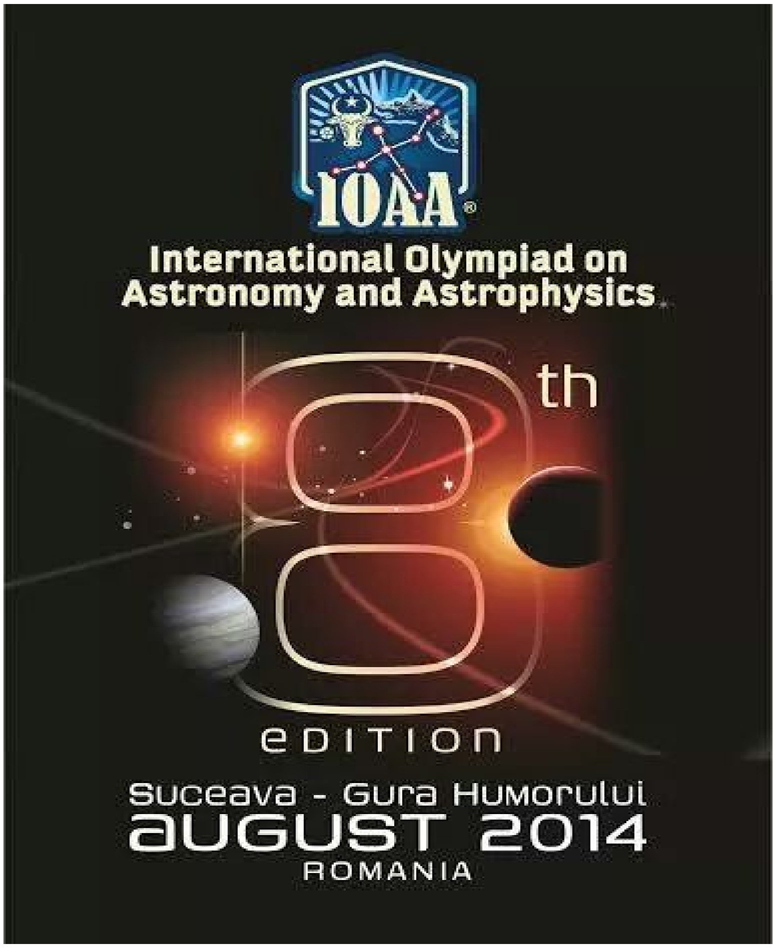
In the name of God



8^{TH} IOAA TEAM

FULL PROBLEM SET

2013 - 2014

www.ioaa2014.blogsky.com
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به شما جوانان برومند که با
کسب چهار مدال طلا و شش نقره

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Preface:

The team sent from Iran to the 8th IOAA managed to receive 4 gold medals won by Alireza Alavi, Shahabeddin Mohin, Ali Zeynali and Yazdan Babazadeh and 6 silver medals won by Kayhan Behdin, Soheil Ansarin, Mohammad Nabizadeh, Mohammadreza Hasanpoor, Amirreza Asadzadeh and Shervin Hakimi. Which was the best result *ever* in the history of the Astronomy and Astrophysics Olympiad in Iran.

Other than that, Yazdan Babazadeh's 2nd place, Shahabeddin Mohin's best data analysis score prize and Alireza Alavi's most creative solution prize were also part of the team's accomplishments in the tournament.

According to the scores, Iran's team came in 2nd after the home team, Romania.

This problem set contains all the problems designed by the team, in English, which consists of 4 problem sets.

We hope that this problems set comes in handy for all those who are interested or plan to participate in the Olympiad exams.

By the way, if you face any difficulty in solving the problems, feel free to contact us through the email addresses provided below.

Best wishes,

Iran's 8th IOAA team

August 2014

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8^{th} IOAA team problem set #1

October 2013

Translators: Mohammad Nabizadeh, Soheil Ansarin

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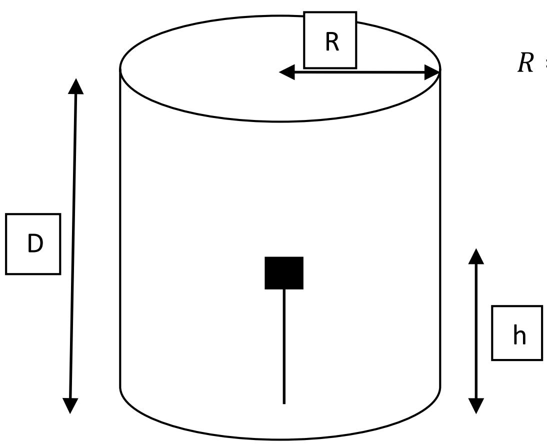
1. Humans have finally succeeded in finding a suitable planet for the future life of the human race and now they plan to go there. But they don't want to leave any traces of the unpleasant past of the human race on earth. So they plan to completely destroy it.
 For this, they have planted a very powerful bomb underground which is controlled through a sensor on the surface. This sensor first receives the location of a specific star in the sky and whenever it sees that star, it triggers the bomb. But there is an important problem to this and that's the sun. When the sensor has even the slightest amount of sunlight in its field of view, it won't work.

 This figure shows how the sensor is planted in the ground:

 (The top of the cylinder is at the surface of the ground and the black box is the sensor)



$R = \sqrt{3}m \quad , d = 6m \quad , h = 3m$

At the vernal equinox day, all humans set off on their journey to the planet. They set the sensor on a star at the following coordinates: $\alpha = 2^h 13^m 7^s$, $\delta = 24° \, 52' 34''$

Assuming that we are at the shortest distance from the sun at the vernal equinox,

I. At what range of latitude should we place this sensor for no explosion to occur?(20 points)
II. Where should we place the sensor for the explosion to occur as soon as possible?(80 points)

$$a_{\oplus} = 1.496 \times 10^{11} m \, , M_{\odot} = 1.989 \times 10^{30} kg \, , e = 0.0167$$

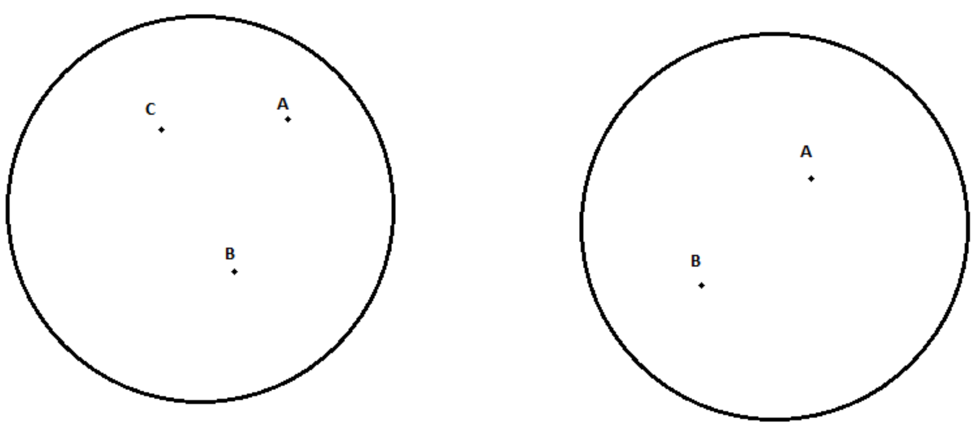
Hint: Take in mind the idea of a small circle being tangential to a great circle.

2. Assume that at this exact moment a satellite in heliocentric orbit is in a solar transit.
   a. Find the distance of the satellite from earth 65 days later.
   b. Find the equatorial coordinates of this satellite 140 days later.

   Assume:

   Satellite orbit inclination = $20°$
   Satellite orbital radius = $0.6\ A.U.$
   Longitude of ascending node = $130°$
   Inclination of the ecliptic relative to the equator = $23.5°$

3. Prove that the angles at the corner of a room when viewed at equal distance from the two walls and roof, will appear to be at 120 degrees. (The same thing we draw on paper for showing a 3 dimensional coordinate system.)
4. The sun makes one complete rotation every 26 days and this is the exact reason why the location of sun spots varies in different photos. The only points stationary in photos are the poles of the sun through which its rotational axis passes.
   In two different days, an observer takes two photos of the sun which shows the location of the spots on it. His assistant (who enjoys sketching) sketches the sunspots in the photos. The photos below show his two sketches of the sun. In the first photo we see three sunspots but in the second one, the assistant forgot to draw one of the three sunspots. Now it's up to you to draw the third sunspot and the location of one of the sun's poles (which is visible in the photo) on the second photo. Also, find the interval between the two photos.



5. A particle is in orbit around an object of mass M. Assume that the object is very massive compared to the particle. Prove that the delta-v required for a single impulsive maneuver to rotate the orbit η degrees around its focus is given by:
$$\Delta V = \frac{2\mu e}{h}\sin(\frac{\eta}{2})$$
6. Assume that the ***lift*** and ***drag*** forces exerted on an airplane as a function of velocity are given by the following equations:
$$F_l = C_l\rho A v^2, F_D = C_D\rho A v^2$$
Where F_l and F_D are respectively the lift and drag forces while C_l and C_D are dimensionless constants which vary by the shape of the airfoil. Also, ρ is the density of the body of air around the plane, A is the wing area and v is the plane's velocity.
   a. If we assume that the engines move the plane with a total force of F,
      I. Find the plane's velocity as a function of time. (Assume the plane's mass to be m)
      II. Find the plane's maximum velocity.
      III. What distance should the plane accelerate on the runway to take off? (The runway is frictionless) How much time would traveling this distance take?
   b. If the plane's engines' force was given by an exponential function of velocity, ($F = F_0 e^{-v/v_e}$) Find the plane's velocity as a function of time. Use the function to find the maximum velocity of the plane.
7. Assume that we have 4 satellites in geocentric orbits with orbital radii of 4 times the earth's radius. Find their distance when every two of them are at maximum distance from each other. (Take the observer to be at the center of earth.)
8. In a practical activity, we have measured the sun's angular diameter. (In hundredth of a degree) The results of our measurements could be found in the table. Fit the best curve possible to the data and find the sun's angular diameter and its error. (Hint: first prove that the best curve is a Gaussian curve)

NO	Sun diameter	NO	Sun diameter	NO	Sun diameter	NO	Sun diameter	NO	Sun diameter
1	53	11	52	21	48	31	53	41	54
2	50	12	54	22	58	32	61	42	48
3	59	13	48	23	51	33	54	43	51
4	62	14	49	24	50	34	45	44	59
5	52	15	44	25	49	35	56	45	60
6	57	16	54	26	58	36	53	46	46
7	57	17	46	27	52	37	51	47	63
8	49	18	47	28	55	38	55	48	66
9	53	19	55	29	53	39	56	49	50
10	56	20	43	30	47	40	53	50	64

(This problem has an attached file.)

9. This problem has two parts.

a. An observer is standing on the midline of an infinite road. The road has two lines each with a width of D. The observer's eyes are h meters from the ground. As you've probably seen in movies, the two sides of the road meet on the horizon. Find the angle between the two sides as viewed by the observer. In the photo below, assuming a line width of 3.2 meters, find the height of the camera tripod.

b. An observer 1.8 meters tall is standing in an 8 meter wide road, 2 meters from the left side. As viewed by the observer, the road side lines meet each other on the horizon. Find the angle between the lines. What's the distance between the observer and the line which appears as the bisector of that angle?

10. For large objects in the sky, the surface brightness is independent of distance. Unless the expansion of the universe effects the intensity of light from the object.
    a. Prove that the surface magnitude of a galaxy is independent of its distance. (Neglect expansion of the universe)
    b. If the blue band surface brightness of a galaxy is $I_B = 27\ mag\ arcsec^{-1}$, show that this is equivalent to $1\ L_{sun}\ pc^{-2}$.
    c. If the infrared band surface brightness of the center of a galaxy is $I_I = 15\ mag\ arcsec^{-2}$, show that this in equivalent to $18000\ L_{sun}\ pc^{-2}$

Required constants:

The visual absolute magnitude of the sun: $M_{v,sun} = 4.83$

$$(B-V)_{sun} = 0.65$$

$$(V-I)_{sun} = 0.72$$

Sun's luminosity: $L_{bol,sun} = 3.85 \times 10^{26}\ W$

$$1\ pc \approx 206265\ A.U.$$

11. In this problem, we plan to calculate the boiling point of liquids. For more information on the concepts in this problem, refer to chapters 5, 8, 9 and 10 of ***An Introduction to Modern Astrophysics***. Recommended time for solving this problem: 2 hours

    The particles of a liquid move with different velocities depending on the temperature and some of these particles find a chance to escape intermolecular forces and earth's gravity (Which has little effect and will be neglected in this problem) and leave the surface of the liquid. They somehow escape the liquid. These particles could transfer their momentum and create pressure on the surface by crashing into their surroundings. This pressure is known as the liquid ***vapor pressure***. The boiling point is the temperature at which the vapor pressure becomes equal to the atmosphere pressure around the liquid.

    a. Using the Maxwell-Boltzmann distribution (shown below) find a relation for the root mean square velocity v_{rms}. (30 points)

    The Maxwell-Boltzmann distribution: $n_v dv = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}} 4\pi v^2 dv$

    $$v_{rms}{}^2 = \frac{1}{N}\sum_{i=1}^{N} v_i{}^2 = \frac{1}{n}\int_0^\infty v^2 n_v dv$$

    b. Now assume that the velocity of the studied particle is equal to this velocity. (v_{rms}) Also, assume that the earth's atmosphere is composed of 80 percent Nitrogen and 20 percent Oxygen and that the studied liquid is water.

    So now we plan to calculate the distance that the escaped particle travels in the atmosphere before a collision. This distance is known as the ***mean free path*** length. Find this length in terms of the number density of the atmosphere and the geometric cross section of the atmosphere particles. (There is no need for complex calculations or simplification. Just write the exact parameters mentioned.) (30 points)

    c. Now we plan to calculate the rate of changes in the momentum of an escaping particle. For this, we just have to divide the change in momentum by the time interval of the process. In this problem, we assume that the liquid particles lose all their momentum by crashing into air molecules.

    The average time it takes for the next liquid particle to make a collision is the mean free path length divided by its velocity.

    Not we know that in this time interval, a momentum equal to the momentum of a liquid particle has been transferred to the air molecule. Using Newton's second law, calculate the force exerted by the liquid particle on the air molecule. (80 points)

    d. As you know, pressure is force exerted on a surface. If we examine this carefully, we realize that the cross section of the force exerted on the air molecules is equal to the geometric cross section of the **liquid** molecules. Now, find an expression for the vapor pressure of a liquid. (20 points)

    Hint:

$$P = n\frac{S_a}{S_i}3kT$$

Where S_a and S_i are the geometric cross section of air molecules and water, respectively. n is the number density of the air molecules near the surface of the liquid and T is the temperature of the liquid.

e. Now by using the hydrostatic equilibrium relation and assuming constant gravitational acceleration and air density and also zero pressure at the top layers of the atmosphere, find a relation for the pressure near the earth's surface. This relation should be written in terms of air density, earth's gravitational acceleration and the thickness of the atmosphere. (40 points)

f. So now we add the condition required for boiling. So we have to set the atmosphere pressure near earth's surface (which was found above) equal to the vapor pressure. Simplify the final result till you get this relation: (50 points)

$$T = mgh\frac{S_i}{S_a \times 3k}$$

Where m is the mean weight of the air molecules, g is the gravitational acceleration which we assumed to be constant, h is the thickness of the atmosphere and the other parameters were described in previous parts.

From now on, we try to find the unknown parameters of the equation in the best way possible.

g. Find the mean weight for the air molecules for the air composition mentioned at the beginning of the problem. (30 points)

h. Now we try to use the concept of the Bohr radius to make a rough estimate of the ratio of the cross sections. Take in mind that some of the assumptions of this part aren't good assumptions but they don't really effect the order of the final answer of the problem.

First assume an electron moving in a circular orbit around a proton. The dominant force is the Coulomb force. Using Niels Bohr's assumption $L = n\hbar$ (Where L is the angular momentum of the electron, n is an integer and $\hbar$ is a constant.) find the distance of the electron from the center of the atom in terms of some physical constants, n and the atomic number Z of the atom. (90 points)

We have to note that Niels Bohr's assumption is just correct for the Hydrogen atom. But in this problem we prefer to assume that this assumption is correct for other atoms. (not a really good assumption) For the calculation of the atomic cross section, we just have to calculate the distance of the farthest electron of the atom to the protons and assume a spherical shape to the atoms. The cross section of these spheres would be the area of their equatorial circle. Keep in mind that the number n is just the first quantum number for the farthest electron which can be looked up from the periodic table of elements. For example for the farthest electron of an Oxygen atom, $Z = 8$ and $n = 2$.

i. Now we can find the cross section of the atoms. We shall calculate the cross section for the air and liquid particles.

For this, assume that each liquid molecule consists of two Hydrogen atoms and one Oxygen atom.

Also, the air particles consist of two Oxygen atoms and two Nitrogen atoms. (Why two?)

So now, find the ratio of the cross sections $\frac{S_i}{S_a}$. (40 points)

j. Take earth's gravitational acceleration to be 9.8 and the height of the atmosphere be 100km.
Now find the boiling point of water. (20 points)

- Note: The designer's idea from this problem is to review your knowledge in astrophysics and introduce a concept of physics which may seem pretty hard to study by use of equations and also remind the fact that we can make interesting calculations at any level of knowledge. At first glance, the problem may seem long, difficult and boring, but you can evaluate your knowledge in astrophysics by taking your time and solving this problem.

8th IOAA team problem set #3

September 2013

Translation by: Mohammad Nabizadeh, Soheil Ansarin

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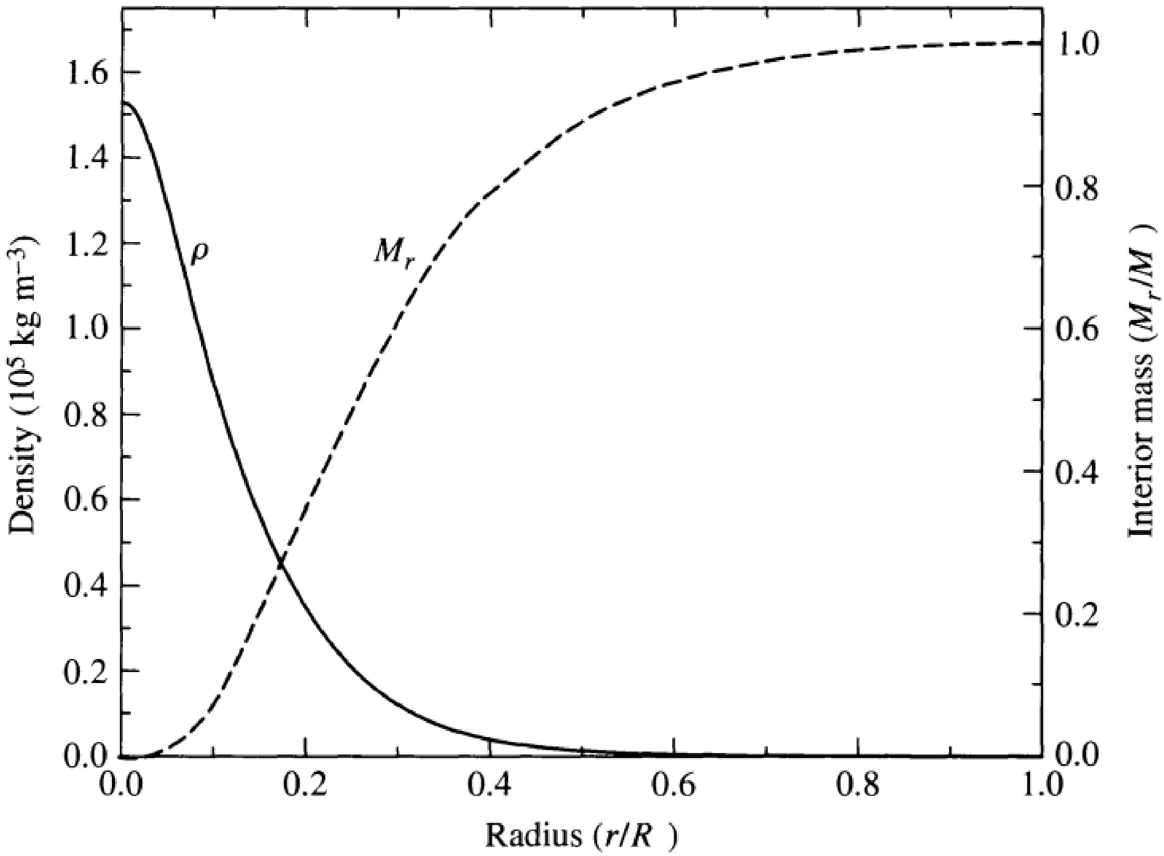
1. Dr. Shahram Abbassi is one of the most recognized Iranian scientists in the field of accretion disks. In one of his recent researches on the giant molecular cloud B32, he found a sun-like star at center of this particular cloud. According to his researches, this cloud has a mass of $10^6 M_\odot$, a radius of $30pc$ and very high viscosity, so great that if the cloud was to collapse, the whole system would collapse with spherical symmetry. The thing of most importance, is calculating the specific heat at constant volume (C_v) for this cloud. According to the problem's data here and using reasonable approximations, find a limit for C_v, so that accretion could be possible.
2. In this problem, you, as a researcher, are going to study the interior structure of the Sun and find the profiles of its density, pressure and interior temperature and finally find the core radius of the Sun.

   There's been some different guesses of the density profile which all had the basis that the profile is monotonically decreasing, so it reaches to almost zero at the star's surface.

   Now answer the parts (a), (b) and (c) by considering and getting data from the figure below which is *The density profile and the interior mass of the Sun as a function of radius*, so that we can successfully find the best profiles. (For enough precision please take at least 10 samples of data from the figure.)



a. One of the suggested profiles is the one below, which you should find the best-fitting line for it and also the unknown parameter n and the regression of the fitted line.(Hint: First you should define appropriate x and y so that the equation could be converted to an equation of the form $y = mx$)

$$\rho(r) = \rho_c(1 - (\frac{r}{R})^n)$$

b. Another suggested profile is the profile below which you should find the unknown parameter n and also the regression of the best-fitting line like the last part. (Hint: Define appropriate x and y so that the line would pass through the origin.)

$$\rho(r) = \rho_c(1 - \frac{r}{R})^n$$

c. Now choose the best profile here according to the regressions found above and use it in the rest of problem.

d. Now that you have the density profile, find the pressure profile (P_r) by using the hydrostatic equilibrium.

e. By just considering the ideal gas pressure and not the radiation pressure, find the temperature profile of the Sun, knowing that the Sun is made of 74% Hydrogen, 24% Helium and 2% metals.

f. For finding the core radius we have to first find the temperature which fusion will start in. You can find the fusion temperature by two methods: First there is the classic method which it will not give us the results that we are looking for and second is the quantum method of finding it which we will be using. In the quantum way, we will assume that the electrostatic force is dominant and use the Heisenberg's uncertainty principle ($\Delta x \Delta x \geq \hbar$) and we will also consider that the proton's uncertainty in momentum (Δp) is at the same order of its momentum.

   Now applying the above information, find the fusion temperature (T_{ign}).

g. Now find the Sun's core radius ($r_{core,\odot}$) using the profile found in part (e) and the temperature found in the last part. Your answer should be reported in terms of $R_\odot$. (In your calculations you will have to solve an equation that you will have to solve it either by Newton's method, or any other method you know of.)

h. Does your answer correspond to the real radius($r_{\mathrm{core},\odot} \approx 0.3\mathrm{R}_\odot$)?

If not which of the pervious assumptions were wrong?

Useful relations for line passing through the origin:

$$m = \frac{\sum_{i=0}^{N} x_i y_i}{\sum_{i=0}^{N} {x_i}^2}$$

$$r = \frac{\sum_{i=0}^{N}\left((x_i - \bar{x})(y_i - \bar{y})\right)}{N s_x s_y}$$

$${s_x}^2 = \frac{\sum_{i=0}^{N}(x_i - \bar{x})^2}{N}$$

$${s_y}^2 = \frac{\sum_{i=0}^{N}(y_i - \bar{y})^2}{N}$$

Newton's method (only usable for converging functions):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3. By the year 25000 A.C., the humans would have acquired the ability to swim at high velocities, but the costal guards should put limits on these speeds for the safety of the divers.

   Divers who go to depths of the sea and take a long time there and will probably die on their way back to the top. Humans are warm-blooded, meaning that their body temperature is almost constant. Then according to ideal gas law ($P.V$) will be constant for the lungs, which P is the gas pressure and V is its volume. Because of the high pressure in seas, the lung volume would decrease. The divers who are making their way to the top, at high speeds, will have their longs expand in a rapid way which will make them explode! In this procedure, if you assume the lungs to be spheres, by their rapid expansion, they will make an empty space between the air particles and lung walls which will allow these particles to move very fast toward the lungs and make them explode.

a. We know that the water's density is constant everywhere under seas. By assuming a constant gravity at every height, find the water pressure at a point with a depth of h.

b. By using the ideal gas law, find a relation between h and the volume of the lung.

c. If we assume the lungs to spheres find their volume as function of the depth height (h).

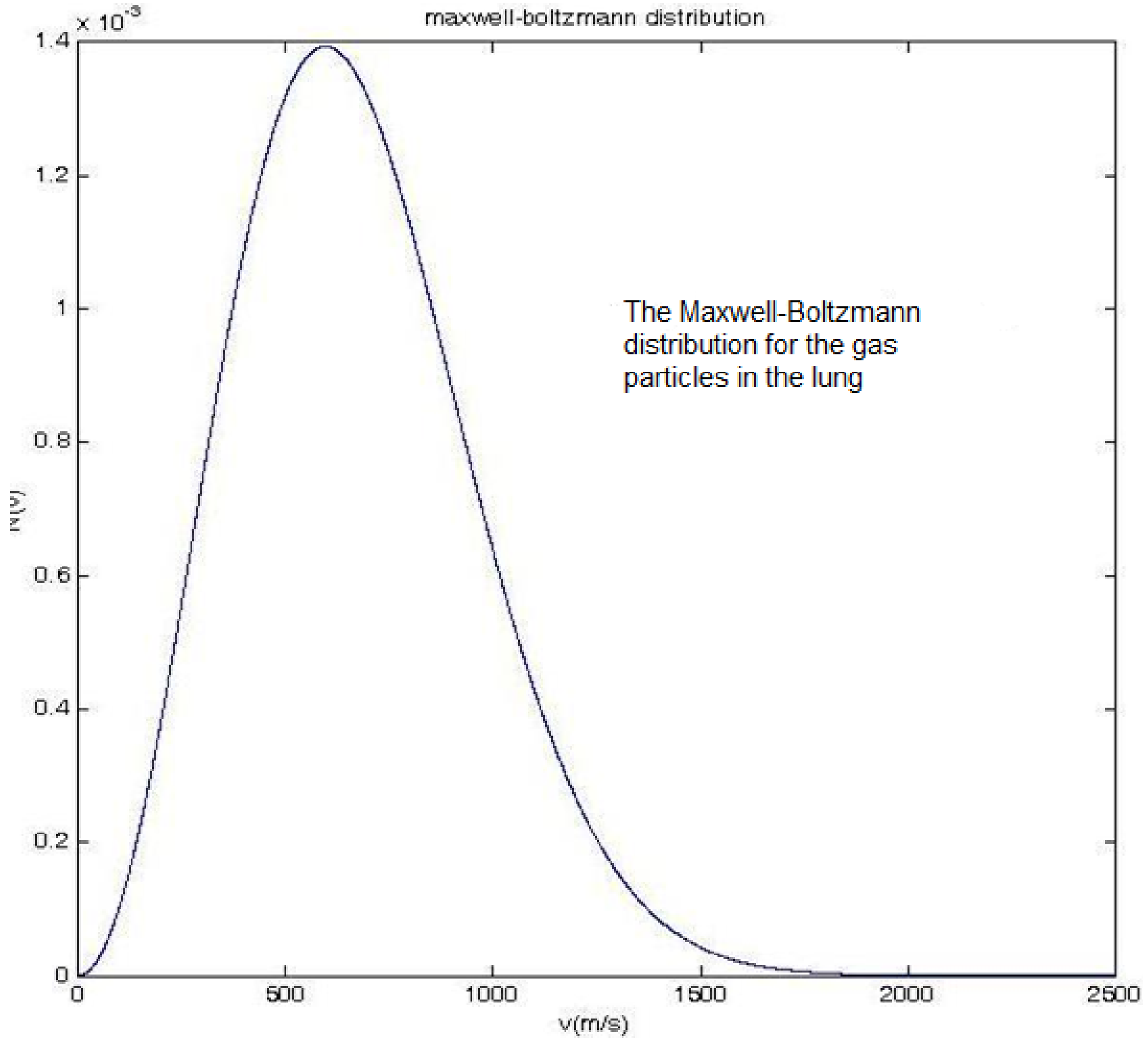
d. Assume that the divers will go up with a velocity of $V_h$and the lungs to expand with a velocity of V_r. Now by differentiating the relation above find V_h in terms of V_r and h.

e. The experiments have shown that if the lunges expand with a relative velocity $10\,m/s$ greater than, more than 60% of the gas particles of surface layer, then the required volume difference will occur and after a few moments the longs will explode. The particles in surface layer are those that will only move in radial direction (moving away from the center). Assume that the gas particles inside the lung obey the Maxwell-Boltzmann distribution, by considering the figure at the end of the question find $V_{r_{max}}$.

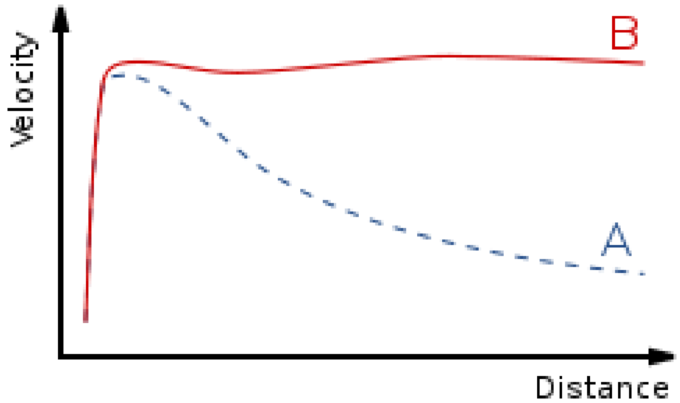
f. Knowing $V_{r_{max}}$, find $V_{h_{max}}$at a height h. This is the velocity restriction which the police are going use.

Assume the gas pressure at the surface to be $1atm$ and humans' lung volume to be $6litre$.

g. What then is the minimum required time for the diver to get to the surface from the depth of $200m$? Take the gravity acceleration to be $9.8\,m/s^2$.



4. One of the hot subjects in the field of galactic dynamics is the rotation curve of the galaxies. The studies show that the rotational velocity curve of an object becomes constant after reaching a maximum which is in a total disagreement with Newtonian mechanics. The figure below shows this. Curve A is what we expect and curve B is observed. One of the suggestions for this problem is MOND (Modified Newtonian Dynamics).This theory describes the second law of Newton in a different way: $f = ma\mu(\frac{a_0}{a})$in which f, m, a, a_0 are force, mass, acceleration and a constant respectively and $\mu(x) = (1 + x)^{-1}$.



a. Find limits for acceleration for far and near distances. (Think about $\frac{a}{a_0}$!!)
b. By considering the last part, find the circular orbit velocity of objects near and far from the center.
c. Do your conclusions explain the everyday behavior of the world? How about the figure above? Do you think that this theory is acceptable?

5. We have in our hands a green laser with a power of $50mw$. Two rather unusual persons want to do some interesting things with it. One of these interesting things is "Astrology". For this purpose we first assume that the atmosphere is present, but the absorption is very weak. The person having the laser in his/her hand will keep the light on in the direction perpendicular to the surface of the Earth. For this interesting thing to happen the fixed observer must always see the star that is at its highest altitude in the first moments; right at point of laser's light disappearance. Find the location of the fixed observer as a function of time and measure the maximum velocity of the other person, according to the assumptions below:

   Assumption 1: The Earth is flat:

   For observers on the equator and in latitude $45°$; for stars with the declination of $0°$, $45°$ and $60°$.

   Assumption 2: The Earth is round:

   For observers on the equator and in latitude $45°$; for stars with the declination of $0°$, $45°$ and $60°$.

6. A spaceship is getting near a red giant star for doing some researches. The mass of the star is M and the luminosity is L.The spaceship falls toward the center of the star in radial path and the sail begin to open in order to stop the ship.

   a. How much must the effective area of the sail be, to completely stop the spaceship? Explain about the equilibrium of the spaceship in this condition.

   The space is going to move in the same radial path after being stopped.

b. If the area of the sail gets any bigger the spaceship will move away from the star. What will be the relation between r and t?
c. If the area of the sail is less that critical area then the spaceship will fall. Now how are r and t related?
d. The stars brightness is measured to be $L = (3 \pm 0.5) \times 10^4 L_\odot$.If the sail opens up to $30 \pm 5m^2$ then the spaceship will stop. What is the approximated mass of the star?
e. The star's radius is $R = (1 \pm 0.4) \times 10^3 R_\odot$ and the spaceship is at a distance of $10^5 R$ from the star's center. If the area of the sail were half its current value, how long would it take for the spaceship to reach the atmosphere of the star?

7. Propellers are tools used for transferring the mechanical energy of an engine to the air around it. Propellers have many different usages such as household fans, airplane propulsion and even car radiators. There are some main parameters in the design of propellers which include: blade length, blade angle of attack, and the blade's aerodynamic shape. The angle of attack x is the angle between the blade and the direction of its movement. In this problem, we neglect the effect of the aerodynamic shape.

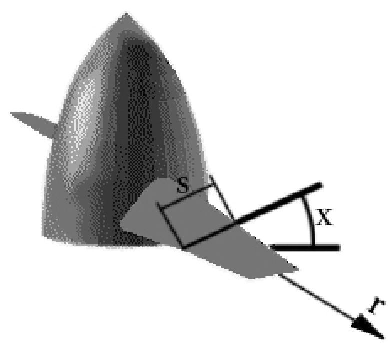


Figure 3, a basic 2 blade propeller

Assume a propeller which has very narrow rectangular blades. In a way that we could assume that differential pieces on a blade at distance r from the middle of the propeller are rectangular. The angle of attack for this propeller is 45 degrees. The blades of this propeller cause no friction. The only force acting against the spinning propeller is the force caused by the changing momentum of the air molecules. The length of the blades on this propeller is $95cm$. Assume that the air molecules leave the blade at the same angle they struck it and that their speed doesn't change at impact.

a. Find the resistance force exerted on the blades in the direction of their movement for when the engine is spinning the propeller at an angular frequency of 100 rounds per second.
b. Find the average velocity of the air affected by the propeller for an angular frequency of $80\ rps$.

c. Find the propulsive force exerted on the propeller for part a. Compare your result with the results of part a. What is your conclusion?

d. If we keep the angular frequency constant, how does the propulsive force change with a variable angle of attack?

8. Three observers are on a planet at locations that are 90 degrees separated from each other. A satellite is orbiting the planet on a circular trajectory. If all three observers see the satellite on the horizon at the same time,

a. Find the orbital radius of the satellite in terms of the planet's radius.

b. If two of the observers are on the planet's equator and the satellite passes just above one of the observers on the equator, find the satellite's orbital inclination.

9. The wave theory of light shows that a perfect focus is not possible because of the diffraction effects associated with the finite aperture of the lens. This lack of perfect focus will not allow us to distinguish the very close objects. You can study this problem in two different points of view. First, classic: The wave theory of light predicts that a lens of diameter D cannot focus a parallel beam of light with wavelength λ to within an angle better than the diffraction limit:

$$\theta_{min} \approx 1.22\,\frac{\lambda}{D}$$

Second, quantum: Consider now the photons which are focused by the lens. Such photons are known to have passed somewhere within one diameter of the lens center. The uncertainty in x-position is associated with an uncertainty in the photon's x-component of momentum. Consequently, a photon which in the absence of this uncertainty would have been brought to the optical axis of the focal plane, may now be deflected through an angle $\theta \ll 1$.Consider the de Broglie wavelength $\lambda = \frac{h}{p}$. Find a limit for θ.

Heisenberg's uncertainty principle: $\Delta x \Delta p_x > h$

10. According to a series of researches the scientists found that the cosmos is rotating with an angular velocity of ω_0.By assuming that the universe is completely made of dark matter find:

a. The Friedmann's first equation

b. The Friedmann's second equation

c. Angular and luminosity distance as a function of z

d. Now by both considering radiation and dark energy do the last parts as much as it is possible to.

11. A model for the hawking radiation:

After the publishing of the general relativity theory in 1916, Carl Schwarzschild solved the general relativity equations for a spherical structure and found a radius limit where every body of matter with a radius smaller than that limit would become a black hole. A black hole is a body that bends space-time around it in a way that nothing could escape its gravity. Since the fastest particles are photons which travel at the speed of light, the escape velocity on the surface of the black hole is somewhat equal to the speed of light.

a. Using the conservation of mechanical energy and Newtonian gravity and assuming that the escape velocity on the surface of the black hole is equal to the speed of light, find a relation between the mass and the radius of the black hole.

The radius you just found is the Schwarzschild radius which has been found from general relativity.

Yet, in 1974, Stephen Hawking was able to show using the quantum theory that for every black hole, it is probable that particles escape the gravity. This phenomenon is called quantum tunneling. Before hawking, other scientists were able to use the laws of thermodynamics to show that a black hole must have entropy and a temperature.

According to the Heisenberg uncertainty principle in determination of the location and momentum of a particle, there must be an uncertainty which is found from the equation below:

$$\Delta x . \Delta p = \hbar$$

In which $\hbar$ is a constant. To make this equation more clear, let me give you an example.

Imagine a football! Assume we have 0.000000001 (m.kg/s) uncertainty in its momentum. Using the uncertainty principle, we have $10^{-25} m$ uncertainty in its location. Now assume that we set a fence with dimensions of $10^{-26} m$ (It's just an example!) around the football and we shall not allow the ball to pass this fence. But according to the uncertainty principle, the uncertainty in the location of this ball is greater than the dimensions of the ball, so this ball can pass this fence, unseen!!! This phenomenon is called quantum tunneling which is very important in the beginning of nuclear fusion in the core of stars.

This also occurs in black holes and photons can use the uncertainty in their location to escape the gravity of black holes. In this problem we plan to find a model for the radiation of black hole which isn't spinning and has a low temperature.

b. According to the assumption above, the temperature of the black hole is low. (As low as the CMB) So no nuclear reactions happen inside. Using this assumption and appropriate boundary conditions, show that the net luminosity of the black hole is zero. (hint: $L(r = 0) = 0$)

c. Now using the photon gas temperature gradient equation, show that the temperature is constant throughout the black hole.

d. And now using the equation for the number density of photons (as a function of temperature) show that the number density of photons is uniform in the black hole.

e. Now we plan to find the probability of tunneling at each point in the black hole. According to the quantum theory, the more the tunneling occurs, the less the probability gets, until we get an effective radius for the tunneling. (Just like the mean free path) This effective radius is equal to the black hole's radius. In other words, we assume tunneling probabilities larger than the radius of the black hole to be zero.

f. Assume a photon which is at distance d from the center of the black hole. Find the set of points on the surface of the black hole which are at a distance of R or less from the photon. Where R is the radius of the black hole.

g. Now using geometry, find a relation for the solid angle of these points as viewed from the photon.

h. According to that said above, the maximum tunneling length is equal to R which means that the solid angle mentioned, is the solid angle in which the photon can tunnel and escape. There is no preferred direction for the movement of the photon. So now find a relation for the escape probability of the photon at a distance d from the center.

i. Using the relations that you found in parts ***b***, ***c*** and ***d***, find a relation for the probability of having a photon at a distance d from the center.

j. Now, if you multiply the two probability functions you found, you get a new function which gives us the tunneling probability for a random photon. Using the concept of mathematical expectation,

$$\langle x \rangle = \int x.p(x).dx$$

and also using the probability functions you found, find the mean value for the tunneling radius of the photons. (Be careful when setting the bounds of the integral!!!)

Now for simplicity, we assume all the photons which want to tunnel are at the same radius you found above and their tunneling direction is radial. (You will eventually realize the reason for this assumption.) Photons which are located in this radius follow the Planck distribution function.

k. Using the Planck function find a relation between the blackbody temperature and the wavelength at which the light intensity is maximum. This relation is called the Wien Law.

l. Now among all the photons which are in this radius, assume a photon which has a wavelength equal to the one found in the previous part. Now, by using the uncertainty principle, the Schwarzschild radius and Wien's law, find the uncertainty in the temperature of tunneling photons.

m. Now, intuitively show that when the temperature is low, the uncertainty in the temperature should be approximately equal to the temperature itself. (Notice that we are sure that the temperature is not negative!!!)

n. With some tidying and simplifying, find a relation for the temperature and compare it to the actual relation which is found by solving quantum fields:

$$T = \frac{hc^3}{8\pi GKM}$$

Now assume that the distribution for the photons which have escaped follows the Planck function. This means that the output power follows the Stephan-Boltzmann law.

o. By substituting the temperature and radius of the black hole from the relations found, find the output power in terms of the black hole's mass and physical constants.

p. Now using special relativity, find a relation between the changes in the relativistic mass as a function of time and the luminosity.

Hint: $E = Mc^2$!!!

q. So, now, by substituting the luminosity in the Stephan-Boltzmann law, find the time required for the black hole to vaporize.

(Why, do you think, did we assume a radial path for the photons in part ***k***???)

Relations required:

Planck function: $B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}$

$$n_{photon} = bT^3 \; where \; b = constant$$

8th IOAA team problem set #5

October 2013

Translation by: Mohammad Nabizadeh, Soheil Ansarin

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1. At 11:00 o'clock in the morning of the 8th of June, someone in Tabriz located at the coordinates of $(\varphi, l) = (38°, 46°)$ sets a mirror facing toward a point in the sky with the following coordinates: $(a, A) = (70°, 105°)$ Towards what point in the sky does the sunlight reflection point? What are the equatorial coordinates of that point? (Assume a circular orbit for earth. The longitude of the Tehran standard time meridian is 52.5 degrees east.)
2. At the 8th of October, an observer sees the beautiful scene of the conjunction of the Moon and Venus just 1 hours 10 minutes after sunset. This sight was such that the line connecting the two ends of the Moon Crescent passed over Venus. This observer decides to find some parameters of Venus. He approximated the altitude of the Moon, 13 degrees and its phase 15%. He also measured the separation between the Moon and Venus, 4 degrees. Having this data and neglecting the orbital inclination of the Moon and Venus, find:
   a. The altitude and azimuth of Venus.
   b. The equatorial coordinates of Venus.
   c. The elongation angle and the phase of Venus.
3. A binary system with circular orbits has been observed. Each of the components have a mass equal to the Sun's mass and the distance between the two is $10\ A.U.$ and the center of mass velocity is zero. By accident, a random star similar to the Sun traveling at $105\ km.s^{-1}$ passes through the system's center of mass with its velocity vector in line with the angular momentum vector of the binary system.
   a. Find the maximum velocity of the star passing through the system.
   b. Find the maximum eccentricity for the system's orbits while the star passes.
   c. Find the minimum distance of the two component of the binary. (If you can't find the exact amount, report it with an error.)
4. An observer at a latitude of $35.5°$ sees the two stars A and B are both on the same meridian. Find the minimum speed required for the stars to stay on a single meridian? What's the direction of movement?

$$A: (\delta, RA) = (45°, 5^h)\ , B: (\delta, RA) = (30°, 2^h)$$

5. It is a fact that Earth isn't a perfect sphere and is actually an ellipsoid. For this ellipsoid, $a = b = 6378\ km$ and $c = 6357\ km$. Take the z axis in line with Earth's rotational axis.
   Hint: The Cartesian equation for an ellipsoid: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

a. Find the range of declination of circumpolar stars for an observer at a latitude of $60°$.
b. What if the observer in part a was $1km$ above the ground?
c. For the observer in the previous part, what percentage of the stars are observable in an instance?

6. An astronomer in Tabriz $(\varphi, l) = (38°, 46°)$ notices a bright star on his horizon at sunrise. After some research, he realizes that the bright star was actually $Capella$.
a. Find the Sun's ecliptic longitude $\lambda_\odot$ in terms of the ST of sunrise.
b. Find the ST for the rise of Capella. The equatorial coordinates of Capella:
$$\begin{cases} \delta_C = 45°0'32.1'' \\ \alpha_C = 5^h 17^m 43.56^s \end{cases}$$
c. Now, find the date of the event.

7. A satellite is orbiting the Earth in an elliptical orbit. Of the people, who have seen the satellite, 14 of them have reported the data about the satellite, when it was passing their zenith.

14 others have also seen the satellite but they did not report any datum of it.

The data are as listed below:

Data from the observing group

Longitudes	Latitudes		Longitudes	Latitudes
75.5	35.6		58.7	2.5
76.8	39.2		64.5	10.1
78.4	45.0		67.6	15.3
79.4	48.7		70.2	20.8
80.1	51.4		72.2	25.6
80.6	53.2		73.9	30.1
81.4	56.4		75.1	33.8

Data from the observing group

$\omega \times 10^4$	h(km)		$\omega \times 10^4$	h(km)
166	437.8		259	302.1
164	448.6		242	323.4
159	453.2		225	346.8
157	466.8		206	358.3
156	468.9		189	388.4
153	473.9		180	412.4
152	488.6		173	428.7

h & ω in the 2nd figure, are the distance from the observer to the satellite and the rotational velocity of the satellite when passing the zenith of the observer, respectively.

a. Calculate the inclination of the satellite's orbit.
b. Find the longitude in which orbit's node is on the zenith of an observer.
c. Calculate the error in calculating the inclination of the orbit and the longitude of the node.
d. Find the minimum value of the semimajor axis of the orbit.

These expressions may prove to be useful:

For fitting a line, that does not pass the origin, through a set of dots (x_i, y_i) , which the dots obey the relation$y = mx + c$, we have:

$$\bar{x} = \frac{1}{n}\sum x_i \qquad D = \sum(x_i - \bar{x})^2$$
$$\bar{y} = \frac{1}{n}\sum y_i \qquad c = \bar{y} - m\bar{x}$$
$$d_i = y_i - mx_i - c \qquad m = \frac{1}{D}\sum(x_i - \bar{x})y_i$$
$$(\Delta m)^2 \approx \frac{1}{D}\frac{\sum {d_i}^2}{n-2} \qquad (\Delta c)^2 \approx \left(\frac{1}{n} + \frac{\bar{x}^2}{D}\right)\frac{\sum {d_i}^2}{n-2}$$

8. We know that the Earth, orbits the sun in an elliptical orbit; moreover the Earth's equator is inclined relative to the elliptic plane and these two factors cause the Sun to have a rather irregular motion relative to the stars so, the duration of the apparent solar day (the time intervals between two successive lower transition of the Sun from observer's meridian.).In this problem, we are going to study the two factors which change the duration of the apparent solar day:
   a. the inclination of Earth's orbit:

   For studying this factor, consider two hypothetical suns which both are in circular orbits. One of them, in the plane of Earth's equator and the other, in the ecliptic plane. We will name the first sun (The sun in the Earth's equator plane), sun A and the other to be sun B; moreover we will define a parameter named Equation of Time (E) like below:

$$E = \alpha_A - \alpha_B$$

In which α_A and α_B are the right ascensions of sun A and B, respectively.

Considering that the equator and the ecliptic are inclined by $\varepsilon = 23.5^{\circ}$ and that both are at the position of vernal equinox on March 20th, find the equation of time (E) as function of the days past from March 20th (n) andE_{max}.

Also mention the date in which equation of time is at its maximum and minimum.

b. The ellipticity of Earth's orbit:

Consider the two hypothetical suns which both are in the Earth's equator plane. One, in an elliptical orbit, having an eccentricity of e, and the other in a circular orbit. Name the sun in the circular orbit A and the other to be B and define a parameter E as before.

Find E_{max} if:

I. $e = 0.0167$(In this part $e \ll 1$ so can use appropriate approximations.)
II. $e = 0.8$ (You cannot use any approximation here like the last part.)

Mention the dates in which the equation of time is at its maximum and minimum.

Assume that both suns are at vernal equinox on March 20th and sun B to be at its perihelion on this date. Consider the rotational period of the Earth to be 365.25 days in whole problem.

9. In this problem we will almost study all the problems regarding binaries.

A group of Iranian scientists are studying a binary system. According to their observational data they have found that: $R_1 = 10R_\odot$, $R_2 = 1R_\odot$ and $d = 0.1\ AU$ in which R_1, $R_2$and d are the radius of stars 1 and 2 and the distance between them, respectively.

a. What is the probability that the system will undergo an eclipse but not a total one, as seen by the scientists?

After days of hard work and putting effort in it, they recorded a series of data regarding the radial speeds of the stars in the binary system in different times but for some unknown reasons they were not able to record some of the data.

NO	JD-2443900	v1	v2
1	65.13237403	-44.95554079	$Error$
2	65.14380016	-44.13839845	$Error$
3	65.19258095	-48.14314475	$Error$
4	65.23630787	-51.30049276	$Error$
5	65.44695883	-40.8247788	$Error$
6	65.56480911	-22.88348057	41.64446341
7	65.56649373	-21.78292221	40.3223495
8	65.56854457	-21.64084183	39.55573231
9	65.59945372	-17.31274551	37.13080136
10	65.65585192	-5.186841318	29.60175859
11	65.81552477	31.82027378	7.840987034
12	65.84951018	38.95424193	2.597827295
13	65.85793329	40.05533283	1.549978952
14	65.87338786	44.22448928	-0.877250698

15	65.98310801	62.93219954	$Error$
16	65.99519333	64.86718315	$Error$
17	66.06726585	75.4886431	$Error$
18	66.10974761	80.10575953	$Error$
19	66.26055788	84.01545035	$Error$
20	66.40660726	72.02024887	$Error$
21	66.5722129	43.86609751	-0.40399415
22	66.67321696	21.96037577	12.86614866
23	66.74902493	5.96250682	23.01450124
24	66.76587115	2.771156674	25.54021056
25	66.86233406	-17.35689063	37.46919099
26	66.97747429	-36.58635787	$Error$
27	66.98516495	-39.00355787	$Error$
28	67.20592364	-52.05996879	$Error$
29	67.27843562	-48.85594015	$Error$
30	67.30011597	-46.60893756	$Error$
31	67.42221442	-29.50956755	$Error$
32	67.4258034	-30.00603441	45.75074734
33	67.48769494	-19.67072281	38.2406469
34	67.49377422	-17.6569178	38.2516601
35	67.4953856	-18.36959348	37.22412011

b. Draw the diagram of the data above.

These scientists want to approximate the masses of these stars but for this to be done they need the inclination(i) of the binary system. Because of not knowing it they have to come up with a way to guess it.

In this procedure they have reached to a point where they need $sin^3 i$; one of the scientists suggests its mean value in order to approximate it, but none of them knows the distribution function.

c. Help these scientists by finding the distribution function of $sin^3 i$, ω_i.Also show that: $\int_{-\infty}^{+\infty} \omega(i) di = 1$

Now for finding the mean value they know that they must find the intervals where i is defined. Considering that, there are studying an eclipsing binary system, the intervals should begin from a particular angle of inclination like i_0.

d. Now calculate $< sin^3 i >$ as a function of i_0 for these scientists, to help them move to the next level of their calculations.

Now they have all the information they need for approximating the stars' masses.

e. Find a lower limit the second star's mass and guess its type. (Either a white dwarf, neutron star on a black hole)

From the data they had before, they found that the first star, is a main sequence star.

f. With approximating the second star's mass by a mass equal to its lower limit and finding their surface temperatures determine their spectral type.
g. Carefully draw the light curve of this eclipsing binary system, considering that $i = < i >$.
h. Give these scientists reasonable answers, to show them why there was some data missing from records of the second star radial speeds.

Some points that may help you:

For finding the mean value of the functions we have: $< f(\tau) > = \int_{-\infty}^{+\infty} f(\tau)\omega(\tau)d\tau$

Which: $\int_{-\infty}^{+\infty} \omega(\tau)d\tau = 1$

10. A pulsating star, having a period of P, is under the effect of lensing. Assume: The angular separation from the black hole to the star to be β, the black hole's mass to be M, the distance from star to the observer to be D_S and the distance from the black hole to the observer to be D_L. What is the time interval of the recorded pictures caused by lensing?
11. Note: All the scientific phenomena, quotes, and data used in this problem are real and based on investigated documents.
Recommended time for solving: 240 minutes
London, 19th century

When night falls, horror casts its shadow on the city and an unwritten law is established throughout the city; families don't allow their children to go outside; restaurants and bars close their blinds; and even the toughest of men wouldn't dare go outside.

The city of London, for a long time, has been a place where various crimes, robberies and murders occur. Criminal gangs would become active at night and there was no one to stop them as people weren't able to counteract. The effects of horrifying atmosphere and a weak police force were the main reason for individuals like Sherlock Holmes to appear from the darkness and use their brilliant mind beyond the police force to bring criminals to custody.

At that time, London had the highest rate of crime in all Europe and in this problem we plan to study the cause.

The main reason for such horrifying atmosphere was an almost permanent fog throughout the city. Fog decreases visibility, especially at night and with some historical studies, we realize that the weapons at the time had very low accuracy which made arrests very difficult to make for the police. But the interesting point is that among all the cities that have the same latitude as London, such conditions only exist in this city.

For example take a look at the map below:

According to the map above, London's latitude is almost the same as Berlin, Kiev and Russia's southern cities. By reviewing history, we realize that winter is very cold and severe in these cities. During the last 4 years, over 200 people died of freezing in Ukraine. Hitler's army surrendered to the destructive winter of this area while closing in on Moscow through the south west of Russia and lost most of his troops there due to freezing. We can say with confidence that there is no weather as fair as England's anywhere in Europe. Now we plan to use an approximate model to explain the weather conditions in this country.

First part:

a. Assume two coordinate systems with same origin point. The only difference is that one is spinning with an angular velocity vector of $\vec{w}$ in line with the z axis. Now assume a particle with an acceleration. Using the differentiation operator and some

vector operators like the cross, find a relation between the particle's acceleration in the two coordinates. The spinning coordinate system is called the non-inertial frame. (100 points)
Final answer: $\overrightarrow{a_n} = \overrightarrow{a_r} - 2(\vec{w} \times \overrightarrow{v_n}) - (\vec{w} \times (\vec{w} \times \vec{r}))$
Where $\overrightarrow{a_n}$ and $\overrightarrow{a_r}$ are the accelerations in the non-inertial and inertial frames, respectively, and $\vec{r}$ is the location vector of the particle. In this part, we plan to make some interesting calculations using the second expression from the right which is called the Coriolis acceleration.

b. Assume that there is a particle with a mass of m with random velocity vector on the ground. Assume that there is no force other than the Coriolis force. (The particle will stay on the ground throughout its path) Assume that your coordinate system is spinning around the z axis, which passes the North Pole, with an angular velocity equal to that of earth. Now using vector multiplication, Newton's second law, the differentiation operator, and simplifying the results, solve the equation of motion. Don't find the mathematical constants of the problem and let them be. Now show that the image of the particle's path on the xy plane is an ellipse. (300 points)
c. Now using the right hand law and the idea you have of the movement, determine the particle's direction of motion on this path. (Clockwise or counterclockwise) (20 points)

Second part:

In the first part, we claimed that if the particle has a velocity in the non-inertial frame, it will start to move in a closed path. Now we plan to find out why the particle had a velocity in the first place.

a. Assume a plane with an area of A which is located at a latitude of φ and is parallel to the ground. For simplicity, assume that the sun has no declination. Find the energy that the plane receives in a day in terms of solar flux, Earth angular velocity, the area of the plane and the latitude of the plane. (70 points)
b. Now assume a cube which has a lateral area of A and height of h and assume there are N particles in this cube which follow the Maxwell-Boltzmann velocity distribution. Find a relation for the total pressure exerted on the walls of this cube in terms of the total energy of the particles, h and A. (70 points)
c. Now we throw the above-mentioned cube into the Atlantic! Assume that the depth is constant throughout the ocean (Which is a good assumption. This depth is $3332m$.) and now assume an element just like the one mentioned above. Using the energy you calculated in part a and by considering an albedo of a for the water surface, find the pressure of a layer of water located at a latitude of φ. (Don't include the pressure of the weight of water in your calculations, yet. You will find out why!) (100 points)

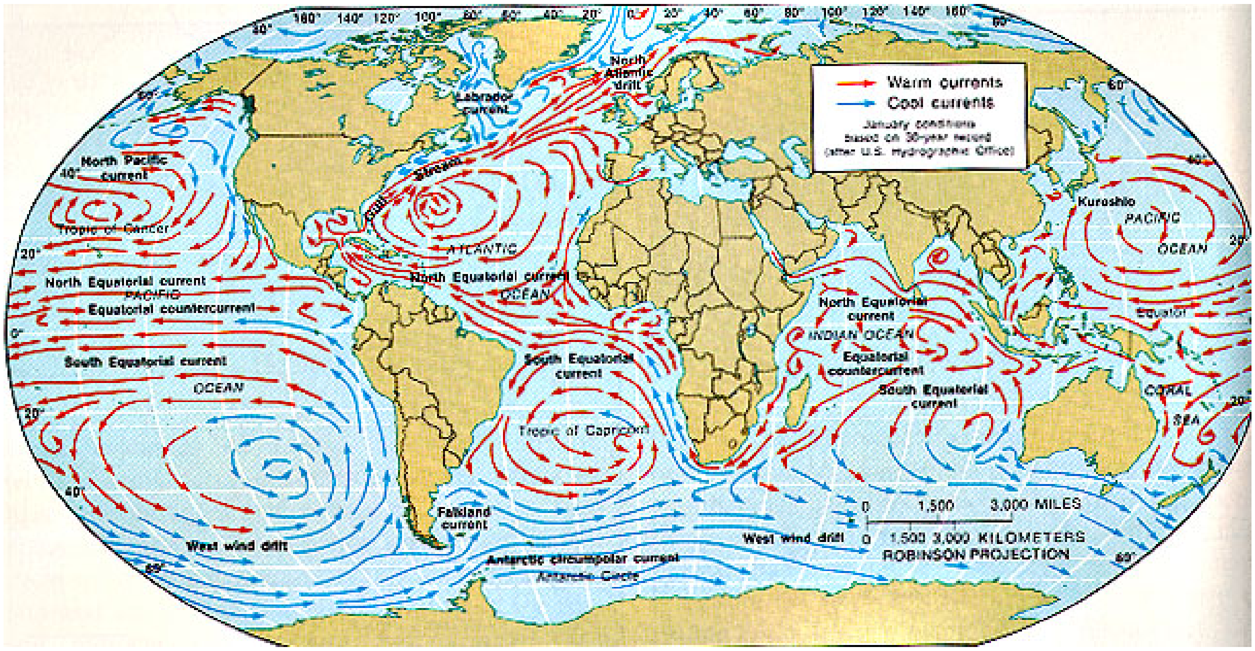
d. Now take an element with the same shape but in a random location. Put a spherical coordinate system on the center of earth. Intuitively we notice that the element will not move radially despite the all the forces exerted on it. (Did you notice the last sentence of the previous part? :D) Now write Newton's second law in a polar coordinate system (Since the problem physics is not related to the latitude due to the symmetry, we can solve the problem in 2D.) and considering the tangential forces exerted on the element (The difference of the pressure of the top and bottom layers!) find the differential equation $\varphi(t)$. (No need to solve the differential equation.) (150 points)

e. Now take a look at the map on the next page.

Now according to everything you've learned here, what's your guess about the reason the bodies of water at the north of Russia were named the Arctic *frozen* ocean? (20 points)

Third part:

According to what we showed in previous parts, it is predictable that the water currents that form in the Atlantic have a closed path of movement and the reason for the initial velocity is the difference in temperature between different layers. Now take a look at the figure on the next page.



The figure above shows the water currents in the oceans on Earth. As our equations predicted, there is a closed water current in the Atlantic which fulfills the task of transporting heat from equatorial regions to the northern areas. This enormous aqueous current is called the *Gulf Stream*. The Gulf Stream is the second most important mechanism for balancing the life ecosystems on Earth. The amount of energy carried by this current is approximately one hundred times the total energy consumed throughout the world. The melting of the polar icecaps and an excessive injection of water to this current is the main threat posed to it.

It's quite interesting to know that by reviewing historical documents, we have found that in the Medieval times, people used to put their letters in bottles and by tossing them into the aqueous current, they sent their letters from Denmark to England!

Forth Part:

In this part we are going to study the Earth's atmosphere. For our purposes in this part, we can assume the atmosphere to be plane-parallel. The best functions that we can consider for the temperature and the density in each layer are the exponential functions below:

$$T = T_0 \times e^{\frac{-r}{a}}, \rho = \rho_0 \times e^{\frac{-r}{a}}$$

a. Now consider an element of matter and show that a force is exerted on the element as a consequence of the difference in the pressure on the top and the bottom. This force is called the *Archimedes* force. (70 points)You can assume the gravitational acceleration and the density to be constant, in the order of elemental displacements.

(Hint: Use the hydrostatic equilibrium equation and consider the pressure of the top layer to be zero.)

b. Now we will also consider the gravitational acceleration in our study. Show that the element's density must be less than the atmosphere density for the element to move up and vice versa. (50 points)
c. Now we are going to find the density of the clouds as a function of its temperature. I must say that solving the equations of earth clouds is very difficult and almost impossible (Just take a look at the weather forecasts!) so we will only take into consideration the experimental phenomena and proceed with our logic! The phenomenon is lightening. (200 points)

By looking at the cumulus clouds we can understand that the clouds will remain in equilibrium for a long time, unless they have an encounter with one another or change their height. But we also know that intermolecular and interatomic forces are similar to the Coulomb force. The lightening phenomenon also confirms our theory of the clouds having electrical charge. Meaning that there are some particles in each cloud that have the same amount of electrical charges but the number of positive charges are not equal to the negative ones which means that each cloud has a total electrical charge of its own. But we can logically conclude that the total amount of electrical charges of all cloud is zero. (Why? Think about the causes and the procedure of the clouds being charged.)We will further our model by assuming that all the clouds are the same and the only different thing between them is whether they are positively or negatively charged. Also assume that the number of positively charged and negatively charged clouds to be equal. Now by applying the virial theorem and assuming the distribution of the particles to be uniform and taking the cloud to be spherical, find the equilibrium equation for both the positively charged cloud and the negative one.

d. Now find a relation between the cloud's temperature and its mass density. (Leave the problem’s constants to be.)(50 points)
e. The cloud will isotherm with the atmosphere's temperature at every height. Now use this assumption to find a height, so if the cloud gets there, it will not change its height and remains fixed. (70 points)

Fifth part:

Now we know that with the existence of the Gulf Stream current, the waters around England are much warmer than they should be. There is a limit to the speeds of the particles of water and if they, the particles, surpass this limit they will escape the water. This speed is called, the evaporation speed limit.

a. Intuitively and by using the Maxwell-Boltzmann figure show that with increasing the water's temperature, the amount particles which escape will increase. (50 points)

So we can conclude that Golf Stream current will increase moisture in England's weather, but the main reason of the thick fog in this country, according to our calculations in part four, is the height in which the steam particles cannot pass.(This phenomenon is called the atmospheric inversion, the main reason of air pollution in Tehran is a similar phenomenon but by the vehicle pollutants!)So a substantial amount of moisture is going to be confined in a rather small volume, which causes the steam's density to increase.

b. By using the definition of optical depth and mean free path, find a relation between the number of incidents between the particles and the photon while covering its path, and the optical depth. Define the optical depth of the observer to be zero. (100 points)
c. Show that we can ultimately look through the cloud till an optical depth of 1. (50 points)
d. Now show that the distance corresponding to an optical depth of 1 will decrease as the density increases. (This phenomenon is called mist!)(50 points)

OK, now we have found the answer we have been looking for! We have been able to explain the existence of the thick fog covering the city of London with our rudimentary models. Now it would be a shame not to do some more calculations!

a. The height, found in the "e" section of the fourth part, depended on the surface temperature of the Earth. Now show that any slight change in surface temperature of the Earth will cause a huge change in the density of the confined clouds. (70 points)
b. The weather condition of London changes in a highly rapid way and the time for these changes to occur is reported to be 2.5^h averaged over a year. So after approximately 2.5^h the weather will change from being sunny to rainy!

Now by using what you have found in section "a", explain this situation. (50 points)

(Let us not forget that all the things we have learnt in Olympiad are ultimately used to explain our surrounding events and phenomena, and may I also remind you that we can make interesting calculations at any level of knowledge.)

Total points 1640

8^{th} IOAA team problem set #7

October 2013

Translation by: Mohammad Nabizadeh, Soheil Ansarin

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10. Assume we have a galaxy consisting of stars with masses ranging from $0.5M_\odot$ to $5M_\odot$. This galaxy is 10 billion years old and since its age was 500 million years, star formations started inside but for some reasons, the star formation rate decreases proportional to $\frac{1}{t}$ where t is the age of the galaxy. This galaxy has 5×10^{10} living stars. We also know that for the main sequence we have the relation, $L \propto M^{3.5}$. Neglect the protostar phase. In this galaxy, the probability of the formation of a star of mass M is proportional to $M^{-2.5}$.

    a. Find the total number of stars. (alive or defunct)

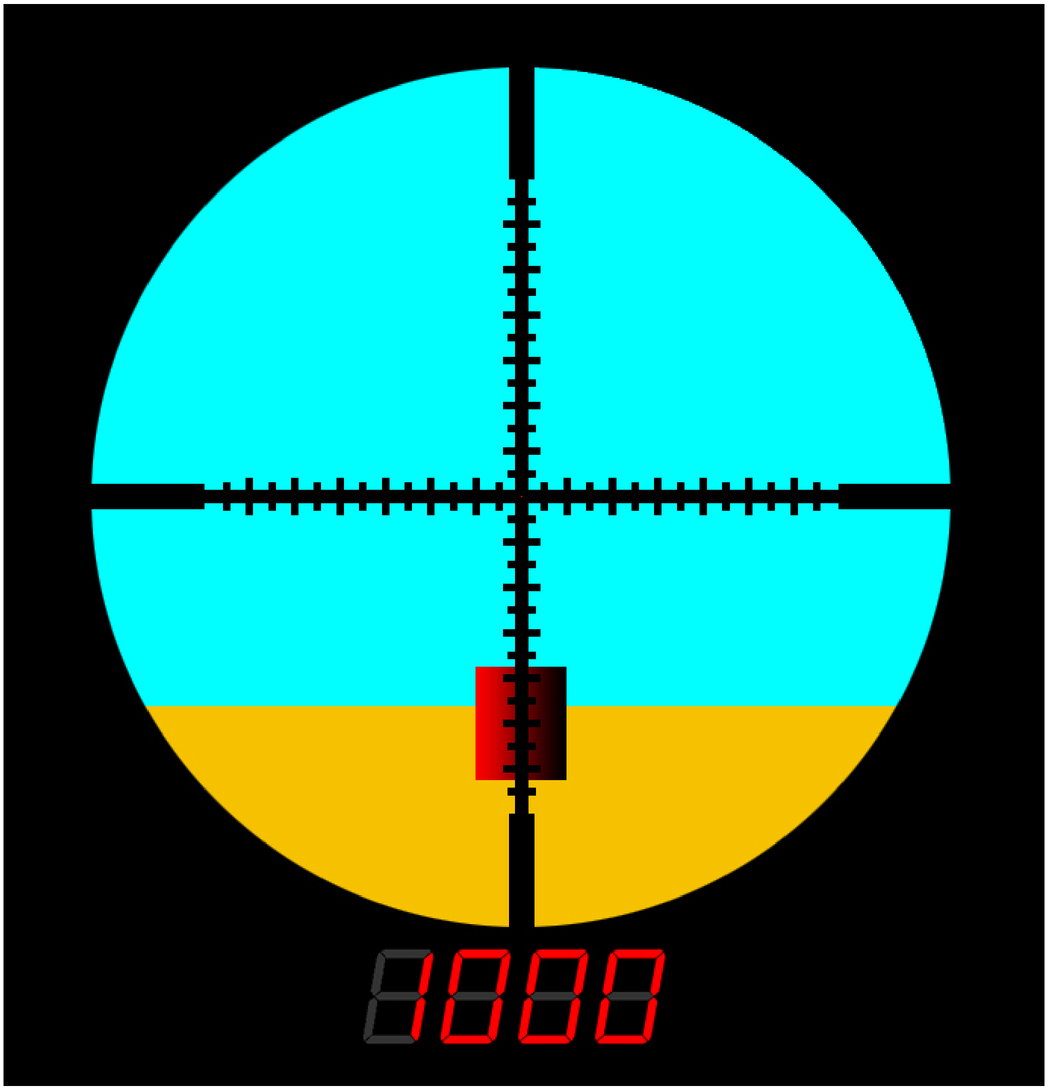
    b. Find the total luminosity of the galaxy.

11. A professional sniper with a Barrett M82A3 rifle, is going to participate in a military operation. His duty in this operation is spotting enemy forces and shooting them. For this operation, new bullets and a new scope have been given to him. He wants to test the scope and the bullets in order to calculate his rifle's specs with the new equipment. For this, he puts an explosive target on a $1.45m$ tall, upright barrel and he moves out to a specific distance from the target. His new scope benefits from an advanced metering system which shows the distance from the **target** on a display inside the scope.
    The figure on the next page shows the barrel as viewed through the sniper's scope.
    The red dot in the middle of the figure shows the exact location the rifle is pointing at.
    The sniper hits the top of the barrel by shooting at that direction. If the gravity acceleration of the location is $g = 9.78\ m.s^{-2}$,

    a. Find the bullet velocity (A.K.A. Muzzle velocity) when leaving the rifle. (Assume the ground is flat)

    b. If the lines on the crosshair are angle lines, what angle does the distance between two lines represent?

    c. Find the scope's field of view.



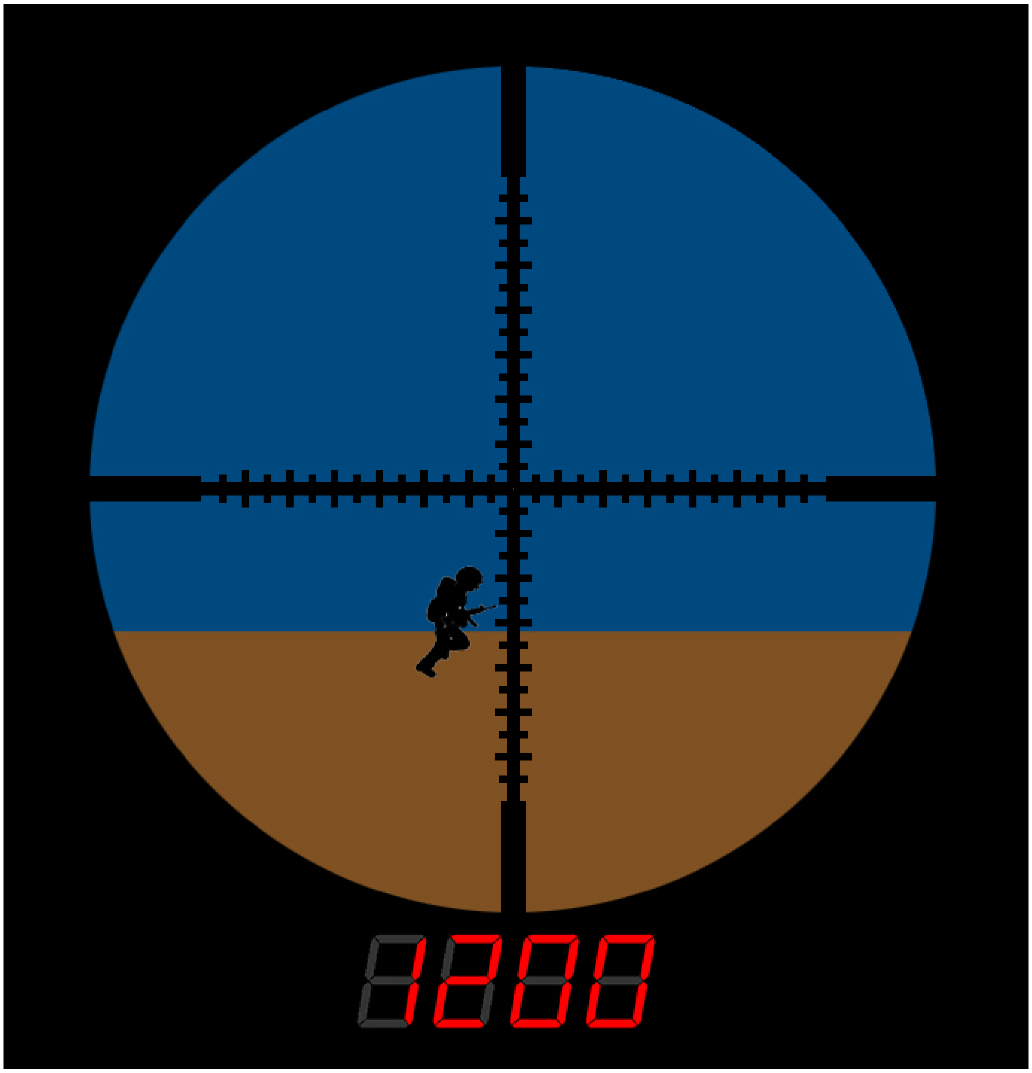
Now, it’s time for the operation. The sniper gets in position and gets his rifle ready. During the operation, he spots a hostile trooper running perpendicular to his line of sight. (Figure on next page)

The sniper has a device which is able to measure the angular speed of the target. He quickly sets the device on the target and finds that his angular velocity is 14.32 arcminutes.

d. Find the trooper’s running speed.

e. How many seconds does it take for the projectile to hit the trooper?

f. Mark the direction that the sniper has to shoot at for a headshot. (Mark it on the dedicated figure.)



12. In this problem we are going get more familiar with the effects of dark energy on the evolution of cosmos and better understand the reason we have it (dark matter) in our calculations.
    a. First derive that the deceleration parameter ($q = -\frac{a\ddot{a}}{\dot{a}^2}$) for a N factor universe is given by the equation below, where $\omega_i = \frac{P_i}{\rho_i c^2}$ and $\Omega_i = \frac{\rho_i}{\rho_c}$.

$$q = \frac{1}{2}\sum_{i=1}^{N} \Omega_i(1 + 3\omega_i)$$

    b. One of the speculations that there is for the reason of the universe acceleration being positive, is the existence of a quantum field named quintessence which has a negative pressure with an equation-of-state parameter $\omega_Q = -\frac{1}{2}$. Now considering that the deceleration parameter in present is $q_0 = -0.55$ and that the curvature of the universe is approximately flat, find Ω_{m_0} and Ω_{Q_0}. (Neglect the effects of radiation and consider the universe to be two-component.)

c. Now using the quantities found above and assuming the present Hubble parameter to be $H_0 = 67.8 \frac{km}{s.Mpc}$, find the age of this universe (t_0).

d. Did you encounter any contradiction in parts (b) and (c)? Can you use these contradictions to deny the existence of *quintessence* or at least say it contributes little of the universe?

According to the last parts, it seems that "quintessence" is not a suitable answer to the positive acceleration of the universe and we have to look for a better answer which cosmologists have named it dark energy $(\Lambda)(Lambda)$.

e. Cosmologists have reasoned that the density of dark energy is constant over the passage of time by using various specific scientific justifications. Now by considering this cosmological assumption and using the fluid equation show that $\omega_\Lambda = -1$.

f. Now again by assuming $q_0 = -0.55$ and a flat universe consisting only matter and dark energy, find Ω_{m_0} and Ω_{Λ_0}.

g. Using the data found above and assuming $H_0 = 68 \frac{km}{s.Mpc}$ find the **age** of the universe. Does the universe have a limited **lifetime** in this condition? Why? (Pay attention to the difference between the two words ***lifetime*** and ***age***!!)

h. Now compare your results with the Benchmark model's results (the best model yet presented) that are listed in the table below and check the accuracy of a flat universe consisting only matter and dark energy.

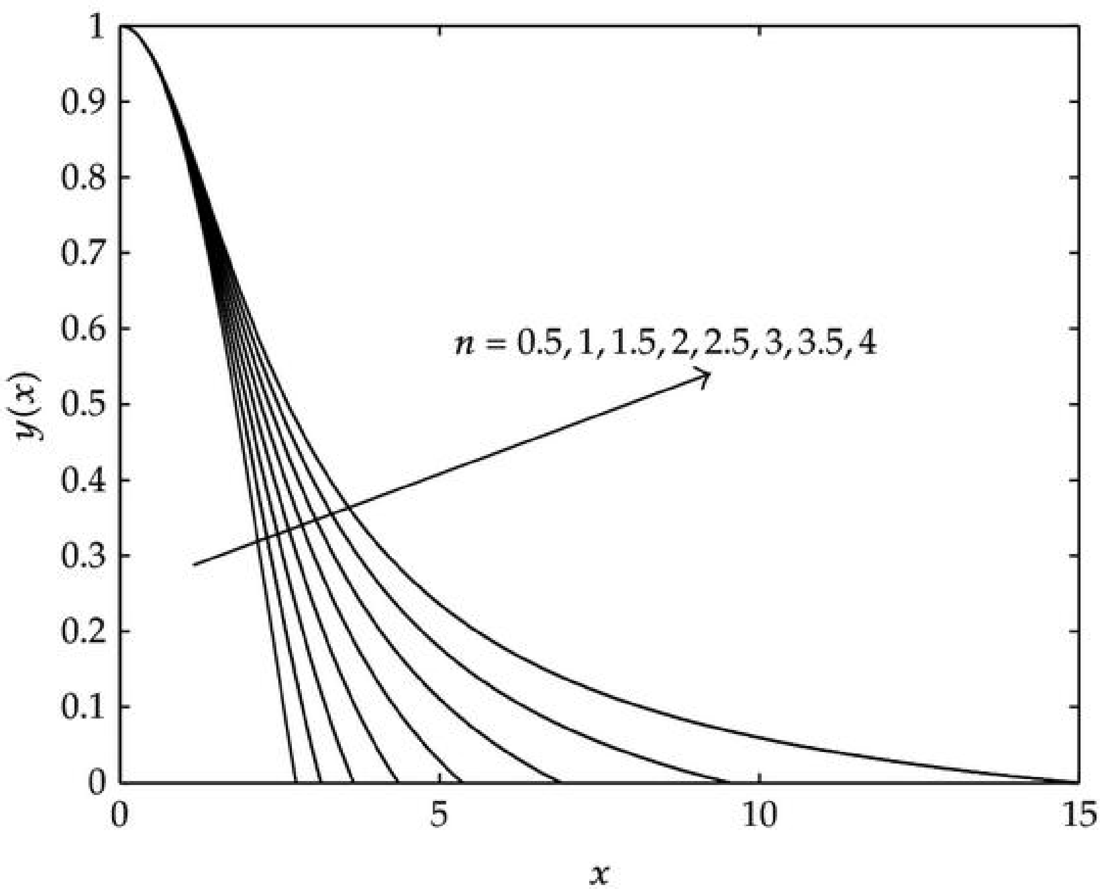
Do you think that dark energy is an appropriate answer to the positive acceleration of universe? Why?

13. Consider a main sequence star with mass $0.4M_\odot$ and radius $1R_\odot$ that is made of ideal monoatomic gas. In this low mass star, the dominant process in energy transport is completely convection; so we can assume the temperature gradient to be adiabatic. We can also assume the interior gas of the star to be polytrophic.

    The Lane-Emden equation for star consisting of polytrophic gas ($P = K\rho^{1+\frac{1}{n}}$) is as below:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{dD_n}{d\xi}\right] = -{D_n}^n$$

    Below is a figure of showing D_n as function of ξ for different n.



    Using the Lame-Emden equation and assuming the star to be made completely of ionized hydrogen find the radius of the star's core.

    The star's core is a region of star where the temperature is high enough for fusion to occur. Fusion will start at adiabatic temperature gradient of $-28\, {}^{k}/_{m}$.

14. Meteor showers are caused by the collision of the dust particles left from comets. These particles have speeds and they orbit the Sun. That is why the convergent points of the showers are not always on the ecliptic.

Part one: Assume the observed shower to have a convergent point (c) with the coordinates of (α_0, δ_0). The peak of this shower is in the N^{th} day of the year. Establish a coordinate with one axis perpendicular to the ecliptic and the other in the direction of the spring equinox.

a. Assuming that the Earth's orbit is circular and that the Sun's coordinate is (0, 0) at the spring equinox, measure the Earth's velocity vector. Take the Earth's velocity to be v_0.
b. Find the right ascension and declination of the point the Earth's is moving toward as a function of N.Name that point F.
c. Find the direction of the meteor particles' real velocity before their collision with the Earth. Your answer should contain only the coordinates of F and c.Note that if the real velocity becomes zero the direction of F and c will be the same.
d. Measure the orbital inclination of the mother comet.
e. Calculate the right ascension of the ascending node of the mother comet.

Part two: In this part the familiar Olympiad approximating begins.

The number density of the remaining meteor particles from the mother comet depends on the comet's size, its substance and its velocity at that point.

By observing the comet we can measure its size, but if observing was possible we would not be doing any of these calculations!!

But assume that this number density is constant in the entire motion and take it to be n_0 (which is a completely wrong assumption.)

Now we are going to measure the relative velocity of the meteor particles.

We will use an observer who the convergent point will pass his/her meridian at some time and we will put him/her on a tower for certainty. Assume that the observer's atmosphere to be at its best condition and that the sky is thoroughly clear.

The observer will see almost all the meteors from the shower, because its convergent point is on the meridian at those moments. (Why?)

Take R to be the collision radius of the Earth with the meteors.

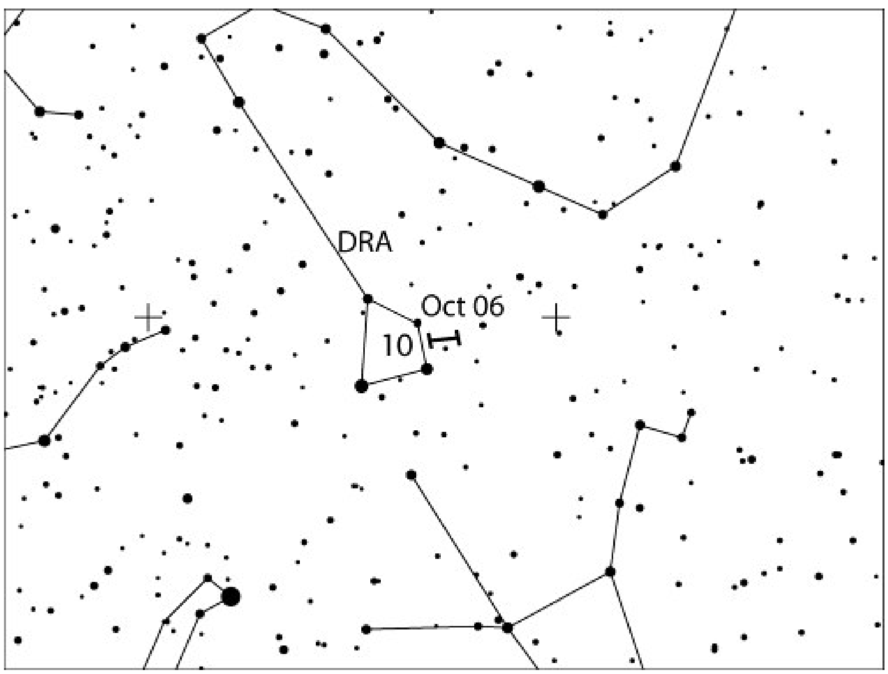
f. According to our assumptions, find our relative velocity if the observer sees k meteors in t seconds.
g. Knowing the relative velocity and the direction of the relative and the real velocities in the first part find the direction and the magnitude of the particles' real velocity.

h. The velocity found in the last part is equal to the mother comet's velocity in good approximations. Now calculate the comet's right ascension of the ascending node, eccentricity, angular momentum vector and argument of perigee.

i. Do parts "d", "e" and "h" for the draconic meteor shower.

Some of the draconic meteor shower properties are listed below:

Declination of the shower's convergent point	$54°$
Right ascension of the shower's convergent point	262°
Date of the shower's peak	October 4th
Approximated number density of the particles	$5 \times 10^{-18}\ \#/m^3$
Approximated ZHR	100



Assume the Earth collision's radius is $6.6 \times 10^6 m$.

15. We are going travel to the Earth's center. Assume the Earth to be an ellipsoid with an eccentricity of 0.3 and a semi-major axis of 6378km.From a location in latitude 35.5 degrees we will travel to the center in the direction of a plump line. What is your distance

to the Earth's center when you reach the Earth's equator plane? (Latitude is measured from the Earth's center.)

16. An equatorial satellite with an orbital period of 3^h59^m and a radius of 3.59m is a complete absorbent of sun's radiation. And it radiates this energy in this way: It will absorb all the energy radiated at different times in one period from the sun and reradiate it an isotropic way and in all directions in a specific period. Measure this satellite's magnitude for an observer in Tehran, located at latitude 35.9°N, as a function of time using appropriate assumptions. Also find the maximum and the minimum of its magnitude and the time intervals between these two phenomena. (Assume the Earth's albedo to be zero)
17. On November 3rd 2013 an eclipse has happened which we are going to do some calculations on it.

    In a point of zero latitude and longitude, the eclipse will start at 11^h44^m according to the local time. At this time the Sun is at its upper transit. The data below are about the positions of the Moon, the Sun and the orbital node of the Moon (relative to the center of the Earth) at the beginning of the eclipse.

    $$\lambda_{node} = 217.378°\,, \theta_{moon} = 3.298°, \lambda_{sun} = 221.225°$$

    a. Find the duration of the total eclipse (from the first contact to fourth) for this observer.
    b. Calculate percentage of the sun's disk, which is darkened for an observer at center of the Earth.
    c. An observer in Tehran (Latitude: 35.75°, Longtitude: 51.5°) will see a partial eclipse. Calculate the time of the eclipse's beginning.

    Take the angular diameter of the sun to be 32 arc minutes and the orbit's inclination to be 5.1°. Neglect the eccentricity of the Moon's orbit. Be careful not to neglect the effects of eye parallax.

9. One of the commonly used parameters in fitting a line is regression and the more the squared regression is the more linear is the function. In this problem we are going to gain an intuitive understanding of regression! Because we can almost convert any function to a linear one, we will use linear fitting method, so that the regression will be $r = \frac{\overline{xy}-\bar{x}\bar{y}}{s_x s_y}$.

    Assume that we have a series of data in the form of ordered pair (x_i, y_i) and by considering the concept of ordered pair, we can assume that only y_i has error.

    a. Assuming that $y_i = ax_i + b$, prove that $r^2 = 1$.
    b. Assuming that $y_a = ax_i + b$ and that y_i has the error of $\Delta y = cte$ meaning that $y_\mathrm{i} = y_a \pm \Delta y$, prove that the error of the ordered pairs is given by $\Delta y = s_y\sqrt{1 - r^2}$.

c. Calculate Δy for $\{r = 1, r = -1, r = 0\}$. Explain the concept of standard deviation according to the calculations you have done here.
d. Assuming that $y_a = ax_i + b$ and that y_i has the error of $\frac{\Delta y}{y_a} = \alpha = cte$ which α is the relative error so $y_\mathrm{i} = y_a \pm \Delta y$, prove that the error of the ordered pairs (x_i, y_i) is given by $\alpha^2 = \frac{2r^2 s_{\bar{y}}^2 - 3s_{\bar{y}}^2 - \bar{y}^2 + \sqrt{9s_{\bar{y}}^4 + \bar{y}^4 - 8r^2(s_{\bar{y}}^4 + s_{\bar{y}}^2 \bar{y}^2) + 10 s_{\bar{y}}^2 \bar{y}^2}}{2(r^2 s_{\bar{y}}^2 + \bar{y}^2)}$.
e. Calculate α for $\{r = 1, r = -1, r = 0\}$.
f. Assuming that $y_a = ax_i + b$ and that y_i has the error of Δy which obeys a Gaussian distribution having an average of zero and a standard deviation of σ so $y_\mathrm{i} = y_a \pm \Delta y$, find σ as function of r.
g. Assuming that $y_a = ax_i + b$ and that y_i has the error of Δy which obeys Poisson distribution, find r.
h. Assuming that $y_a = ax_i + b$ and that y_i has the error of Δy which obeys a uniform distribution having a minimum of $-E$ and a maximum of E so $y_\mathrm{i} = y_a \pm \Delta y$, now find E as a function of r.

10. [A word with the reader,

In previous series, we were able to model and simulate interesting phenomena with minimum knowledge. From now on, we plan to go further and follow more ambitious goals. This goal is to model phenomena from which we have no initial info! For modeling this type of problems, we use a method which was first used by Galileo. The only tool required for using this method is logics! Yes, that's right. The method is based on thinking and proceeding on the basis of logics. Which means that there will be some problems at first. For example, at some point of a theory, we might encounter inconsistency or lack of evidence. That's when an individual would start step by step simulation of the event or inconsistency and find the reason for the problem or make adequate evidence for the hypothesis.

The usage of this method is when we don't have access to any sort of experimentation or we don't have the tools for an experiment and for this reason, we have to experiment in our own mind! (In Einstein's general relativity article, the term *Thought Experiment* has been used. For example the elevator experiment of the principle of equivalence is one of the most famous thought experiments in physics.)

Examples of the usage of this method are numerous in history from which the most famous are Galileo's theories of motion, Newton's laws, Einstein's general relativity and theoretical cosmology.

On the other hand, there is another applied method in physics which is the complete opposite. This method is completely based on experimentation and the application is when we don't have much access to correct theory of a phenomenon! And by experiment and suggesting a model which is in most agreement with the experiment results, they get the desired results.

The most famous usages of this method in physics are the Ptolemy theory, the ether theory, quantum mechanics, dark matter density distribution, observational cosmology and etc. (Some physicists believe, the reason for the lack of unity between general relativity and quantum mechanics, is the difference in their essence.) Our goal in this series and later ones is to model using the first method.

Here is an interesting quote from Albert Einstein,

Before the observation of the 1919 solar eclipse by Eddington who confirmed general relativity by observing the bent light path, (This experiment along with Galileo's experiment of two objects falling from the top of Pisa tower, are known by some to be the most significant experiments of all time.) Einstein was asked what he would do if the results of the experiment failed to match general relativity. Einstein replied, "Then I would feel sorry for the good Lord. The theory is correct."

In reply, he was asked, "You haven't made such an experiment before. How are you so sure?"

Einstein then replied, "For 3 years all I did was sit at a desk and think. About a phenomenon that could not be experimented simply. If your thoughts are logical and in good order, **nature should obey** your results!]

And we shall follow the goal in this section,

For a start, we begin with a non-astronomical topic. (Recommended time for solving the problem: 4 hours)

(Note that all the quotes, claims and scientific info are based on truth and investigated documents.)

Evolution, is a biological process which was first found by the great biologist, Charles Darwin. He managed to explain the formation of advanced creatures like us using the Evolution theory. According to a worldwide survey in 2009, Evolution was the biggest human discovery of all time. At first, this theory was just used in biology but as time went by, it was required that philosophical and intellectual revolutions form throughout the world. Evolution is a new way for explaining our life on planet earth which is in contrast with some religions. The Vatican Basilica has always declared that many great biologists are heretics. Every year, many anti-evolution protests are held at Darwin's birthday throughout the world but despite all the protests, evolution is the most logical way to explain life.

In short, evolution is based on three rules: Genetic Mutation, Reproduction and Survival. The procedure is that if living creatures in an environment are going through some biological process (Like the life of humans! Or photosynthesis!) and then they mutate, (The formation of new, random properties in the next generation) and after reproduction, these new properties make life easier, then the mutant species will survive in this environment and with reproduction, less number of them will die and slowly, the previous species would diminish and completely disappear. This is called the evolving of a species. The best example of evolution (Which historical documents show that they were Darwin's inspiration) are Giraffes. According to Evolution, giraffes are evolved versions of another creature which used to live around India and China. At first, they were short and because of foothill vegetation, they had no problems in feeding. (The skeleton of one of these animals has been found in the glaciers of Tibet.) But then, for unknown reasons, they migrated to the Middle East and then to Africa. Because of the special vegetation of Africa, (tall trees) these animals faced problems in finding food. (Because they were short) but then because of a mutation, tall giraffes appeared. These tall giraffes got food much more easily and by reproduction, the population of the tall giraffes increased rapidly and the previous generation disappeared. (This process took 30 million years!)

We humans are no different. We are anticipated to have mutated from a specific type of monkeys. (It may seem interesting that this type of monkey has more memory than humans!!) You may think that the evolution process from unicellular organisms to creatures as advanced as us humans, is a very unlikely process but the calculated time for this process is about 5 billion years and geological studies of the old rocks of Earth show that the age of Earth and the Solar System is about 5.5 billion years!!

Some biological studies show that the origin of life on Earth must have come from somewhere outside the planet. (Several theories confirm this statement. We will avoid discussing them since they are longsome.)

On one hand, we know that the best mechanisms for transporting anything in the interstellar or interplanetary medium are meteors, comets, asteroids or dwarf planets. On the other hand, by studying theories on the atmosphere of Earth and zoology we realize that Earth's weather and atmospheric conditions (Mean temperature, mean height and pressure on the planet's surface) have remained the same since the formation of the atmosphere. (About 5 billion years ago) This means that the distance to the sun hasn't changed much since the formation of Earth. This prohibits the collision of any type of massive space object with Earth. Because the collision of a massive object would significantly change the distance from the sun. So we assume that our space object is a point with very low mass. And at last we know that surface of Earth is 70% water. So we can predict that the mass must have fallen into the water and studies on the fossils left from that time confirms this assumption. (Have you ever noticed that most bacterial illnesses are related to contaminated water??) So life begun from water!!!

Our ancestors form!!

The unicellular organisms were more primitive than the most primitive modern bacteria, but they became more advanced over time till the first photosynthetic organisms took shape. This event was one of the most significant factors in the continual of life on Earth. The formation of these creatures caused an increase in oxygen that gradually changed the methane-saturated atmosphere of Earth to the atmosphere we have today!

Basically, our main reason for designing this problem was finding an answer to one of the interesting steps of evolution for which we didn't have a specific answer for a very long time. As you know, most of the life on Earth is on land! And on one side we know that life first formed in the water. So logically, at some time in the past, life must have come to the land from the seas.

On the other hand, since creatures in the sea know by instinct that they shouldn't be out of the water, they don't come out! So a mechanism should throw them onto land. Then if a genetic mutation makes life on land easier for the next generation compared to the previous one, the new species would survive on land and by reproduction life appears on land! And then by the evolution of the creatures living on land, we came to existence!

For a long time, biologists were searching for a suitable mechanism for the transportation of life from sea to land. But they didn't know that the best possible mechanism is actually an astronomical phenomenon. A very interesting phenomenon!

The moon joins the party!

Yes! That's right! Our desired mechanism is tides! (I really enjoyed realizing this fact!) Now we know what to do and what to model!

1.

a. Assume two coordinate frames that have parallel axes and none are spinning. Assume a particle. Using the properties of vectors and the derivative operator, find a relation between the acceleration in the two frames and the acceleration of one frame relative to the other. (20 points)

b. Now assume the Earth-Moon system. (Here, we want to show a small part of what we called *thought experiment*!) Until the end of this section, take the Moon's orbital plane parallel to the equator and assume a circular orbit for the moon and also neglect the effects of the Sun.

   Using what you found in **a**, show the forces exerted on any part of the Earth's surface, as viewed by an observer on Earth, by drawing a figure. (Thought experimentation is such that you picture the system in your mind and assume that for a moment, the Earth accelerates toward the Moon. According to what you have learned of acceleration in city buses (!) what, do you think, will happen to Earth's surface water??!) (50 points)

2.

Now we plan to find the exact shape of the water bodies on Earth in the presence of the Moon.

a. A point that will help us in solving this section, is that the potential energy should be constant throughout the surface of a liquid. (Why?? Hint: look closely at the surface of the water in a glass!) The potential energy from the forces exerted on a mass element on the surface of the bodies of water on Earth can be easily found and written. Then by playing around with the equations, find the polar function of the distance from the surface of the water to the center of Earth in terms of latitude. (100 points)
b. Now assume that the altitude of Earth's water is very low and neglect the second degree of the ratio of Earth's radius to the distance from Earth to the Moon. Now simplify the potential energy relation. (50 points)
c. Assume a 2 dimension shape for now! As you probably have realized, the shape isn't really famous. But if you like, you can draw the shape in a drawing software and enjoy! Here, we plan to use a prevalent method in physics. Notice that if the Moon wasn't there, the shape of the water on Earth would be spherical. Now we can take the effect of the Moon to be a sort of disorder and assume that a slight disorder is caused in our sphere. (In two dimensions, our circle!) The best approximation for a disturbed circle is an ellipse with a small eccentricity. Now start with the Cartesian equation of an ellipse and replace the Cartesian components in terms of polar ones and approximate the equation to the second degree of the eccentricity. Then find a relation for the polar distance from the center in terms of the latitude. Keep in mind that your relation should be as simple as possible! (100 points)
d. Now find the eccentricity in terms of the other parameters. (70 points)

3. Now, in this section, we plan to find the maximum amount of altitude variations on Earth's surface in the presence of the Moon.
   Assume that the Moon's orbital plane is parallel to the equator. Now take a point on the ground that the Moon is at Zenith. Intuitively, it is clear that the tidal force is maximum at that location and bodies of water are pulled radially. But in the locations that are on the circle which this point is its pole, the tidal force is minimum and water is pushed towards the center of Earth. So we just have to calculate the variations of water levels at the two locations.
   a. Assume an element of water which has the Moon at Zenith and is at distance r from the center of Earth. Take r_m to be the distance from the center of Earth to the center of the Moon. Now find a relation for the tidal forces exerted on this element. You may assume that the ratio of the distances of the element to the center of Earth and the element to the center of the Moon, is very small and neglect the second degree of the term. (70 points)
   b. Now repeat the same calculations for a water element on the circle which has that point as the pole. You may assume that the distance of the element to the moon is

equal to the distance of the center of Earth to the Moon. Keep in mind that for small angles we have $\sin x \approx x$. (80 points)

c. Now we plan to get closer to that wanted by the problem by using another thought experiment and make you more familiar with this tool in physics. This experiment is one of the most famous and oldest experiments in history which was first proposed by Newton and is known as Newton's well.
Imagine the two elements of water we proposed in the two previous sections and assume that we dig out a well towards the center of Earth from the two locations. These two wells intersect. Intuitively, it is obvious that the pressure of water from the two wells is equal at the center of Earth (Why?!) and on one hand, we know that we can assume that the pressure on the surface of these two wells is zero. On the other hand, since liquids are non-compressible, the density of water in the two wells is equal. Now using the hydrostatic equilibrium and integrating, find a relation between the pressures at the center of earth in the two wells. And using the tips above, set the two relations equal. Hint: You must get this equation:
$$\int_0^{h_1} g(r)dr - \int_0^{h_2} g(r)dr = \int_0^{h_1} 2C \times rdr + \int_0^{h_2} Crdr$$
Where $C = \frac{GM_m}{r_m^3}$. Now by using this mathematical theorem:
$$\int_0^{a} f(x)dx + \int_a^{b} f(x)dx = \int_0^{b} f(x)dx$$
simplify the integral on the left hand side and for solving the integrals on the right hand, you can approximately take h_1 equal to h_2 and equal to the radius of Earth. (These approximations are pretty good ones in physics!) (150 points)

d. Now find a relation for the altitude variations. Hint: $\Delta h = \frac{3M_m}{2M_e}\left(\frac{R_e}{r_m}\right)^3 R_e$ (70 points)
Using real values, the result would be $53.5cm$.
Ok now. We were able to find a suitable mechanism for the transfer of life to the land!!!
But… Here, we face the same problem that great biologists faced. $53.5cm$ is very very small for such a move and has virtually no effect on this transfer!
It seems as if our model has failed! But no!! Another astronomical phenomenon can save us…

Gravitational Locking

If you look at the sky sometimes, you would know that just one side of the moon faces the earth. (Regardless of the *liberation* phenomenon.) In other words, the Moon's rotational and orbital periods are the same. This isn't just some random event, but a very interesting mechanism which is called *gravitational locking* or *tidal* *locking* and in this section, we plan to model this event with much precision. First we must gain some info about this phenomenon.

Imagine a binary system that consists of two stars, one much more massive than the other. Both of the components rotate and the lighter star orbits the massive star. Pay attention that for now, none of the specified frequencies are equal. Now we generalize the system to two newly formed planets. Just like Earth and the Moon at the time of their formation. Then, since the planets were at high temperatures, they were made of hot molten metal but because of the presence of pressure, (We know from high school chemistry that the melting temperature increases with increases in pressure.) Solid metal was mostly in the inner parts of Earth and the Moon and near their surfaces, there was molten metal. We assume that the Moon had a circular orbit around Earth. Now we add the tidal forces to our model. The tidal forces exerted on the two planets, cause the magma on the surfaces to start rotating with an angular velocity equal to the **angular velocity of the Moon's orbit around Earth**. On one hand, since the Moon itself is rotating with an angular velocity different to that of the magma, a relative velocity is created between the molten surface and the solid core. This relative speed causes friction which between the two surfaces and wastes energy. (The calculation of the friction between liquid and solid matter is related to fluid mechanics. In the first version, we were planning to model that as well, not considering the difficulty. But we eventually decided against it, considering the size of the problem!) The friction between the two surfaces gradually wastes their kinetic energy and finally, the rotational frequency of the Moon becomes equal to the rotational frequency of the molten matter on its surface, which is equal to the orbital frequency. Because when the two have equal frequencies, there will be no more friction. (This has happened to the Moon but not to Earth.)

For one more time, imagine the Earth-Moon system. We can show that if no net external torque is exerted on our system, the total angular momentum of its particles would remain constant. To continue, we assume that the work done by the total net external torque on the system is zero. (Think about why this is a logical assumption!!) So the total angular momentum of our system's components is constant. Now imagine a frame with the origin at the center of Earth and two of the axes on the equatorial plane (Which is the Moon's orbital plane) and the third axis pointing toward the North Celestial Pole.

1. In this section, we plan to break down the angular momentum.
   a. The total angular momentum: The rotational angular momentum of Earth + The rotational angular momentum of the water on Earth + The Moon's orbital angular momentum + The rotational angular momentum of the Moon + The rotational angular momentum of the Moon's surface molten layer.

Using what was noted in the description of the problem, we know that the rotational frequency of the water on Earth, the Moon's liquid layer, and the Moon's orbit around Earth are equal.
Now by using the imaginary view you have of the system's movement, find the direction of each of the angular momentums. (50 points)

b. Using what you found in the section above, show that we can show each of the angular momentums in the relation $L = I\omega$. (70 points)
c. Now differentiate the total angular momentum relation and put it equal to zero and get an equation in terms of the differential of each of the angular momentums. (50 points)
d. This is where we use effective approximations in our model! You can neglect the ratio of the *inertia of the Moon's liquid layer to Earth's surface water*. You may also neglect the ratio of the *differential of the angular momentum of Earth's water to the differential of the Moon's orbital angular momentum*. Actually, using the approximations above, we were able to reduce our equation from 5 components to 3. (70 points)
e. Since we didn't exactly model the effect of friction, we can't calculate any numbers from our equations. So we attempt to get our required info qualitatively. Biology needed higher tides for the transfer of life from water to land. According to the equations we found for the variations in altitude, it is obvious that the less the distance from Earth to the Moon, the stronger the tides get. If we could just show that gravitational locking causes the Moon to move away from Earth, then we could conclude that the Moon was closer to Earth in the past and this would solve the biologists' problem. On one hand, according to the info that was given about gravitational locking, we know that the kinetic energy of Earth and the Moon is being wasted, so their angular velocity is decreasing. Now using the equation you found in part **d**, show that the variation in the orbital radius of the Moon is positive! (100 points)

So now we were able to find almost everything we needed! We were able to explain one of the most significant contradictions in the Evolution theory using our simple model. We may have left our perimeters of knowledge a few times and used high levels of knowledge at some sections but still this level of knowledge is very little compared to the greatness of what we found at last!

As you saw, very simple points which can be found in any mechanics book can explain our life on earth and this might be just a very small bit of the beauty of **science**!

Let us not forget that whatever we learn in life, should and will be used for explaining what happens around us and Olympiad contents are no exception.
Total: 1100 points

GOOD LUCK