated the scale factor *a* as a distance – it is a *dimensionless constant* relating one distance to another distance measured at a given reference time (convention: reference time is current epoch)
- Many did not understand the density parameter Ω – it is the ratio of the mass density of a component to the critical density at current epoch (for radiation and dark energy, converted using Einstein's *mass-energy equivalence*)
- Participants did not understand that the total density of the Universe today has been measured to be very close (virtually equal) to the critical density at current epoch

d)	Since a_0 = 1 by definition, $H^2 = H_0^2\left[\frac{\rho}{\rho_{c,0}} + \frac{1}{a^2}\left(1 - \frac{\rho_0}{\rho_{c,0}}\right)\right]$ Then, substituting in the definition of the density parameter Ω_X, we get $\Omega_X = \frac{\rho_X}{\rho_{c,0}}\ where\ X = m, r, \Lambda$ $H^2 = H_0^2\left[\Omega + \frac{1}{a^2}(1 - \Omega_0)\right]$ where a subscript 0 indicates the value of the density parameter now. Since the density of the Universe ρ is equal to the sum of the density of its three components, thus $\Omega_X = \Omega_r + \Omega_m + \Omega_\Lambda$ $\Omega_{X,0} = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$ Award this 1m if <u>at least one</u> of the two equations above are stated. Hence, we get $H^2 = H_0^2\left(\Omega_r + \Omega_{m,} + \Omega_\Lambda + \frac{1}{a^2}(1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0})\right)$ However, as the densities of the components change with time, so do the density parameters, i.e. $\Omega_m = \Omega_{m,0}a^{-3}$ $\Omega_r = \Omega_{r,0}a^{-4}$ (extra factor of a^{-1} is due to redshift) $\Omega_\Lambda = \Omega_{\Lambda,0}$ (density of dark energy remains constant as Universe expands) $H^2 = H_0^2\left(\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0} + \frac{1}{a^2}(1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0})\right)$ Award this 1m for expressing how the densities of the components change with a and correct use of density parameters to obtain <u>all</u> $\Omega_{r,0}a^{-4}$, $\Omega_{m,0}a^{-3}$ and $\Omega_{\Lambda,0}$. Substituting in the spatial curvature equation from the formula list, since $\Omega_{total,0}$ = 1 (corresponding to a flat critical Universe with zero curvature), $\Omega_{k,0} = 1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0}$ $H^2 = H_0^2(\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0} + \Omega_{k,0}a^{-2})$ Award this 1m for substituting spatial curvature equation to obtain $\Omega_{k,0}a^{-2}$.	**M1** **B1** **M1**

	By definition, the Hubble parameter, $H \equiv \frac{\dot{a}(t)}{a(t)}$ $\therefore \frac{\dot{a}^2}{a^2} = H_0^2\left(\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0} + \Omega_{k,0}a^{-2}\right)$ Award this 1m for correct definition of the Hubble parameter and its substitution into the Friedmann equation. Multiplying a^2 across and taking square-root on both sides, we obtain the desired equation $\dot{a} = H_0\sqrt{\Omega_{r,0}a^{-2} + \Omega_{m,0}a^{-1} + \Omega_{k,0} + \Omega_{\Lambda,0}a^2}$ Award this final 1m only if no severe mistakes were made in manipulation.	**B1** **B1**
	Common Mistakes: • The *Hubble constant*, is in fact, the value of the *Hubble parameter* at the current epoch; its value changes with the evolution of the Universe • Most candidates did not understand the Universe is made up of radiation (photos and neutrinos), matter (baryonic and dark) and dark energy • Lack of understanding and misuse of density parameter and critical density • Not simplifying the $(1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0})$ term • Lack of understanding that the Universe is essentially critical and flat • Poor mathematical manipulation • Attempting to differentiate LHS does not spawn out an $\dot{a}$!	
e)	When $\ddot{a} = 0$, the Universe switched from decelerating (where $\ddot{a} < 0$) to accelerating expansion (where $\ddot{a} > 0$), thus $2\Omega_{r,0}a^{-3} + \Omega_{m,0}a^{-2} - 2\Omega_{\Lambda,0}a = 0$ Since $\Omega_{r,0}$ is negligible (order of 10^{-5}), this further reduces to $\Omega_{m,0}a^{-2} - 2\Omega_{\Lambda,0}a = 0$ $\therefore \Omega_{m,0}a^{-2} = 2\Omega_{\Lambda,0}a$ $a^3 = \frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}$ For the standard model of the Universe (ΛCDM), $\Omega_{m,0} \sim 0.31$ and $\Omega_{\Lambda,0} \sim 0.69$, $a = \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{m,0}}{2(1-\Omega_{m,0})}\right)^{1/3} = \mathbf{0.61}$ Award this final 1m only if $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$.	**A1** **M1** **B1**

	Participants are to understand that if a represents the scale factor (*ratio of a distance at time t relative to a reference*), then ȧ is akin to velocity (as it shows how that distance changes with time) and thus ä to acceleration. Participants are allowed to solve the quartic equation $$2\Omega_{\Lambda,0}a^4 - \Omega_{m,0}a - 2\Omega_{r,0} = 0$$ by any means and will be awarded full credit provided $\Omega_{m,0}$, $\Omega_{\Lambda,0}$ and $\Omega_{r,0}$ follow the criteria above and the other unreal solutions ($a < 0$) are rejected. Note solving the quartic will also give $a = \mathbf{0.61}$. Common Mistakes: • Considering $\dot{a} = 0$ as the transition condition instead of $\ddot{a} = 0$.	
f)	**Non-calculus derivation** Considering the hydrogen cloud of mass m, then the gravitational force is $$F_g = \frac{GMm}{R^2}$$ The radiation flux at a distance R is $$F = \frac{L}{4\pi R^2}$$ Then, consider the radiation pressure at this distance, **since $\boldsymbol{p = E/c}$**, $$P_{rad} = \frac{L}{4\pi R^2}\frac{1}{c}$$	**B1**
	Then, the radiation force on the nuclei, $$F_{rad} = P_{rad}A = P_{rad}\sigma_T$$ Award this 1m only if the opacity κ has been considered (correct dimensions).	**M1**
	Equating the two and substituting $\kappa = \sigma_T/m_p$, $$\frac{GMm}{R^2} = \frac{L\sigma_T}{4\pi R^2}\frac{1}{c}$$ $$GM = \frac{L}{4\pi c}\frac{\sigma_T}{m_p}$$	**M1**
	$$\therefore \boldsymbol{L_{Edd} = \frac{4\pi GMc}{\kappa}}$$ *Note that this derivation assumes spherical symmetry, which realistically only applies to newly formed accretion clouds as they eventually collapse to a disk.	**B1**

	<u>Calculus derivation</u> The radiation energy flux at a distance R is $$\frac{dE}{dtdA} = \frac{L}{4\pi R^2}$$ For a photon, ***p* = *E*/*c***, thus the momentum flux is $$\frac{dp}{dtdA} = \frac{L}{4\pi c R^2}$$ Hence the momentum transfer rate to an electron with Thomson cross-section σ_T is $$\frac{dp}{dt} = \frac{L}{4\pi c R^2} dA = \frac{L}{4\pi c R^2} \sigma_T$$ This must be less than the gravitational force, hence $$\frac{L}{4\pi c R^2} \sigma_T \leq \frac{GMm_p}{R^2}$$ Hence, rearranging, $$\boldsymbol{L_{Edd}} = \frac{4\pi GMcm_p}{\sigma_T} = \boldsymbol{\frac{4\pi GMc}{\kappa}}$$	**B1** **B1** **M1** **A1**
	The derivation is considerably simplified by considering the gravitational and radiation force on just a single electron-proton pair. Common Mistakes: • Dimensionally inconsistent equations and expressions – *do not underestimate dimensional analysis* • Incorrect expression relating the momentum and energy of a photon • Introduction of new variables not described in the question without proper definition • Many failed to substitute the opacity κ in the final expression as required • Some brought in attenuation relation and optical density/depth, introducing unnecessary dimensions that complicated matters • No need to consider Hawking radiation from the black hole in the quasar due to its negligible power output, in absolute terms	

<table>
<tr>
<td>g)</td>
<td>
By observation, the quasar must be radiating away a portion of the GPE of the accreted rest-mass energy, which can be expressed as a fraction ϵ = 0.1 of RME.

$$L_{Edd} = \epsilon \dot{M} c^2$$
$$\dot{M} = \frac{L_{quasar}}{\epsilon c^2}$$
$$= \frac{6.3 \times 10^{13} \times 3.85 \times 10^{26}}{0.1 \times (3.00 \times 10^8)^2}$$
$$= 2.7 \times 10^{24}\ kg\ s^{-1}$$
$$\approx \mathbf{43\ solar\ masses\ year^{-1}}$$

Award this 1m even if the Eddington luminosity was not derived in part (f).
There is no need to express $\dot{M}$ in solar masses per year.

$$M_{quasar} = \frac{L_{quasar}\sigma_T}{4\pi G c m_p}$$
$$= \frac{(6.3 \times 10^{13} \times 3.85 \times 10^{26})(6.65 \times 10^{-29})}{4\pi(6.67 \times 10^{-11})(3.00 \times 10^8)(1.67 \times 10^{-27})}$$
$$= 3.84 \times 10^{39}\ kg$$
$$= \mathbf{1.9 \times 10^9\ solar\ masses}$$

The estimated mass is in close agreement with the reported mass.

Award final 1m for logical conclusion provided the previous 1m was awarded.
</td>
<td>**B1**

B1

A1</td>
</tr>
<tr>
<td></td>
<td>
Common Mistakes:

- Many did not read the question thoroughly and left out the estimate of the *Eddington-limited accretion rate*
- The mass estimated from the backhand estimate, whilst off by ~5%, is actually in close agreement (within experimental error).
</td>
<td></td>
</tr>
</table>

Q6 Celestial coordinate systems and geometric astronomy (16m)

1: Calculating Threa's Coordinates

In this solution, the subscript R for Rams' orbital elements are omitted as they are used very often.

Additionally, the following parameters are used:

- Rams' semi-minor axis, b
- Linear eccentricity of Rams' orbit (i.e. distance between centre and either foci), $f = ae$

Define a right-handed coordinate system C_0 such that

- The origin is in the centre of Rams' orbit.
- The x-axis points in the direction of Rams' aphelion.
- The z-axis points perpendicular to Rams' orbit.
- When facing the negative z-direction, Rams orbits in the anti-clockwise direction.

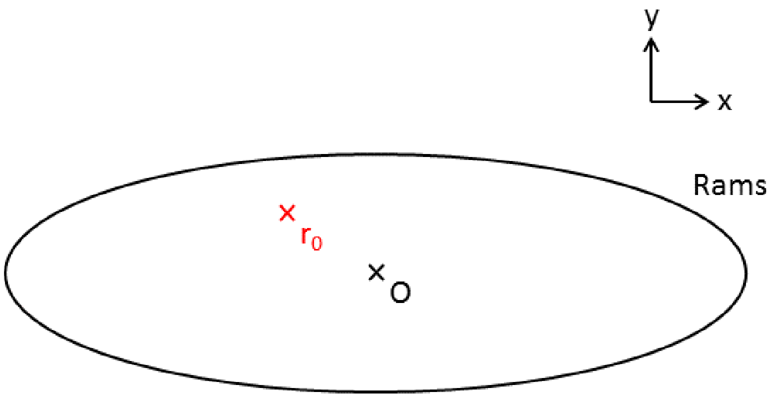


Figure 1: The coordinate system C_0

In this system, Threa will occupy the same coordinates on every Thimssarc day. This system will turn out to be the most useful system to work in later. We will obtain Threa's coordinates in this system, $\vec{r}_0$ through a series of coordinate transformations.

Then, define another right-handed coordinate system C_1 such that

- The origin is in the centre of **Threa's orbit**.
- The y-axis points in the direction of **Rams' descending node**.
- The z-axis is perpendicular to **Threa's orbit**.
- When facing the negative z-direction, **Threa** orbits in the anti-clockwise direction.

At vernal equinox, Threa is at the point in its orbit diametrically opposite the Vernal Point ♈, such that at local noon the Vernal Point is directly overhead.

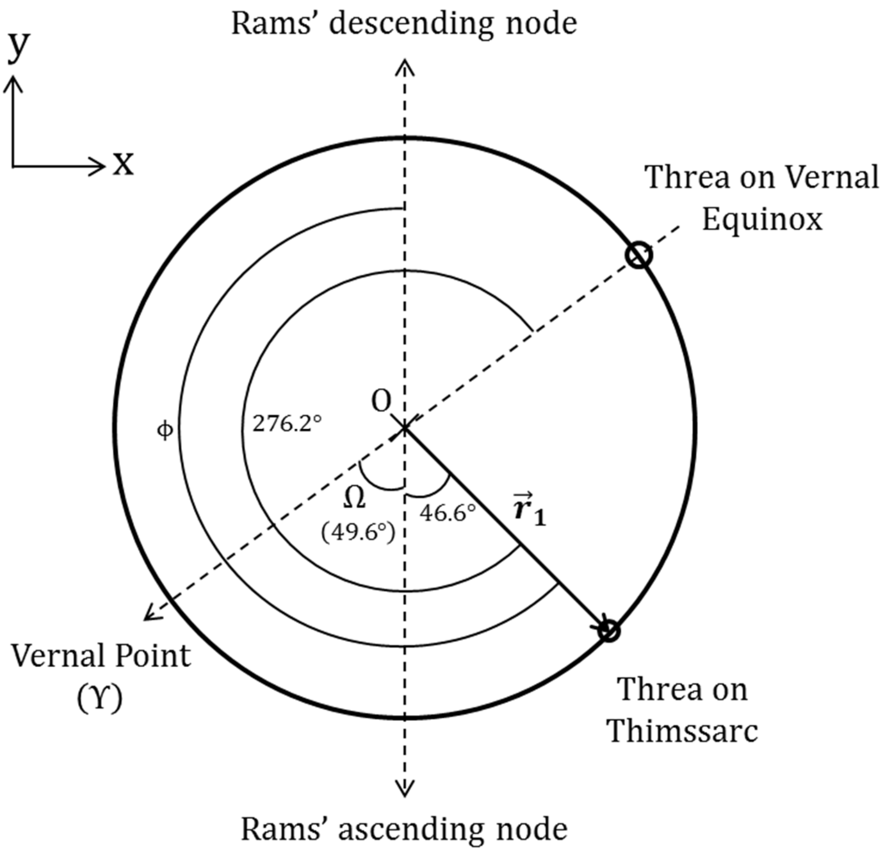


Figure 2: The coordinate system C_1

On Thimssarc day, Threa is

$$\vartheta = \frac{280}{365} \times 360° = 276.2°$$

anticlockwise from the point in its orbit diametrically opposite the Vernal Point, and thus an angle

$$\phi = 276.2° - 49.6° = 226.6°$$

anticlockwise from the positive y-axis.

Hence, Threa's coordinates $\vec{\boldsymbol{r}}_1$ in C_1 are

$$\vec{\boldsymbol{r}}_1 = a_T \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

Now, define the right-handed coordinate system C_2 such that

- The origin is in the centre of Threa's orbit.
- The y-axis points in the direction of Rams' descending node.
- The z-axis is perpendicular to **Rams' orbit**.
- When facing the negative z-direction, **Rams** orbits in the anti-clockwise direction.

B1

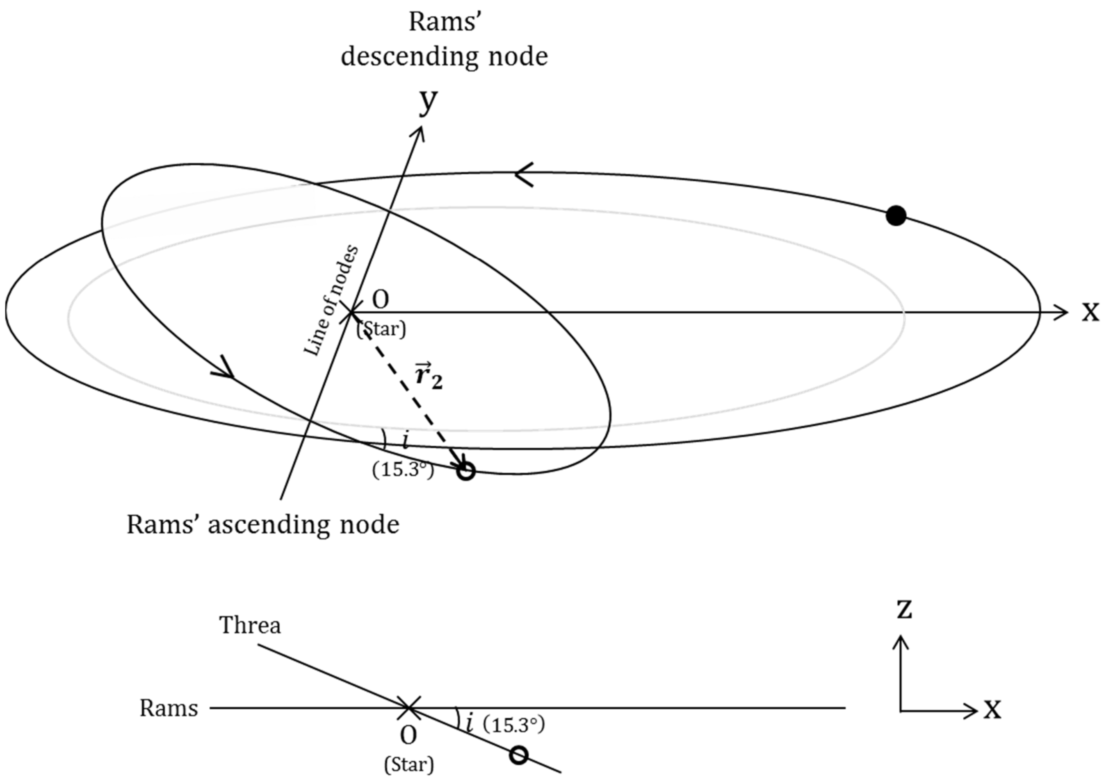


Figure 3: The coordinate system C_2

C_2 is C_1 rotated by $i = 15.3°$ about the y-axis, so the y-coordinate remains unchanged. Projecting to the xz-plane, the length of Threa's position vector remains unchanged but is now an angle 15.3° **below** the x-axis, so the new coordinates are

$$\vec{\boldsymbol{r}}_2 = a_T \begin{pmatrix} -\sin\phi\cos i \\ \cos\phi \\ \sin\phi\sin i \end{pmatrix} = a_T \begin{pmatrix} 0.7008 \\ -0.6871 \\ -0.1917 \end{pmatrix}$$ **B1**

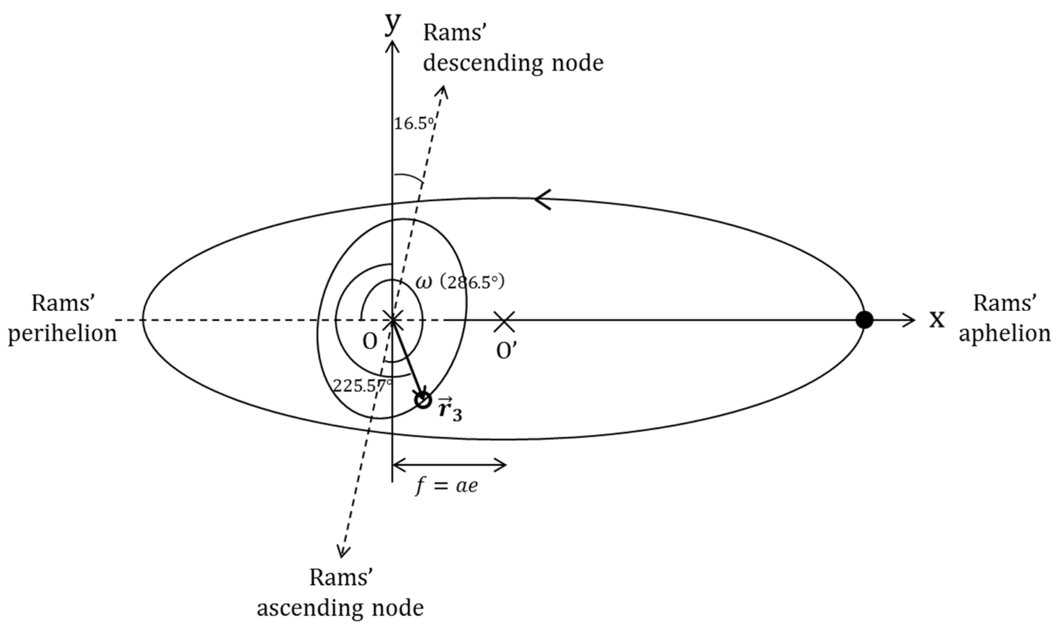


Figure 4: The coordinate system C_3

Lastly, define the right-handed coordinate system C_3 such that • The origin is in the centre of Threa's orbit. • The **x-axis** points in the direction of **Rams' aphelion**. • The z-axis is perpendicular to Rams' orbit. • When facing the negative z-direction, Rams orbits in the anti-clockwise direction.	
C_3 is C_2 rotated by $\omega - 270° = 286.5° - 270° = 16.5°$ about the z-axis, so the z-coordinate remains unchanged. Let $\vec{\boldsymbol{r}}'_2$ and $\vec{\boldsymbol{r}}'_3$ be $\vec{\boldsymbol{r}}_2$ and $\vec{\boldsymbol{r}}_3$ projected to the xy-plane, and transform,	B1
$\lVert\vec{\boldsymbol{r}}'_2\rVert = a_T\sqrt{0.7008^2 + (-0.6871)^2} = 0.9814a_T$	
$\text{Angle of } \vec{\boldsymbol{r}}'_2 \text{ from y} - \text{axis} = \tan^{-1}\left(\frac{0.7008}{0.6871}\right) + 180° = 225.57°$	B1
$\text{Angle of } \vec{\boldsymbol{r}}'_3 \text{ from y} - \text{axis} = 225.57° - 16.5° = 209.07°$	
$\vec{\boldsymbol{r}}_3 = a_T\begin{pmatrix} -0.9814\sin 209.07° \\ 0.9814\cos 209.07° \\ -0.1917 \end{pmatrix} = a_T\begin{pmatrix} 0.4768 \\ -0.8578 \\ -0.1917 \end{pmatrix}$	B1
C_0 is simply C_3 translated in the negative x-direction by $f = ae = 0.456$ AU, so we have	
$\vec{\boldsymbol{r}}_0 = \vec{\boldsymbol{r}}_3 - \begin{pmatrix} ae \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.02080 \\ -0.8578 \\ -0.1917 \end{pmatrix}$	B1
Award 0.5m if the linear eccentricity f is calculated but $\vec{\boldsymbol{r}}_0$ is not.	
<u>2: Removing a Dimension</u> Hence, we want to find the point $\vec{\boldsymbol{r}} = (x, y, 0)$ on Rams' orbit that minimises its distance from $\vec{\boldsymbol{r}}_0 = (x_0, y_0, z_0)$. This is equivalent to minimising its square, which is given by	
$\Delta^2 = (x - x_0)^2 + (y - y_0)^2 + z_0^2$	
Since z_0 is constant, this is equivalent to minimising $(x - x_0)^2 + (y - y_0)^2$, that is, projecting $\vec{\boldsymbol{r}}$ onto the xy-plane and solving the two-dimensional problem of finding the point on Rams' orbit that minimises its distance from $\vec{\boldsymbol{r}'} = (x_0, y_0, 0)$.	
<u>3: Proof of the Lemma</u> Consider an ellipse E and a point P inside the ellipse. We want to find the point Q on E that minimises its distance from P. We have the following:	
PQ is normal to E at Q.	
Using this lemma, we can then find Q by finding the intersection of PQ and E.	

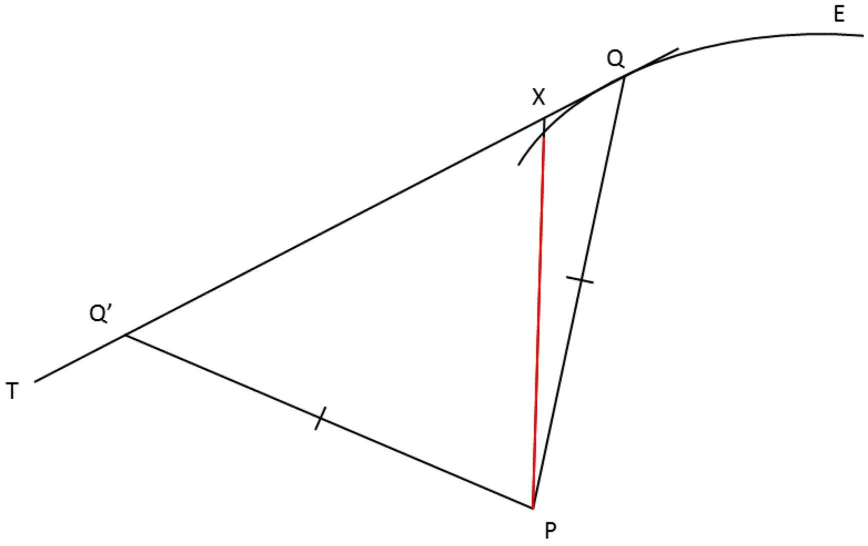


Figure 5: Proof that PQ must be normal to E at Q

To prove this, draw the tangent T to E at Q. Assume, to prove a contradiction, that PQ is not normal to E. Then, it is possible to find another point Q on T such that PQQ' is isosceles. Choose any point X between QQ'.

Since P is inside E and E is convex, the line PX intersects E at a point closer to P than QQ', hence the arc QQ' of the circle centred at P, and hence Q, contradicting the fact that Q is the point on E closest to P. Hence proven by contradiction that PQ is normal to E.

4: Finding the Equation of the Normal

Now we need to find the equation of PQ in terms of the parameters of E and the coordinates of P.

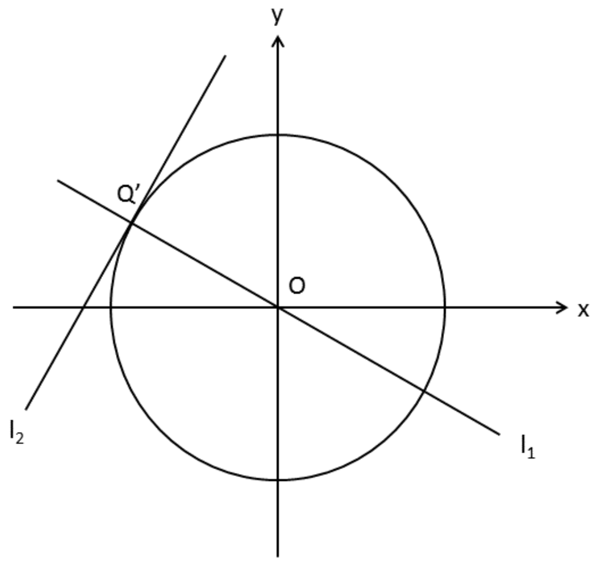
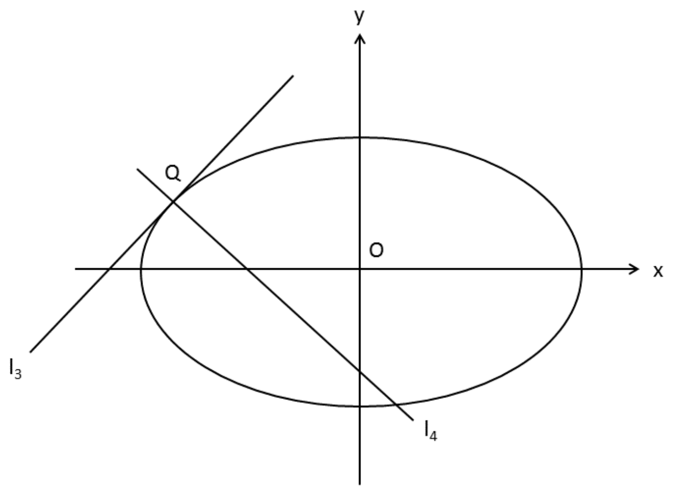


Figure 6: Lines l_1 and l_2

Consider the transformation that scales along the y-axis by a factor of $\frac{1}{b}$ and along the x-axis by a factor of $\frac{1}{a}$. This transforms the ellipse E into the unit circle and Q with coordinates (x_Q, y_Q) into another point Q' with coordinates $(x'_Q, y'_Q) = \left(\frac{x_Q}{a}, \frac{y_Q}{b}\right)$ Let l_1 be the line OQ' and l_2 be the line normal to l_1 at Q'. l_2 would thus be tangent to the circle. Then, for l_1, we have $y = \frac{y'_Q}{x'_Q} x$	**B1**
For l_2, we have, $y - y'_Q = -\frac{x'_Q}{y'_Q}(x - x'_Q)$	**B1**
 Figure 7: Lines l_3 and l_4 Now we transform l_2 back to find the equation of l_3, which is tangent to E at Q, $\frac{y}{b} - y'_Q = -\frac{x'_Q}{y'_Q}\left(\frac{x}{a} - x'_Q\right)$ $\Rightarrow y - y_Q = -\frac{b^2 x_Q}{a^2 y_Q}(x - x_Q)$	**B1**
Then, we find the equation of PQ by finding the equation of l_4, the line normal to l_3, $y - y_Q = \frac{a^2 y_Q}{b^2 x_Q}(x - x_Q)$ $\Rightarrow b^2 x_Q(y - y_Q) = a^2 y_Q(x - x_Q)$	**B1**

5: Forming the Quartic

We now change our notation to what we used in Section 2 by replacing (x, y) with(x_0, y_0) and (x_Q, y_Q) with (x, y). The equation of the line becomes

$$b^2x(y_0 - y) = a^2y(x_0 - x)$$

We want to find the intersection of this line and the ellipse (we take the bottom half since $y_0 = -0.8578 < 0$),

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$
$$\Rightarrow y = -\frac{b}{a}\sqrt{a^2 - x^2}$$

Substituting the equation of the line into the equation of the ellipse E,

$$b^2xy_0 = a^2\left(-\frac{b}{a}\sqrt{a^2 - x^2}\right)(x_0 - x) + b^2x\left(-\frac{b}{a}\sqrt{a^2 - x^2}\right)$$
$$\Rightarrow b^2xy_0 = \left(-\frac{b}{a}\sqrt{a^2 - x^2}\right)[a^2x_0 - a^2x + b^2x]$$

Using $a^2 - b^2 = f^2$,

$$\therefore b^2xy_0 = \left(-\frac{b}{a}\sqrt{a^2 - x^2}\right)[a^2x_0 - f^2x]$$

B1

Award this 1m for substituting the equation of the line into that of the ellipse without any mistakes.

Squaring both sides and expanding, we get a quartic,

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

M1

Award this 1m for formulating the quartic symbolically (mark also awarded if all working is done numerically – provided no mistakes were made in manipulation).

with coefficients

$$A = -f^4 = -0.04324\ AU^4$$
$$B = 2a^2f^2x_0 = 0.01999\ AU^5$$
$$C = -a^4x_0^2 - a^2b^2y_0^2 + a^2f^4 = -3.477\ AU^6$$
$$D = -2a^4f^2x_0 = -0.04617\ AU^7$$
$$E = a^6x_0^2 = 0.005336\ AU^8$$

A1

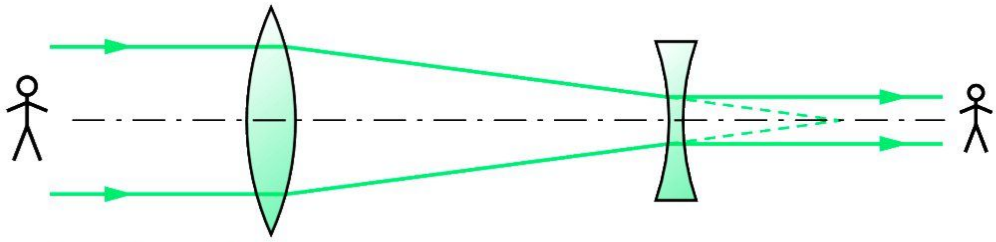
where we also use $b = a\sqrt{1 - e^2} = 1.450\ AU$.

Award this 1m for calculating the values of <u>all five</u> coefficients without mistakes (their quartic may differ but still correct).

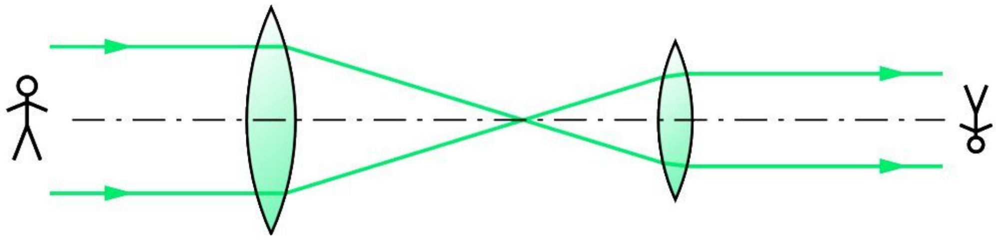
<u>6: Solving the Quartic</u> For convenience, make the first coefficient unity, then $x^4 - 0.4622x^3 + 80.41x^2 + 1.068x - 0.1234 = 0$ We know, intuitively, that the solution should lie between x_0 and a. We can thus use any of the following techniques to solve the quartic: • Using a graphic calculator • Using the secant method with initial bounds x_0 and a • Using Newton's method with initial guess a • Using binary search with initial bounds x_0 and a • Depressing the quartic and factorising into a sum of two squares • Writing the quartic as a product of two quadratics, expanding and comparing coefficients (not recommended) Regardless of which method we use, we get $x = 0.03310$. (The other real root $x = -0.04636$ is rejected.) Award this 1m for correct answer even if no working is shown and solution method is not stated. **<u>7: Computing the Answer</u>** Using the equation of the ellipse, $y = -\frac{b}{a}\sqrt{a^2 - x^2} = -1.450\text{ AU}$ Award this 1m for finding y (or x if quartic was in y). Finally, $\Delta = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z_0^2} = \mathbf{0.623\ AU}$ Accept answers given to accuracy of either 2 s.f. or 3 s.f.	 **A1** **B1** **A1**
Intended to be the most challenging question. Generally few participants made an attempt to answer the question; most attempts were generally short and led to no fruitful progress to solving the question.	

Q7 Optics and detectors (6m)

a) From the given context, the scope is a **Galilean refractor**, as the image and source object are both upright in order for it to facilitate aiming in the battlefield, i.e.



A. Galilean refractor



B. Newtonian refractor

M1

Hence, *the focal point of the objective lens is behind the eyepiece lens*.

From the given magnification (3x), the ratio of the focal lengths of the objective to eyepiece focal lengths is 3, i.e.

$$\therefore \frac{f_o}{f_e} = 3$$

Award this 1m only if the participant has shown consideration of the above.

B1

From a ray diagram, since the distance between the two lenses is the difference between their focal lengths, the objective focal length f_o is

$$\begin{aligned} \therefore f_o &= L \times \frac{3}{2} \\ &= 120\, mm \times \frac{3}{2} \\ &= 180\, mm \end{aligned}$$

Thus, the f-number is

$$\begin{aligned} \mathrm{F} &= \frac{f_o}{D} \\ &= \frac{180\, mm}{25\, mm} \\ &= \mathbf{7.2} \end{aligned}$$

<table>
<tr><td></td><td>Common Mistakes:
Expression for f-number was inverted, giving extremely fast but unrealistic optics
Misuse of terms focal length and aperture, by those with little understanding of optics</td><td></td></tr>
<tr><td>b)</td><td>For point sources (i.e. stars),

$$\begin{aligned}\frac{LGP_{scope}}{LGP_{eye}} &= \frac{A_{scope}}{A_{eye}} \\ &= \left(\frac{D_{scope}}{D_{eye}}\right)^2 \\ &= \left(\frac{25\ mm}{8\ mm}\right)^2 \\ &= \mathbf{9.765}\end{aligned}$$

Hence, the new limiting magnitude is

$$\begin{aligned}m'_{lim} &= m_{lim} + 2.5\log\left(\frac{LGP_{scope}}{LGP_{eye}}\right) \\ &= 3.2 + 2.5\log(9.765) \\ &= \mathbf{5.67}\end{aligned}$$</td><td>B1
B1</td></tr>
<tr><td></td><td>Common Mistakes:
Light Gathering Power/light grasp was often incalculated incorrectly where employed
Incorrectly converting the extra factor in LGP to a decrease in limiting magnitude instead of an increase (ability to detect fainter objects)
Magnification does not affect the limiting magnitude obtained this way for point objects</td><td></td></tr>
<tr><td>c)</td><td>The angular resolution of the scope using the Rayleigh criterion is

$$\vartheta_{lim} = \frac{1.22\lambda}{D}$$

Hence assuming the wavelength of visible light $\lambda = 500nm$, the linear resolution of the scope for small ϑ_{lim} is

$$\begin{aligned}d_{lim} &= \frac{1.22\lambda}{D}R \\ &= \frac{1.22(500\times10^{-9}\ m)}{25\times10^{-3}\ m}(384400\times10^3\ m) \\ &= \mathbf{9.38\times10^3\ m}\end{aligned}$$

Hence, since $d_{lim} < d_{Ross}\ and\ d_{lim} < d_{Rosse}$, the scope can resolve both craters.

Only award full 2m if the wavelength of visible light is reasonably estimated.</td><td>M1
B1</td></tr>
</table>

	Other conclusions based on reasonable estimates of the wavelength of visible light between 300*nm* and 700*nm* are also accepted. Correct conclusions on the basis of incorrect physics are awarded no marks. Common Mistakes: • Guessing an answer without quantitative proof nets zero marks. • Some used the incorrect units for angular resolution • Missing assumption for the wavelength of light in the visible spectrum/wavelength used is not part of visible spectrum	