4th Singapore Astronomy Olympiad

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Question	Score
1	/14
2	/8
3	/20
4	/12
5	/25
6	/16
7	/5

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1	/14
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Rules

- Students may use pen or pencil to answer the paper.
- Any working or answers on the question papers will not be considered.
- Only the use of scientific calculators is allowed.
  Graphing/programmable calculators are not allowed.
- Students are to bring their own stationery.
- Students have 3 hours to complete the paper. If a student is late, no time extension is granted.
- Students should **clearly state their assumptions** in their working, if any.
- Students may leave any time upon submission of their solutions and question paper.
- Cheating or allowing others to cheat is grounds for immediate disqualification.
- No notes are allowed.
- Question papers must be returned together with their scripts. **The cover page and Figure 2 are to be attached to the answer script.**

Constants / Values

Gravitational constant, G: 6.67×10^{-11} m^3 kg^{-1} s^{-2}
Speed of light in vacuum, c: 3.0×10^{8} m s^{-1}
Hubble constant, H_0: 67.8 km s^{-1} Mpc^{-1}
Radius of Earth, R_e: 6.371×10^{6} m
Mass of Earth, M_e: 5.972×10^{24} kg
Solar mass: 2.0×10^{30} kg
Nominal solar luminosity: 3.828×10^{26} W
Thomson-cross section for an electron, σ_T: 6.65×10^{-29} m^2
Mass of a proton, m_p: 1.67×10^{-27} kg
Current density parameter for matter (baryonic and dark), $\Omega_{m,0}$: 0.31
Current density parameter for radiation, $\Omega_{r,0} \simeq 9.1 \times 10^{-5}$
[Errata] Apparent magnitude of the Sun: -26.74

Formulae

Hawking Temperature of a black hole, $T_H = \frac{\hbar c^3}{8\pi G M k_B}$
Spatial curvature density, $\Omega_{k,0} = \Omega_{total,0} - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0}$

Question 1

The Hubble Space Telescope (HST), launched in 1990, will someday be decommissioned. It will fire its retrorockets, fall back into Earth's atmosphere and crash harmlessly into the Pacific Ocean. Let us study the dynamics of this operation. Assume that the HST is currently in a circular orbit around the Earth and that the retrorockets will fire in precisely the opposite direction of its motion. The HST has an orbital period of 95.6 minutes.

a) What is the radius of the HST's orbit? [2m]

b) What is the orbital velocity of the HST? [2m]

c) What is the minimum ΔV required to crash the HST into the Earth's surface? [4m]
Briefly explain why such a trajectory would be unwise. [1m]

d) NASA suggests that the HST should impact the ocean at an angle of at most 75° from the vertical. What would be the ΔV required to accomplish this? Neglect air resistance. [5m]

Question 2

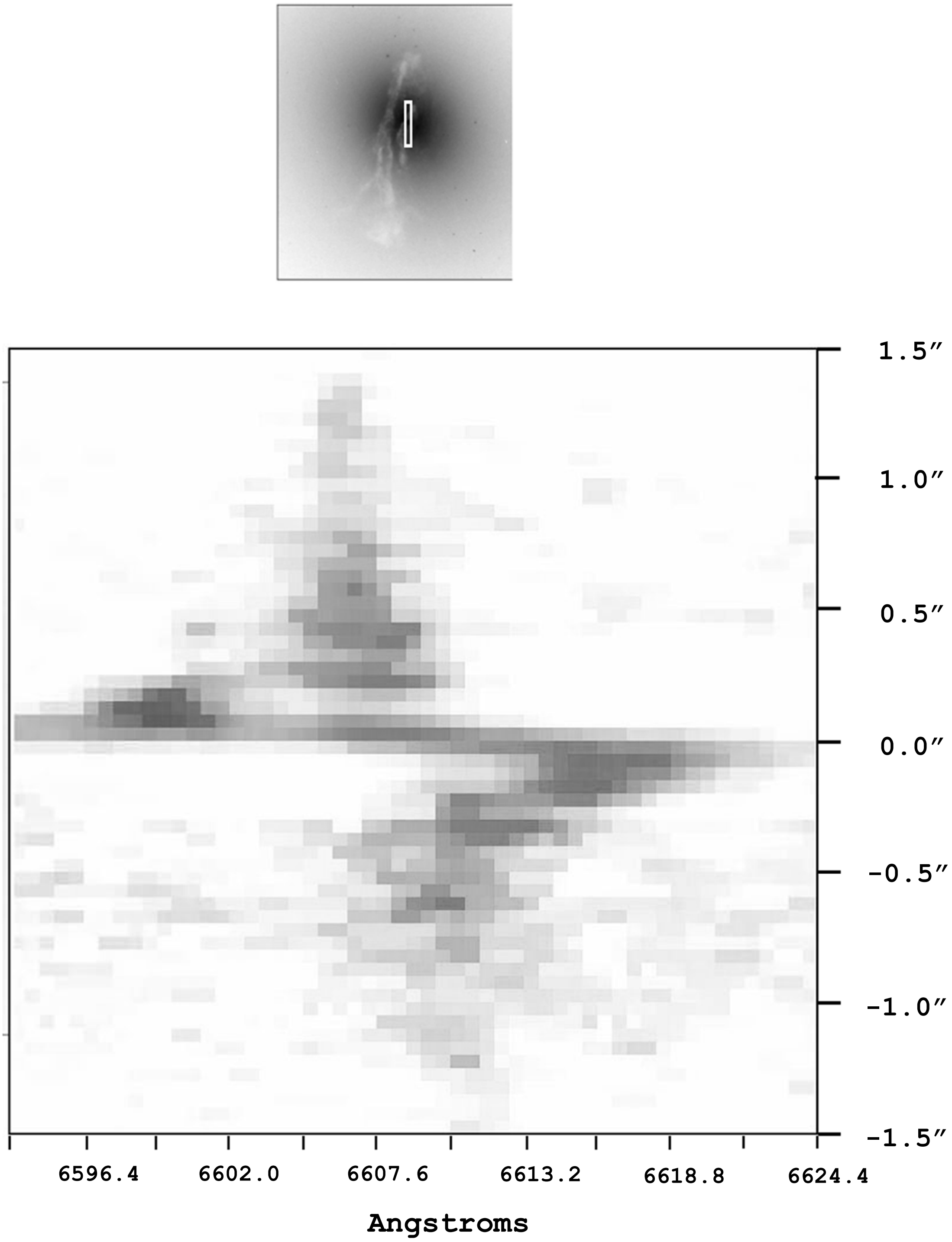
Long-slit spectroscopy was used to observe a gas disk in the central region of the galaxy M84. Refer to Fig. 1 for this problem. On the top there is a small image, which you will not need to analyse to solve this problem; you may just assume that the gas disk is thin, circular, and almost edge-on, and the slit is aligned through the centre of the disk, which coincides with the nucleus of M84. What you will instead need to analyse carefully is the spectrograph data below. It shows part of the data that includes the N II emission line (6583Å). The angular scale is centred to M84's nucleus.

a) Estimate the distance to M84, **in Mpc**. [2m]

b) At this distance, what is the length of the arc subtended by an angle of 1 arcsecond from Earth? [2m]

c) Estimate the total mass (in solar masses) within an 8pc radius of the central nucleus. [4m]

Figure 1. (Data from: G. Bower et al., Hubble Space Telescope, WFPC2, STIS, NASA/ESA)



Question 3

Preamble

A common way of creating a sky chart of the night sky is through a method called stereographic projection, where stars, being points on the celestial sphere, are projected onto a disk and then printed on paper. The edge of the disk represents the horizon, and the four cardinal points (N, S, E, W) are indicated on it.

A recent online news article read:

"*Five planets paraded across the dawn sky early Wednesday in a rare celestial spectacle set to repeat every morning until late next month*"

- The New York Times

Orienting yourself to the sky

The sky chart (Figure 2, printed on page 17), shows the sky at an unknown latitude at 0° Longitude; at 06:15 UTC on 28 Jan, 2016. Do all of the following on the sky chart, in any order:

Note: You may be penalised if the final answers are indecipherable on the sky chart.

a) **Draw out** the famous **Summer Triangle** on the sky chart with solid lines. [2m]

b) **Draw out** the **Little Dipper**, mark out **Polaris**, and determine **the latitude** of the observer. [3m]
Note: In a stereographic projection, constellations near the edge of the sky chart are stretched and appear larger than if they were at the centre of the chart.

c) **Mark out a small arrow** that points to the star **Denebola**. [1m]
The tip of the arrow should be just at where this star should be.
Do the same for the Messier objects **M101** [1m] and **M13** [1m].

d) Draw the **celestial equator (with the label: CE)** on the chart with a solid line. [2m]

e) Using **any** method, **estimate** the **Local Sidereal Time (LST)**. Outline your reasoning as clearly as possible. [3m]

f) **Draw** the **ecliptic (with the label: Ecliptic)** on the chart with a solid line. [2m]

g) **Mark the location of the planets** on the chart. [1m each]
Note that the planets are currently **NOT marked** on the chart. The following hints might be useful:
- Mercury has just risen above the horizon
- Venus is located slightly above (i.e. at a slightly higher altitude from) Kaus Borealis, which is a star in Sagittarius.
- Mars is located slightly west of Alpha Librae (Common name: Zubenelgenubi)
- Jupiter is located slightly west to the point of the Autumnal Equinox
- Saturn is located at a Right Ascension of $16^h50^m38^s$, and a declination of $-20°\ 48.8'$

Question 4

Black holes are objects in space with a gravitational field so large that not even light can escape from its grasp. It has been thought that black holes do not emit radiation until in 1974 when Professor Stephen Hawking gave theoretical proof that black holes do emit blackbody radiation according to the Stefan-Boltzmann law due to quantum effects near the event horizon, and therefore will evaporate its mass away. This radiation was called Hawking Radiation, named after the professor.

Black holes have an equivalent temperature that can be used in the Stefan-Boltzmann law; this is called the Hawking Temperature. In this question, we will attempt some very crude black hole thermodynamics and consider the abnormal behaviour of black holes when it comes to thermodynamics. Consider only non-rotating and non-charged black holes.

a) The Schwarzschild radius of a black hole is the radius at which objects travelling at the speed of light cannot escape. It also doubles as the "radius" of a black hole. Given this, relate the surface area of a black hole, A, to its mass, M. [2m]

b) Given a black hole of mass M, calculate its luminosity. Let the emissivity of a black hole be 1. [2m]

c) Consider a star of absolute magnitude m_0. For some reason it decides to split into N identical stars of the same temperature as the initial star, conserving mass (i.e. each new star has a mass of initial mass/N). What is the final combined absolute magnitude of all N stars? [4m]

d) Consider now, that a black hole of absolute magnitude m_0 also decides to split into N identical black holes, conserving mass only (i.e. each new black hole has a mass of initial mass/N). What is the final combined absolute magnitude of all N black holes? [4m]

Question 5

In June 2011, scientists reported the discovery of the furthest known quasar yet with a redshift beyond 7. Located in Leo, **ULAS J1120+0641** has a luminosity of 6.3 x 10^{13} solar luminosities. Fig. 3 shows its spectrum in infrared wavelengths.

a) In Fig. 3, the Lyman-α line is redshifted from a rest wavelength of 1216Å. Calculate the redshift z of this quasar. [2m]

b) Calculate the apparent magnitude of this quasar as seen from Earth, assuming no intergalactic extinction. [3m]
Hint: The comoving distance can be calculated from $d_C(z) = d_H \chi(z)$ where $\chi(z)$ is a dimensionless factor that is equal to 2.000 for this quasar, d_H is the *Hubble radius*, the distance beyond which objects recede faster than the speed of light due to cosmological (Hubble) expansion.

c) **Estimate** the age of the Universe at the radiation-matter equality, given that the matter-dark energy equality occurred 4.05 billion years ago. [4m]
Hint: In a radiation-, matter- and dark-energy-dominated Universe, $a(t) \propto t^{1/2}$, $a(t) \propto t^{2/3}$ and $a(t) \propto \exp(t\sqrt{\frac{\Lambda}{3}})$ respectively.

d) From this form of the Friedmann equation,

$$H^2 = H_0^2 \left[\frac{\rho}{\rho_{c,0}} + \frac{a_0^2}{a^2} \left(1 - \frac{\rho_0}{\rho_{c,0}} \right) \right]$$

where ρ is the density of the Universe at time t, ρ_0 is the current density of the Universe, and $\rho_{c,0}$ is the current critical density of the Universe, **show** that the equation can be manipulated to obtain

$$\dot{a} = H_0 \sqrt{\Omega_{r,0} a^{-2} + \Omega_{m,0} a^{-1} + \Omega_{k,0} + \Omega_{\Lambda,0} a^2}$$

by considering the various components of our Universe and using the definition of the Hubble parameter $H(a)$, where $\dot{a}$ is the rate of change of the scale factor. [5m]

e) This equation can be differentiated to obtain

$$\ddot{a} = -\frac{1}{2}H_0^2\left(2\Omega_{r,0}a^{-3} + \Omega_{m,0}a^{-2} - 2\Omega_{\Lambda,0}a\right)$$

From this, **estimate** the scale factor $a(t)$ when our Universe switched from decelerating to accelerating expansion. [3m]

Please note the following convention:

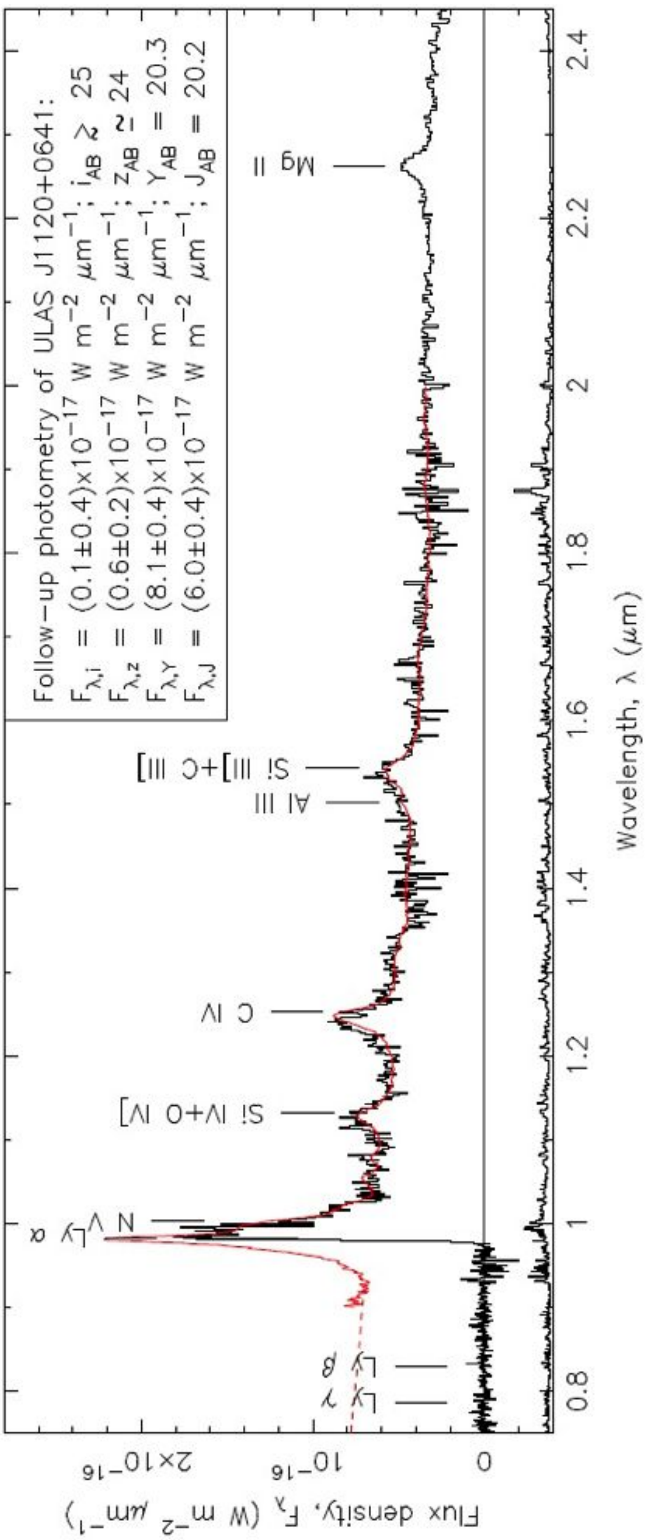
$$\dot{a} = \frac{da}{dt}, \quad \ddot{a} = \frac{d^2a}{dt^2}$$

f) The *Eddington luminosity* is the maximum luminosity a massive body can achieve when there is a balance between the radiation and gravitational forces acting on a particle in the vicinity of the body. Considering an ionised hydrogen cloud of mass m and opacity κ at a distance R from a radiating source of luminosity L and mass M, **derive** the *Eddington luminosity* for this body.

For ionised hydrogen, we can assume that the opacity (cross-sectional area per unit mass) is provided by Thomson scattering such that $\kappa = \sigma_T/m_p$. [5m]

g) Assuming that the quasar ULAS J1120+0641 radiates at its Eddington luminosity, **estimate** its mass and *Eddington-limited accretion rate* for this quasar to maintain this luminosity. It is reasonable to assume that the accretion process is only 10% efficient. Compare this to its reported mass of $M_{BH} = 2 \times 10^9$ solar masses. [3m]

Figure 3: Spectrum of ULAS J1120+0641 at infrared wavelengths, data extracted from http://arxiv.org/pdf/1106.6088v1.pdf



Question 6

Linda is an alien who lives on planet Threa. She likes observing the planet Rams, which is also in her solar system. Sometime in the next 50 years, she would like to conduct an astronomy convention on a Thimssarc day, which is a religious festival significant to Threaean culture. It occurs annually exactly 280 days after Threa's version of the Vernal Equinox. The highlight of the convention would be a Rams observation session, and Linda wants it to be the most epic Rams observation session in Threaean history by ensuring that Rams appears as large as possible in her telescopes. She thus asks the following question: What is the minimum distance from Threa that Rams would ever acquire on a Thimssarc day? [16m]

- You can assume that Threa and Rams follow unchanging Keplerian orbits that do not precess. The following data about the two planets are provided:
- Threa's semi-major axis (a_T): 1.00 AU
- Threa's eccentricity (e_T): 0.00
- Threa's orbital period (T_T): 365 days
- Rams' semi-major axis (a_R): 1.52 AU
- Rams' eccentricity (e_R): 0.300
- Rams' orbital inclination (i_R): 15.3° to the ecliptic
- Rams' longitude of ascending node (Ω_R): 49.6° (from the Vernal Equinox)
- Rams' argument of perihelion (ω_R): 286.5° (from the ascending node)
- Threa and Rams orbit in the same direction, which is the same as the directions in which Rams' argument of perihelion and longitude of ascending node are measured.

Hint: The definitions of the various orbital elements are as in the following diagram:

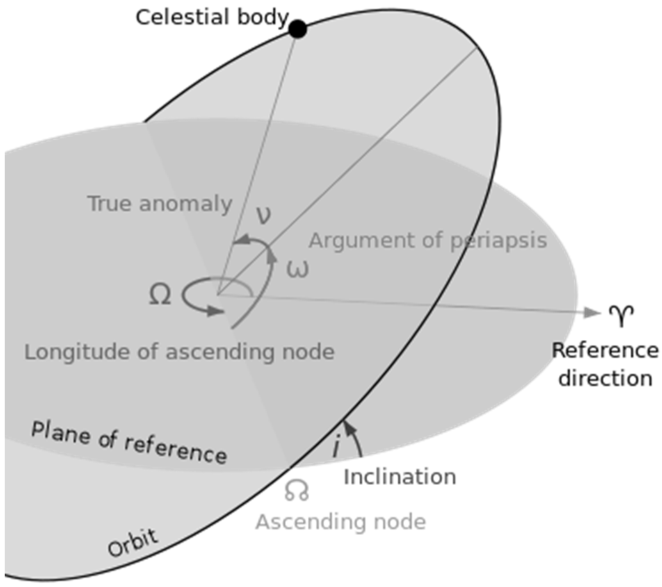


Figure 4. (By Lasunncty, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=8971052)

Specifically:

- Rams' orbital inclination is the angle between Rams' orbital plane and Threa's ecliptic;
- Rams' ascending node is the point at which it crosses from the south to the north of Threa's ecliptic;
- Rams' longitude of ascending node is the angle between its ascending node and Threa's Vernal Equinox;
- Rams' argument of perihelion is the angle between its perihelion and its ascending node.

Another hint: You may encounter the need to find the root of a quartic f(x) while solving this question. You do not need to provide any working to receive credit for solving the quartic. You may solve this using any analytical or numerical method of your choice. If you can't think of any, you may find the following description of the Secant Method (one of the oldest numerical root-finding techniques) helpful:

1. Find any two numbers a and b such that $f(a) < 0$ and $f(b) > 0$. There will definitely be a root somewhere between a and b, so they act as an upper and lower bound for a search range. Since a quartic may have more than one real root, ensure that your desired root exists between a and b, and that that is the only root between a and b.
2. Find the equation of the line l connecting a and b.
3. Find the root c of the line l. This will definitely lie between a and b.
4. If $f(c) < 0$, then set a to c. If $f(c) > 0$, then set b to c. Now you have a smaller search range.
5. Repeat steps 1 to 4 until c stops changing, to the desired accuracy.

Yet another hint: If you're stuck, you might find the following tips useful:

- This problem can be reduced to a problem in coordinate geometry – finding the point on an ellipse closest to a given point.
- The most convenient coordinate system to use for this problem is the right-handed Cartesian coordinate system defined with the origin in the centre of Rams' orbit, the x-axis pointing towards Rams' aphelion and the z-axis perpendicular to Rams' orbit such that Rams orbits anti-clockwise when looking in the negative z direction.
- The problem of finding the closest point on an ellipse can be reduced to the same problem in two dimensions – you just have to find out how.
- The point P' on an arbitrary curve C closest to a given point P is always such that the line connecting P and P' is normal to C.
- Finding the equation of the normal to a point on an ellipse may be difficult, but what about the equation of the tangent? Consider what happens when you compress or stretch the ellipse along the x-axis into a circle.

Question 7

The SAR21 is the Singapore Armed Forces' weapon of choice. A variant of it features a 3x optical scope (called the sharpshooter scope) with a diameter of 25mm, and a length of 120mm. It is also the weapon of choice for an unfortunate astronomer stuck on an island Northeast of Singapore, as he reckons the scope has a greater light gathering power.

a) Find the f-number of the sharpshooter scope. [1m]

b) The limiting magnitude of stars on the island is 3.2. What is the new limiting magnitude when the sharpshooter scope is used? [2m]
The diameter of the pupil of the human eye is 8mm (night adapted).

c) The astronomer points the scope to the Moon. Given the Earth-Moon distance is 384400km, would he be able to resolve these two craters on the moon? [2m]
   a. Ross Crater (D = 24km). Fun fact: This crater is in Mare Tranquillitatis
   b. Rosse Crater (D = 12km). Fun fact: This crater is in Mare Nectaris

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Figure 2. Sky Map