8th International Olympiad on Astronomy and Astrophysics

Suceava - Gura Humorului - August 2014

Content

Indications

1. The problems were elaborated concerning two aspects:
    a. *To cover merely all the subjects from the syllabus;*
    b. *The average time for solving the items is about 15 minutes per a short problem;*
2. In your folder you will find out the following:
    a. *Answer sheets*
    b. *Draft sheets*
    c. *The envelope with the subjects in English and the translated version of them in your mother tongue;*
3. The solutions of the problems will be written down only on the answer sheets you receive on your desk. **PLEASE WRITE ONLY ON THE PRINTED SIDE OF THE PAPER SHEET. DON'T USE THE REVERSE SIDE**. The evaluator will not take into account what is written on the reverse of the answer sheet.
4. The draft sheets is for your own use to try calculation, write some numbers etc. BEWARE: These papers are not taken into account in evaluation, at the end of the test they will be collected separately . Everything you consider as part of the solutions have to be written on the answer sheets.
5. Each problem have to be started on a new distinct answer sheet.
6. On each answer sheet please fill in the designated boxes as follows:
    a. *In **PROBLEM NO**. box write down only the number of the problem: i.e. **1** to **15** for short problems, **16** to **19** for long problems. Each sheet containing the solutions of a certain problem, should have in the box the number of the problem;*
    b. *In **Student ID** – fill in your ID you will find on your envelope, consisted of 3 leters and 2 digits.*
    c. *In page no. box you will fill in the number of page, starting from 1. We advise you to fill this boxes after you finish the test*
7. We don't understand your language, but the mathematic language is universal, so use as more relationships as you think that your solution will be better understand by the evaluator. If you want to explain in words we kindly ask you to use short English propositions.
8. Use the pen you find out on the desk. It is advisable to use a pencil for the sketches.
9. At the end of the test:
    a. *Don't forget to put in order your papers;*
    b. *Put the answer sheets in the folder 1. Please verify that all the pages contain your ID, correct numbering of the problems and all pages are in the right order and numbered. This is an advantage of ease of understanding your solutions.*
    c. *Verify with the assistant the correct number of answer sheets used fill in this number on the cover of the folder and sign.*
    d. *Put the draft papers in the designated folder, Put the test papers back in the envelope.*
    e. *Go to swim*

GOOD LUCK !

16. Long problem 1. Eagles on the Caraiman Cross !

In the Bucegi mountains, part of the Carpathian mountains, after the end of the First World War an iron cross was built by the former King of Romania called Ferdinand the I-st and his wife Queen Maria. The cross is an unique monument in Europe. The monument is an impressive iron cross called „The Heros' Cross" which in 2013 entered in the Guiness Book as the cross build on the highest altitude mountain peek.

The cross was built on the plane plateau situated on the top of the peek called Caraiman, at the altitude $H = 2300\,m$ from sea level. Its height, including the base-support is $h = 39{,}3\,m$. The horizontal arms of the cross are oriented on the N-S direction. The latitude of the Cross is $\varphi = 45^{\circ}$.

A. In the evening of 21st of March 2014, the summer equinox day, two eagles stop from their flight, first near the monument, and the second, on the top of the Cross as seen in figure 1. The two eagles are on the same

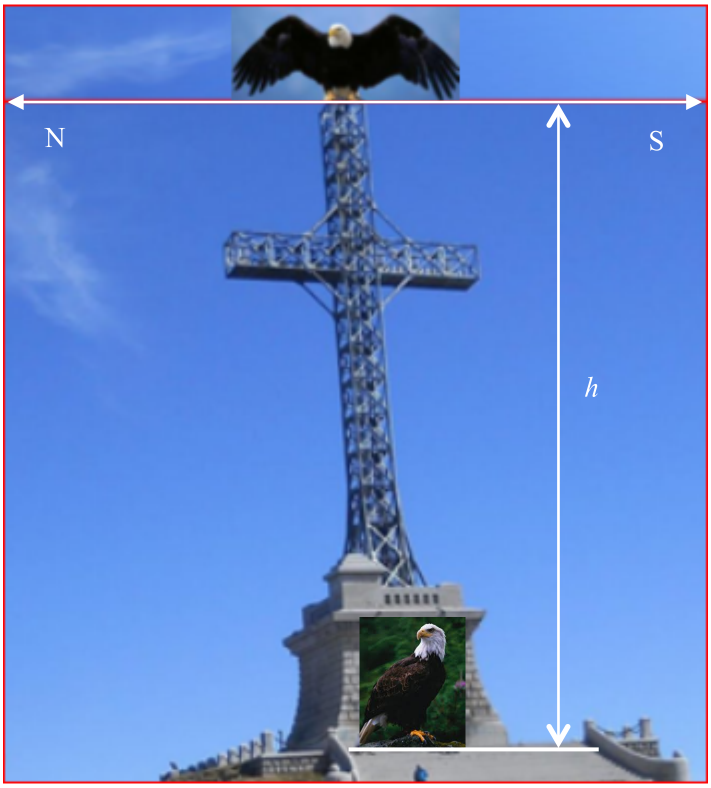


Figure 1

vertical direction. The sky was very clear, so the eagles could see the horizon and observe the Sun set. Each eagle start to fly right in the moment that each of it observes that the Sun completely disappears.

In the same time, an astronomer located at the sea-level, at the base of the Bucegi Mountains. Assume that he is on the same vertical with the two eagles.

Assuming negligible the atmospheric refraction, solve the following questions:

1) *Calculate* the duration of the sunset, measured by the astronomer.

2) *Calculate* the durations of sunset measured by each of the two eagles and indicate which of the eagles leaves first the Cross. What is the time interval between the leaving moments of the two eagles.

The following information is necessary:

The duration of the sunset measurement starts when the solar disc is tangent to the horizon line and stops when the solar disc completely disappears.

The Earth's rotation period is $T_E = 24\,\mathrm{h}$, the radius of the Sun $R_S = 6{,}96 \cdot 10^5\ \mathrm{km}$, Earth – Sun distance $d_{ES} = 15 \cdot 10^7\ \mathrm{km}$, the local latitude of the Heroes Cross $\varphi = 45^\circ$.

B) At a certain moment of the next day, 22nd March 2014, the two eagles come back to the Heroes Cross. One of the eagles lands on the top of the vertical pillar of the Cross and the other one land on the horizontal plateau, just in the end point of the shadow of the vertical pillar of the Cross.

1) Calculate the distance between the two eagles, if this distance has the minimum possible value.

2) Calculate the length of the horizontal arms of the Cross l_b, if the shadow on the plateau of one of the arm of the cross has the length $u_b = 7\ \mathrm{m}$

C) At midnight, the astronomer visit the cross and, from the top of it, he identifies a bright star at the limit of the circumpolarity. He named this star „Eagles Star". Knowing that due to the atmospheric refraction the horizon lowering is $\xi = 34'$, calculate:

1) The "Eagles star" declination;

2) The "Eagles star" maximum height above the horizon.

Long problem 1. Marking scheme - Eagles on the Caraiman Cross

1)	**10**
2)	**10**
B1)	10
B2)	10
C1)	5
C2)	5

A. 1 The following notations are used: D_S the diameter of the Sun, d_{ES} Earth-Sun distance, θ angular diameter of the Sun as seen from the Earth:

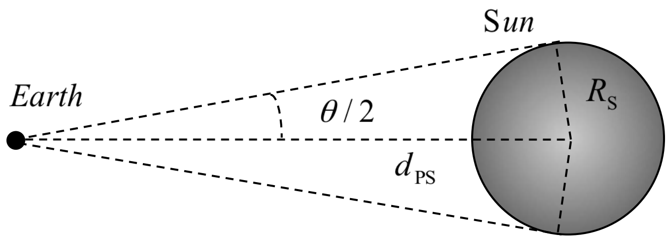


Fig. 1

According the fig. 1 the angular diameter of the Sun can be calculated as follows

$$\sin\frac{\theta}{2} = \frac{R_S}{d_{PS}} \approx \frac{\theta}{2};$$

$$\theta = \frac{2R_S}{d_{PS}} = \frac{D_S}{d_{PS}} = \frac{2 \cdot 6{,}96 \cdot 10^5 \text{ km}}{15 \cdot 10^7 \text{ km}} = 0{,}00928 \text{ rad}.$$

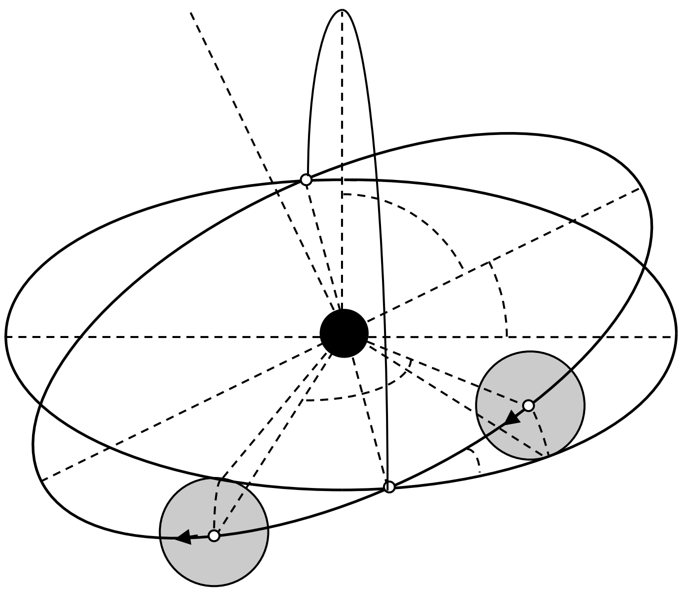
The figure 2 presents the Sun's evolution during sunset as seen by the astronomer. In an equinox day the Sun moves retrograde along the celestial equator. There are marked the following 3 positions of the Sun:

T_{dos} - The solar disc is tangent to the equatorial plane above the standard horizon – the start of the sunset;

S_{dos} - The center of the solar disc on the celestial equator in the moment of the sunset starts;

T_{sos} - The solar disc is tangent to the equatorial plane bellow the standard horizon – the end of the sunset

S_{sos} - The center of the solar disc on the celestial equator in the moment of the sunset ends;



The duration of the sunset is τ. During this time the center of the Sun moves along the equator from S_{dos} to S_{sos}. The vector-radius of the Sun rotates in equatorial plane with angle ϕ and in vertical plane with angle θ. i.e. the angular diameter of the Sun as seen from the Earth.

Considering that the Sun travels the distance $2x$ along the equatorial path with merely constant i.e. during time τ and that the spherical right triangle $S_{dos}T_{dos}V$ can be considered a plane one the following relations can be written:

$$\sin\gamma = \frac{R_S}{x};\ x = \frac{R_S}{\sin\gamma};\ \ 2x = \frac{2R_S}{\sin\gamma} = \frac{D_S}{\sin\gamma};$$

$$\tau = \frac{2x}{v} = \frac{2x}{\omega\cdot d_{PS}} = \frac{\dfrac{D_S}{\sin\gamma}}{\dfrac{2\pi}{T_P}\cdot d_{PS}} = \frac{\dfrac{D_S}{d_{PS}}}{\dfrac{2\pi}{T_P}\cdot\sin\gamma} = \frac{\theta\cdot T_P}{2\pi\cdot\sin\gamma};$$

$$\sin\gamma = \sin\left(90^\circ - \varphi\right) = \cos\varphi;$$

$$\tau = \frac{\theta\cdot T_P}{2\pi\cdot\cos\varphi};$$

$$\tau = \frac{0{,}00928\,\text{rad}\cdot 24\,\text{h}}{2\cdot 3{,}14\,\text{rad}\cdot\cos\left(45^\circ 21'\right)} = \frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}707}\,\text{min} \approx 3\,\text{min}.$$

2) If the atmospheric refraction is negligible the eagle on the top of the cross V_1 on figure 3 is on the same latitude (φ), as the astronomer but at the altitude H. Thus from the point of view of the V_1 the horizon line is below the standard horizon line with an angle $\Delta\alpha_1$,

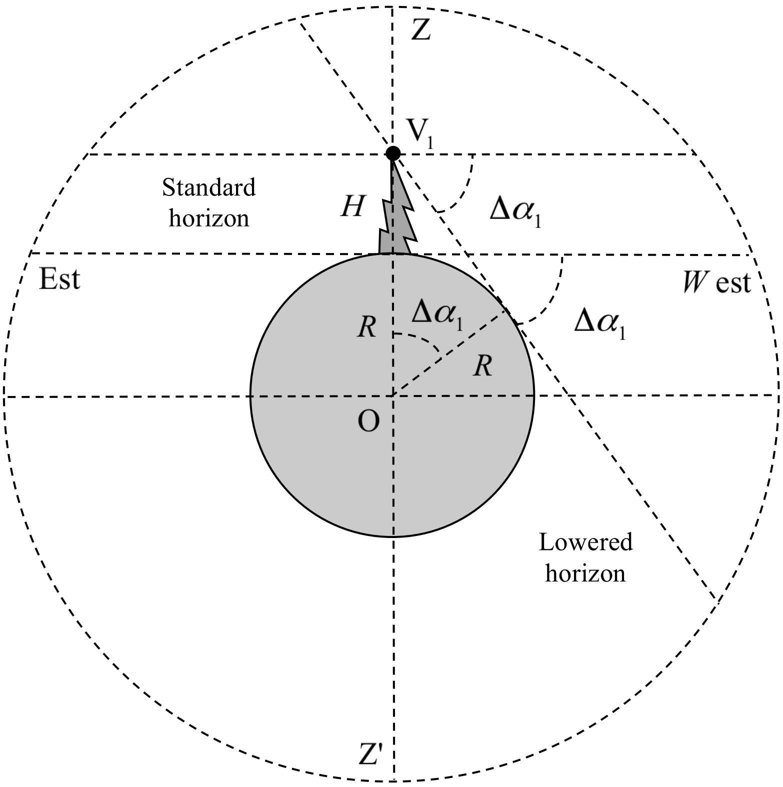


Fig. 3

$$\cos\Delta\alpha_1 = \frac{R}{R+H};$$

$$\sin \Delta\alpha_1 = \frac{\sqrt{(R+H)^2 - R^2}}{R+H} = \frac{\sqrt{2RH + H^2}}{R+H} \approx \frac{\sqrt{2RH}}{R} = \sqrt{\frac{2H}{R}} \approx \Delta\alpha_1;$$

$$\Delta\alpha_1 = \sqrt{\frac{2 \cdot 2{,}3\ \text{km}}{6380\ \text{km}}} \approx 0{,}02685\ \text{rad} \approx 1{,}54^\circ.$$

For the observer V_1 the Sun will go below the lowered horizon after moving down under the standard horizon with angle $\Delta\alpha_1$ and moving along the equator with an angle $\Delta\beta_1$, as seen in fig. 4

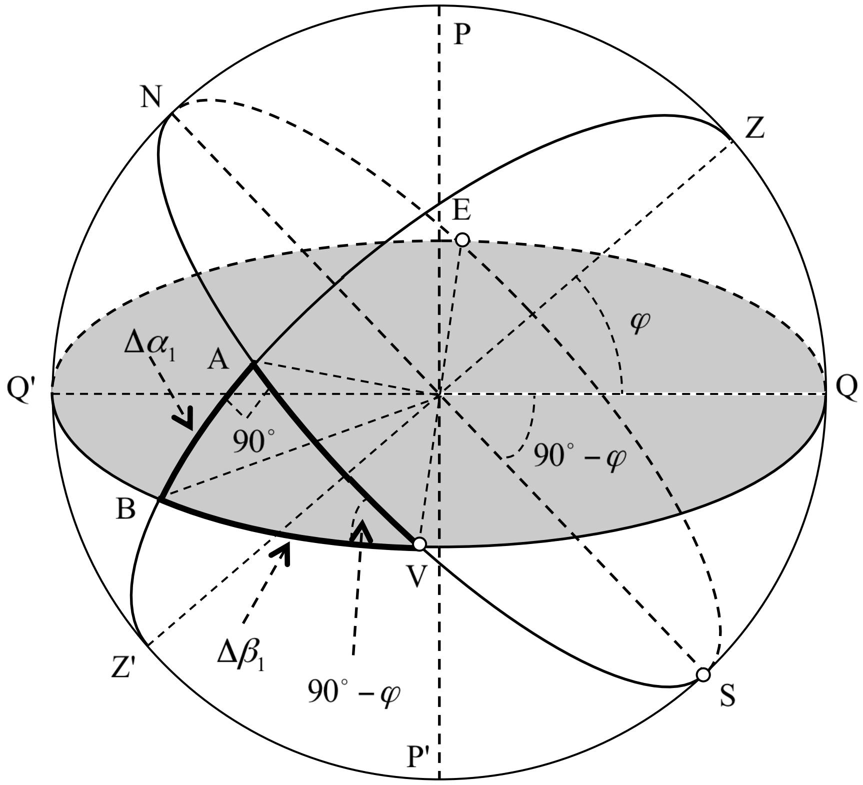


Fig.

In the right spherical triangle ABV by using the sinus formula :

$$\frac{\sin(90^\circ - \varphi)}{\sin \Delta\alpha_1} = \frac{\sin 90^\circ}{\sin \Delta\beta_1};$$

$$\frac{\cos\varphi}{\Delta\alpha_1} = \frac{1}{\Delta\beta_1};\ \Delta\beta_1 = \frac{\Delta\alpha_1}{\cos\varphi};$$

$$\Delta\beta_1 = \omega \cdot \Delta\tau_1 = \frac{2\pi}{T_P} \cdot \Delta\tau_1;$$

$$\Delta\tau_1 = \frac{\Delta\alpha_1}{\cos\varphi} \cdot \frac{T_P}{2\pi} = \frac{1{,}54^\circ}{\cos(45^\circ 21')} \cdot \frac{24 \cdot 60\ \text{min}}{360^\circ} \approx 8{,}71\,\text{minute},$$

Which represents the delay of the start of the sunset from the point of view of V_1 regardless to the astronomer due to the V_1 observer's altitude.

The altitude effect on the total time of sunset can be calculated by using the fig. 5

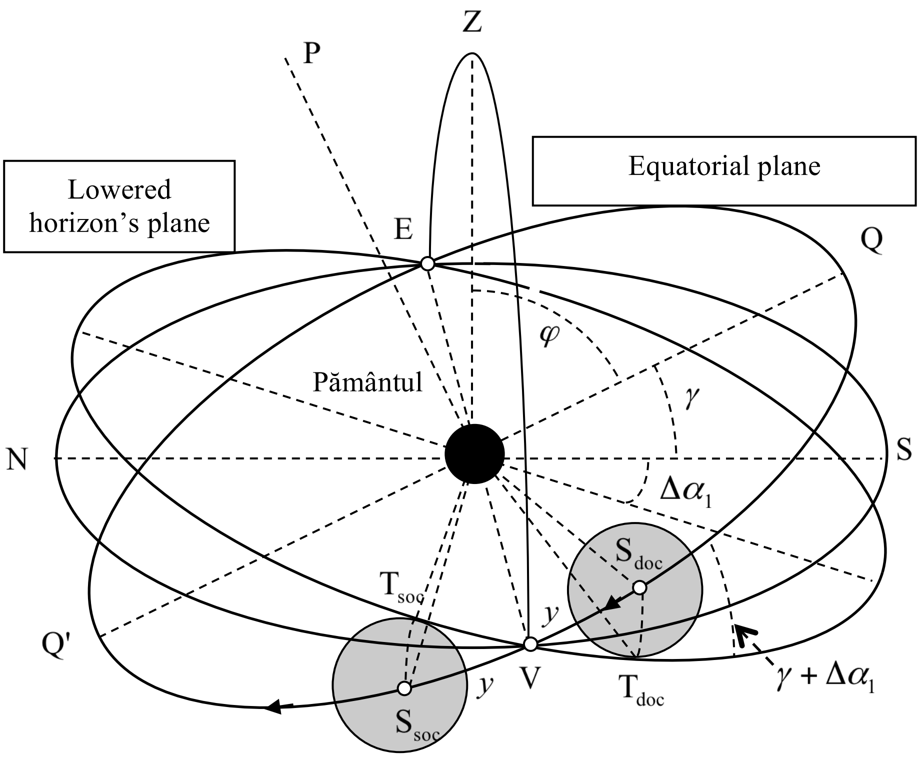


Fig.5

$$\sin(\gamma + \Delta\alpha_1) = \frac{R_S}{y};\; y = \frac{R_S}{\sin(\gamma + \Delta\alpha_1)};\; 2y = \frac{2R_S}{\sin(\gamma + \Delta\alpha_1)} = \frac{D_S}{\sin(\gamma + \Delta\alpha_1)};$$

$$\tau_1 = \frac{2y}{\mathrm{v}} = \frac{2y}{\omega \cdot d_{PS}} = \frac{\dfrac{D_S}{\sin(\gamma + \Delta\alpha_1)}}{\dfrac{2\pi}{T_P} \cdot d_{PS}} = \frac{\dfrac{D_S}{d_{PS}}}{\dfrac{2\pi}{T_P} \cdot \sin(\gamma + \Delta\alpha_1)} = \frac{\theta \cdot T_P}{2\pi \cdot \sin(\gamma + \Delta\alpha_1)};$$

$$\gamma = 90^\circ - \varphi;$$

$$\sin(\gamma + \Delta\alpha_1) = \sin(90^\circ - \varphi + \Delta\alpha_1) = \sin[90^\circ - (\varphi - \Delta\alpha_1)] = \cos(\varphi - \Delta\alpha_1);$$

$$\tau_1 = \frac{\theta \cdot T_P}{2\pi \cdot \cos(\varphi - \Delta\alpha_1)};$$

$$\tau_1 = \frac{0{,}00928\,\text{rad}\cdot 24\,\text{h}}{2\cdot 3{,}14\,\text{rad}\cdot\cos\left(45^\circ - 1{,}54^\circ\right)} = \frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}725}\,\text{min} \approx 2{,}9350\,\text{min},$$

Which represents the total duration of sunset for V_1 at altitude *H*.

Similarly for eagle V_2 at the same latitude (φ), but altitude $H + h$ (the top of the cross), the lowering effect on the horizon is measured by angle $\Delta\alpha_2$ thus

$$\cos\Delta\alpha_2 = \frac{R}{R+H+h};$$

$$\sin\Delta\alpha_2 = \frac{\sqrt{(R+H+h)^2 - R^2}}{R+H+h} = \frac{\sqrt{2R(H+h)+(H+h)^2}}{R+H+h} \approx \frac{\sqrt{2R(H+h)}}{R} = \sqrt{\frac{2(H+h)}{R}} \approx \Delta\alpha_2;$$

$$\Delta\alpha_2 = \sqrt{\frac{2\cdot(2{,}3+0{,}0393)\,\text{km}}{6380\,\text{km}}} \approx 0{,}02707\,\text{rad} \approx 1{,}55^\circ;$$

$$\frac{\sin\left(90^\circ - \varphi\right)}{\sin\Delta\alpha_2} = \frac{\sin 90^\circ}{\sin\Delta\beta_2};$$

$$\frac{\cos\varphi}{\Delta\alpha_2} = \frac{1}{\Delta\beta_2};\quad \Delta\beta_2 = \frac{\Delta\alpha_2}{\cos\varphi};$$

$$\Delta\beta_2 = \omega\cdot\Delta\tau_2 = \frac{2\pi}{T_P}\cdot\Delta\tau_2;$$

$$\Delta\tau_2 = \frac{\Delta\alpha_2}{\cos\varphi}\cdot\frac{T_P}{2\pi} = \frac{1{,}55^\circ}{\cos\left(45^\circ 21'\right)}\cdot\frac{24\cdot 60\,\text{min}}{360^\circ} \approx 8{,}77\,\text{minute},$$

Which represents the delay of the start moment of the sunset for V_2 due to the altitude $H + h$.

Similar the total duration of the sunset for the observer V_2 :

$$\tau_2 = \frac{\theta\cdot T_P}{2\pi\cdot\cos(\varphi - \Delta\alpha_2)};$$

$$\tau_2 = \frac{0{,}00928\,\text{rad}\cdot 24\,\text{h}}{2\cdot 3{,}14\,\text{rad}\cdot\cos\left(45^\circ - 1{,}55^\circ\right)} = \frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}726}\,\text{min} \approx 2{,}9309\,\text{min},$$

We may note the following:

- the horizon- lowering $\Delta\alpha$ is increased by the increase of the altitude;

$$\left(H < H+h \rightarrow \Delta\alpha_1 < \Delta\alpha_2;\, H\uparrow \rightarrow \Delta\alpha\uparrow\right)$$

- the delay of the moment of sunset start is increased by the increase of the altitude:

$$\left(H < H+h \rightarrow \Delta\tau_1 < \Delta\tau_2;\, H\uparrow \rightarrow \Delta\tau\uparrow\right)$$

- the total duration of sunset is reduced by the increase of the altitude:

$$\left(0 < H < H+h \rightarrow \tau > \tau_1 > \tau_2;\, H\uparrow \rightarrow \tau\downarrow\right)$$

Conclusions:

If we consider t_0 the moment of sunset star for the astronomer

- for V_1 the sunset starts at $t_0 + 8{,}71\,\text{min}$ and ends at $t_0 + 8{,}71\,\text{min} + 2{,}9350\,\text{min} = t_0 + 11{,}6450\,\text{min}$
- for V_2 the sunset starts at $t_0 + 8{,}77\,\text{min}$ and ends at $t_0 + 8{,}77\,\text{min} + 2{,}9309\,\text{min} = t_0 + 11{,}7009\,\text{min}$
- Thus eagle from the plateau leaves first the cross;

- The time between the leaving moments is:

$$\Delta t = t_0 + 11{,}7009\,\text{min} - t_0 - 11{,}6450\,\text{min} = 0{,}0559\,\text{min} = 3{,}354\,\text{s}.$$

b)

As seen in fig. 6 the length of the cross on the plateau will be minimum when the Sun passes the local meridian, i.e. the height of the Sun above the horizon will be maximum:

$\left(h_{\max} = \gamma = 90^\circ - \varphi\right)$.

Thus the shadow of the horizontal arms of the cross is superposed on the shadow of the vertical pillow.

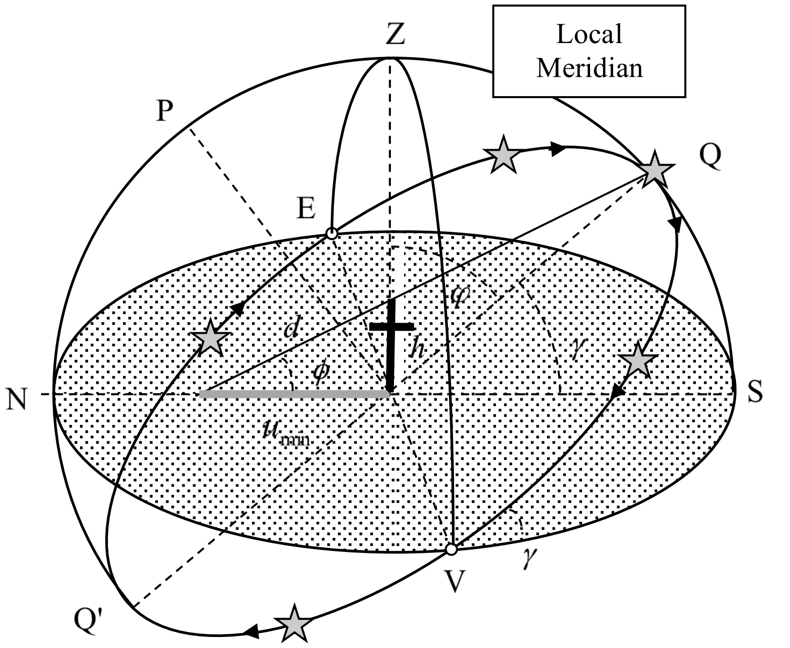


Fig. 6

In this conditions :

$$\sin\phi = \frac{h}{d};\ \phi \approx \gamma = 90^\circ - \varphi;$$

$$d = \frac{h}{\sin\phi} \approx \frac{h}{\sin\gamma} = \frac{h}{\sin(90^\circ - \varphi)} = \frac{h}{\cos\varphi} = \frac{39{,}3\,\text{m}}{\cos 45^\circ} = \frac{39{,}3}{0{,}707}\,\text{m} \approx 55{,}58\,\text{m};$$

The distance between the two eagles is

$$u_{\min} = h \cdot \cot\phi \approx h \cdot \cot\varphi = h \cdot \cot(90^\circ - \varphi) = h \cdot \tan\varphi = 39{,}3\,\text{m}.$$

2) In the above mentioned conditions the shadow of the arm oriented toward South is on the vertical pillow of the cross, as seen in fig. 7:

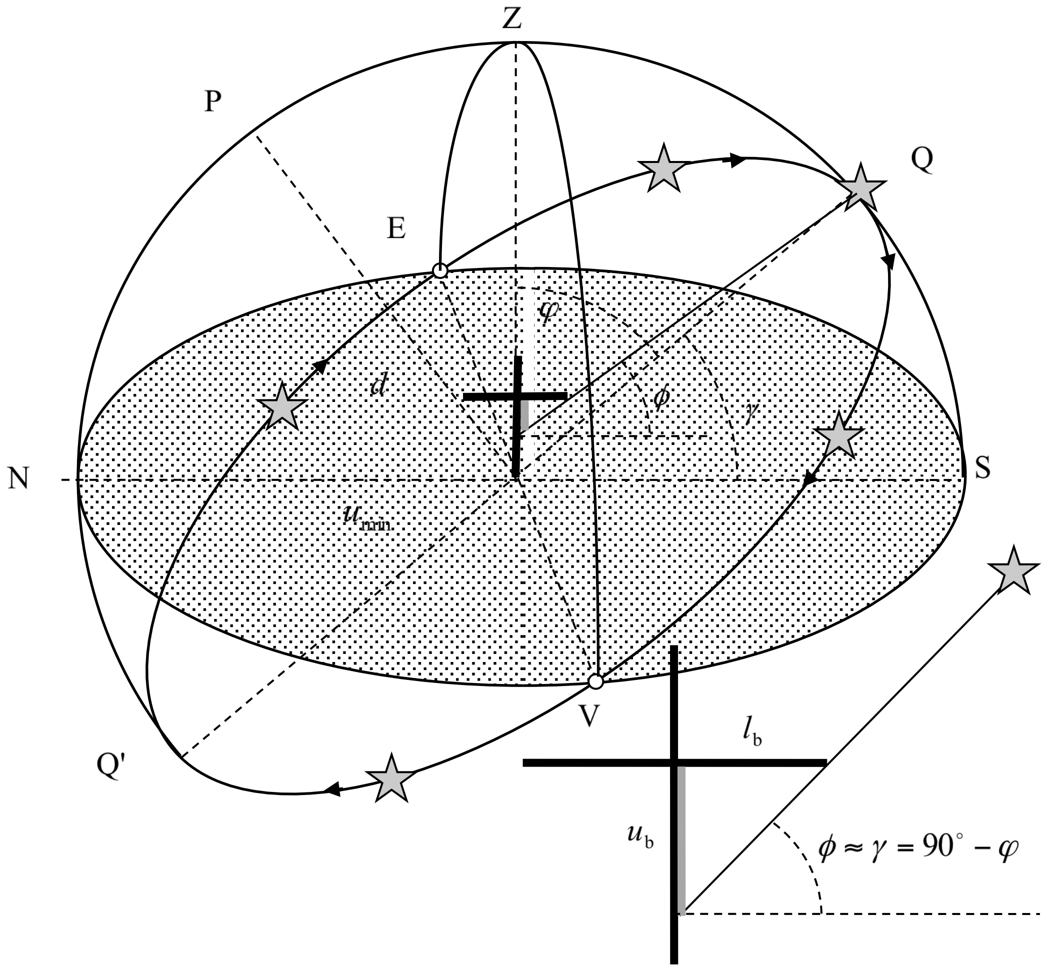


Fig. 7

$$\tan\varphi = \frac{l_\mathrm{b}}{u_\mathrm{b}};\ l_\mathrm{b} = u_\mathrm{b}\cdot\tan\varphi = 7\ \mathrm{m}\cdot\tan 45^\circ = 7\ \mathrm{m},$$

Which represents the length of the cross arm.

C)

1) For an observer situated in the center O of the celestial topocentric sphere, at latitude φ, at sea level, al the stars are circumpolar ones see fig. 8. Their diurnal parallels, parallel with the equatorial parallel, are above the real local horizon $\left(\mathrm{N_0S_0}\right)$. The star σ_0 is at the circumpolar limit because its parallel touches the real local horizon in point $\mathrm{N_0}$ but still remains above it. Thus σ_0 is a marginal circumpolar star. Without taking into account the atmospheric refraction:

From the isosceles triangle $\mathrm{O}\sigma_0\mathrm{N_0}$ results the σ_0 declination:

$$\delta_{0,\min} + 90^\circ + \left(\varphi - \delta_{\min}\right) = 180^\circ;$$

$$\delta_{0,\min} = 90^\circ - \varphi.$$

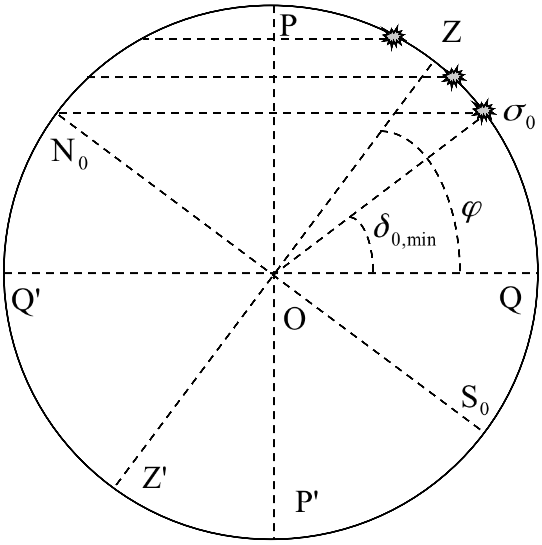


Fig. 8

By taking into account the atmospheric refraction the horizon line changes to line $N'S'$, with angle $\theta' \approx 34'$, as seen in fig. 9. The star σ_0 remains a circumpolar one but above the limit. In this conditions the star σ' gather the limit conditions its declination been $\delta'_{\min} < \delta_{\min}$. In this conditions for an observer situated in the center of the topocentric celestial sphere, at latitude φ and altitude zero, the star σ', with declination $\delta'_{\min} < \delta_{0\min}$ is on the limit of the circumpolarity.

From the isosceles triangle $N'O\sigma'$ the declination σ' will be :

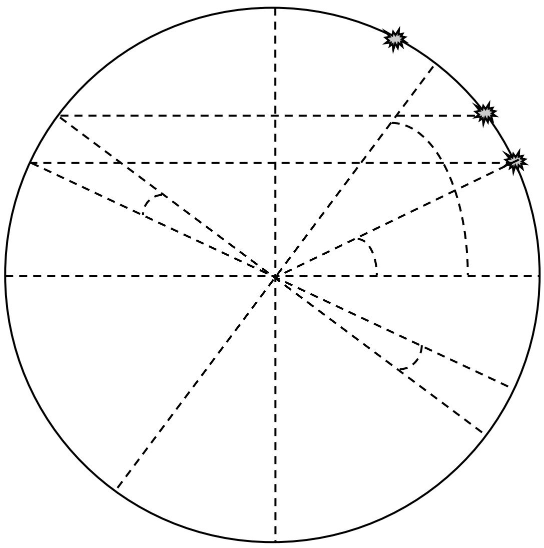


Fig. 9

$$2\delta'_{\min} + (\varphi - \delta'_{\min}) + (90^\circ + \theta') = 180^\circ;$$
$$\delta'_{\min} = 90^\circ - \varphi - \theta'.$$

For the observer at latitude φ, but at height h, taking into account the effect of lowering of the horizon the star σ'' will meet the problem requirements see figure 10. The new horizon is N"S" and declination of σ'' is $\delta''_{\min} < \delta_{0\min}$. Star σ_0 remains a circumpolar one but above the limit.

From the isosceles triangle N"Oσ'' the declination of star σ'' will be:

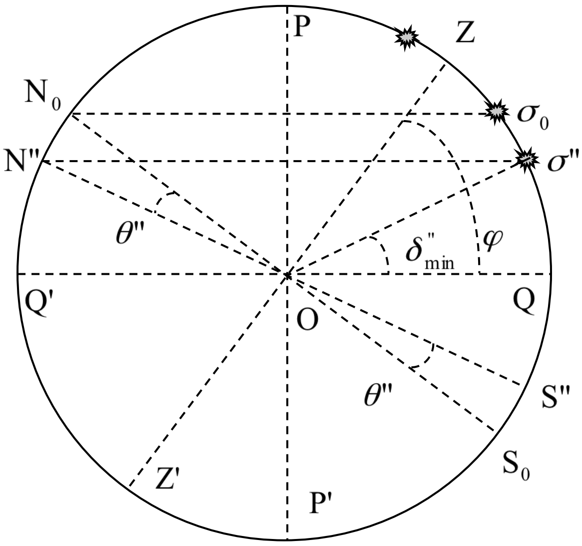


Fig. 10

$$2\delta''_{\min} + \left(\varphi - \delta''_{\min}\right) + \left(90^\circ + \theta''\right) = 180^\circ;$$
$$\delta''_{\min} = 90^\circ - \varphi - \theta''.$$

By taking into account the refraction effect and the altitude effect, from triangle NOσ in figure 11, the declination will be

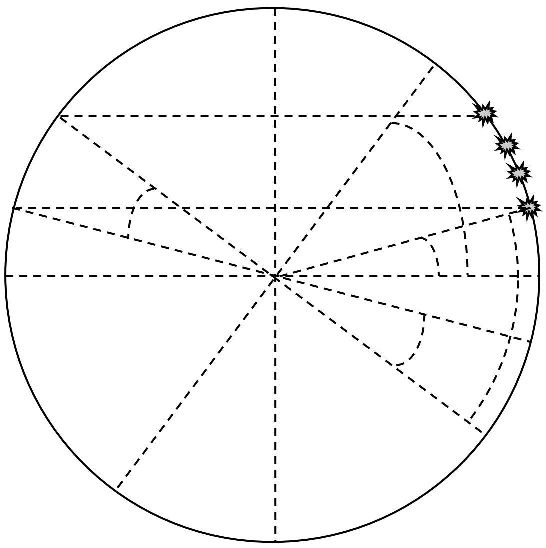


Fig. 11

$$2\delta_{\min} + \left(\varphi - \delta_{\min}\right) + \left(90^\circ + \theta\right) = 180^\circ;$$
$$\theta = \theta' + \theta''; \ \theta' = \xi = 34': \theta'' = \Delta\alpha_2 = 1{,}55^\circ;$$

$$\delta_{\min} = 90^\circ - \varphi - \theta' - \theta'' = 90^\circ - \varphi - \theta' - \Delta\alpha_2;$$
$$\delta_{\min} = 90^\circ - 45^\circ - 0{,}56^\circ - 1{,}55^\circ \approx 42{,}9^\circ.$$

2) The maximum height above the horizon will be

$$h_{\max} = 90^\circ + \delta_{\min} - \varphi = 90^\circ + 42{,}9^\circ - 45^\circ = 87{,}9^\circ.$$

17. Long problem 2. Cosmic Pendulum

A space shuttle (N) orbits the Earth in the equatorial plane on a circular trajectory with radius *r*. From the spaceship. The shuttle has an arm designated to place satellites on the orbit. The arm is a metallic rod (negligible mass) with length $l << r$. The arm is connected to the shuttle with frictionless mobile articulation. A satellite **S** is attached to the arm and let out from the shuttle. At a certain moment the angle between the rod and the shuttle's orbit radius is α, see figure 1. You know the mass of the Earth –**M** , and the gravitational constant **k**.

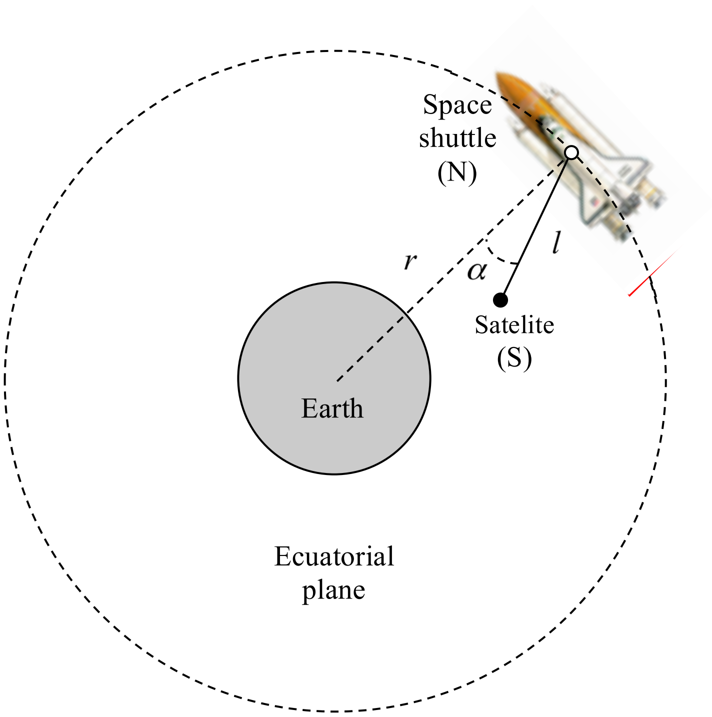


Fig.2

a) Find out the values of angle α for which the configuration of the system shuttle – rod –satellite remains unchanged regardless to Earth (the system is in equilibrium), during orbiting the Earth. For each found value of angle α, specify the type of the system equilibrium i.e. stable or unstable.

You will take in to account the following assumptions: the initial orbit of the shuttle is not affected by the presence of the satellite **S**, all the external friction-type interactions are negligible, the satellite – shuttle gravitational interaction is negligible too. The following data are known: , m_1 - the mass of the shuttle, the mass of the satellite $m_2 << m_1$.

b) In the moment of one stable equilibrium configuration, the rod with the satellite attached is slightly rotated with a very small angle in the orbital plane and then released. Demonstrate that the small oscillations of the satellite **S** relative to the shuttle are harmonically ones. Express the period T_0 of this cosmic pendulum as a function of the orbiting period T of the shuttle around the Earth.

It is known the linear harmonic oscillator equation:

$$\frac{d^2\beta}{dt^2} + \omega_0^2\beta = 0;\ \omega_0 = \frac{2\pi}{T_0},$$

Where : β – the instantaneous angular deviation; T_0 – the period of the linear harmonic oscillator.

c) If we consider that the mass of the satellite S , m_2 is not negligible by comparison with the shuttle's one m_1 in the conditions from point a) the evolution on the orbit of the shuttle would be influenced by the presence of the satellite S rigid attached to the shuttle by the rod. Identify and determine the consequences on the shuttle's movement after one complete rotation around the Earth.

d) Propose a special technical maneuver which can cancels the influence of the non negligible mass satellite S on the shuttles movement.

a)	15
b)	15
c)	10
d)	10

Long problem 2. Marking scheme - Cosmic Pendulum

a) In figure 1 are represented the forces in the system.

$\vec{F}_1$ – the gravitational attraction force acting on the shuttle due to the Earth;

$\vec{F}_2$ – the gravitational attraction force acting on the satelite due to the Earth;

$\vec{F}$ – the tension force in the suspention rod.

For a value of $\alpha \neq 0$ the movement equations are:
For the shuttle on circular orbit around the Earth:

$$m_1\omega^2 r = K\frac{m_1 M}{r^2} + F\cos\alpha;$$

For the satelite on circular orbit around the Earth:

$$m_2\omega^2\left(r - l\cos\alpha\right) = K\frac{m_2 M}{\left(r - l\cos\alpha\right)^2} - \frac{F}{\cos\alpha}.$$

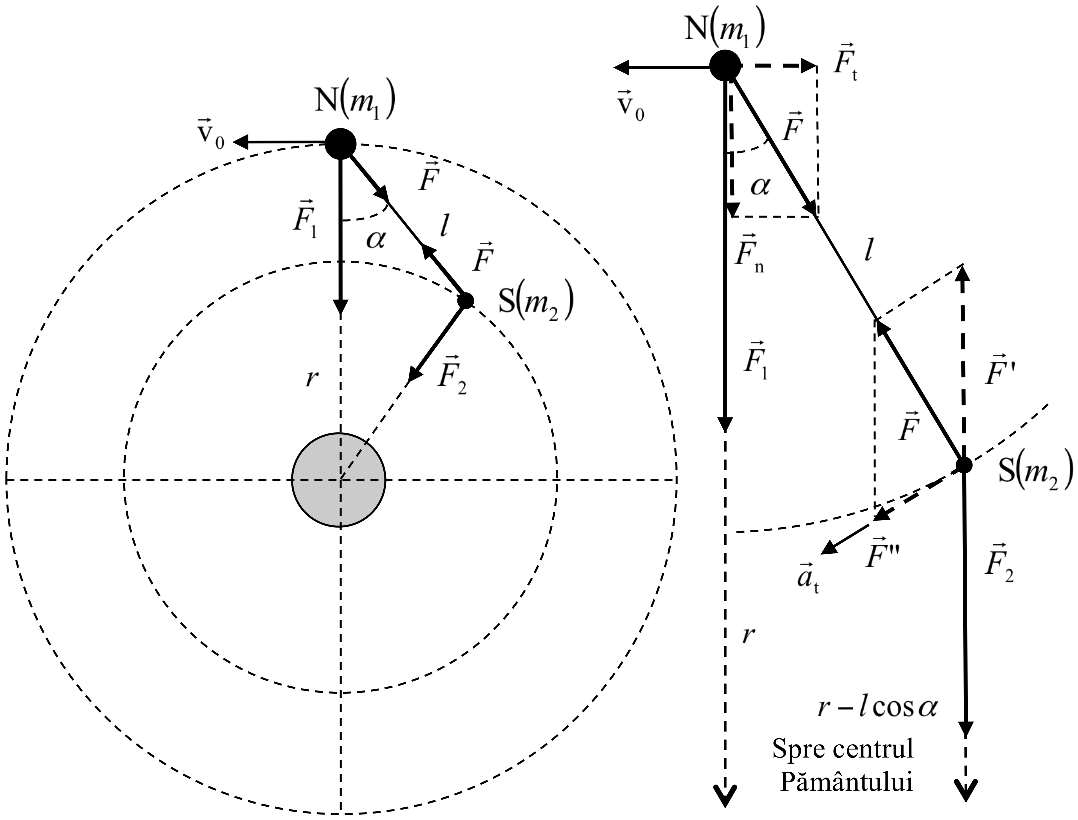


Fig.1

If the gravitational interaction between the shuttle and the satellite is negligible results:

$$F\cos\alpha << K\frac{m_1 M}{r^2};$$

$$m_1\omega^2 r \approx K\frac{m_1 M}{r^2};\ \omega^2 = \frac{KM}{r^3} = \frac{4\pi^2}{T^2};$$

$$T = 2\pi\sqrt{\frac{r^3}{KM}},$$

The revolution time of the shuttle around the Earth:

$$m_2\frac{KM}{r^3}\left(r - l\cos\alpha\right) = K\frac{m_2 M}{\left(r - l\cos\alpha\right)^2} - \frac{F}{\cos\alpha};$$

$$m_2\frac{KM}{r^3}(r-l\cos\alpha)^3 = Km_2M - \frac{F}{\cos\alpha}(r-l\cos\alpha)^2;$$

$$m_2\frac{KM}{r^3}r^3\left(1-\frac{l}{r}\cos\alpha\right)^3 = Km_2M - \frac{F}{\cos\alpha}r^2\left(1-\frac{l}{r}\cos\alpha\right)^2;$$

$$m_2KM\left(1-\frac{l}{r}\cos\alpha\right)^3 = Km_2M - \frac{F}{\cos\alpha}r^2\left(1-\frac{l}{r}\cos\alpha\right)^2;$$

$$m_2KM\left(1-3\frac{l}{r}\cos\alpha\right) = Km_2M - \frac{F}{\cos\alpha}r^2\left(1-2\frac{l}{r}\cos\alpha\right);$$

$$3Km_2M\frac{l}{r}\cos\alpha \approx \frac{F}{\cos\alpha}r^2;$$

$$F \approx 3Km_2M\frac{l}{r^3}\cos^2\alpha,$$

The tension force in the rod.

The satellite movement regardless to the shuttle is non uniform circular one, described by the equation:

$$m_2\vec{a}_t = \vec{F}'';$$

$$m_2a_t = m_2\varepsilon l = m_2\frac{d^2\alpha}{dt^2}l = -F'' = -F\tan\alpha = -3Km_2M\frac{l}{r^3}\sin\alpha\cos\alpha;$$

$$\frac{d^2\alpha}{dt^2} + 3\frac{KM}{r^3}\sin\alpha\cos\alpha = 0,$$

The solutions of this equation is $\alpha(t)$.

If during the system evolution the configuration of the system remains the same results:

$$\alpha = \text{constant}; \ \frac{d^2\alpha}{dt^2} = 0;$$

$$\sin\alpha\cdot\cos\alpha = 0;$$

$$\sin\alpha = 0;\ \alpha_1 = 0;\ \alpha_2 = \pi;\ \text{(echilibru stabil)};$$

$$\cos\alpha = 0;\ \alpha_3 = \pi/2;\ \alpha_4 = 3\pi/2;\ \text{(echilibru instabil)},$$

As seen in the pictures .

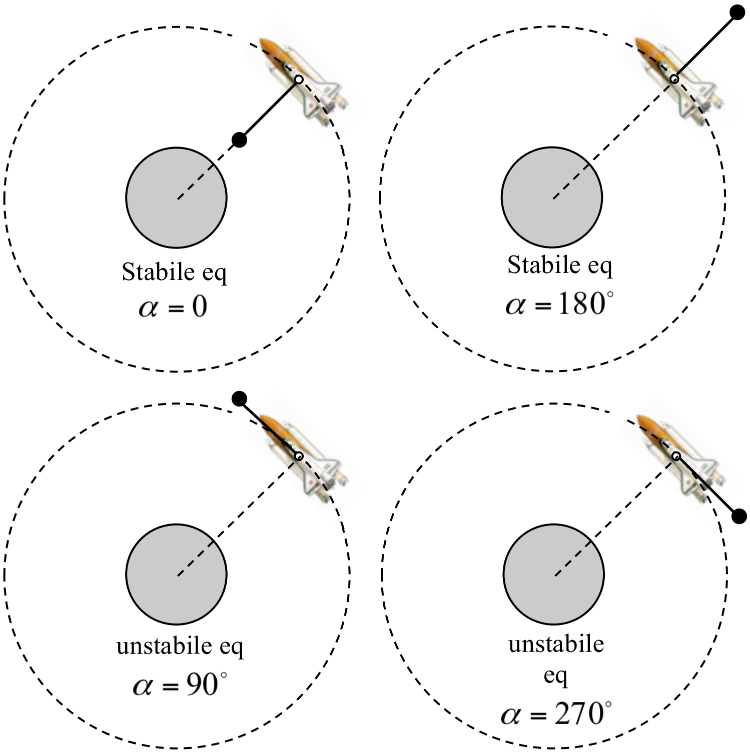


Fig.

b) Coresponding to the position with $\alpha = 0$, the movement equation of the satellite displaced from equilibrium position with a very small angle β:

$$\frac{d^2\beta}{dt^2} + 3\frac{KM}{r^3}\sin\beta \cdot \cos\beta = 0;$$

$$\sin\beta \approx \beta; \cos\beta \approx 1;$$

$$\frac{d^2\beta}{dt^2} + 3\frac{KM}{r^3}\beta = 0,$$

Which represents the linear harmonic oscillator;

$$\frac{d^2\beta}{dt^2} + \omega_0^2\beta = 0;$$

Thus the period of the small oscillations will be

$$\omega_0^2 = 3\frac{KM}{r^3} = \frac{4\pi^2}{T_0^2};\ T_0 = 2\pi\sqrt{\frac{1}{3}\frac{r^3}{KM}} = \frac{T}{\sqrt{3}},$$

c)

$$m_1\omega^2 r = K\frac{m_1 M}{r^2} + F\cos\alpha;$$

$$m_2\omega^2(r - l\cos\alpha) = K\frac{m_2 M}{(r - l\cos\alpha)^2} - \frac{F}{\cos\alpha};$$

$$\omega^2 = \frac{KM}{r^3} + \frac{F\cos\alpha}{m_1 r};$$

$$m_2\omega^2\left(r - l\cos\alpha\right)^3 = Km_2M - \frac{F}{\cos\alpha}\left(r - l\cos\alpha\right)^2;$$

$$m_2\omega^2 r^3\left(1 - \frac{l}{r}\cos\alpha\right)^3 = Km_2M - \frac{F}{\cos\alpha}r^2\left(1 - \frac{l}{r}\cos\alpha\right)^2;$$

$$m_2\omega^2 r^3\left(1 - 3\frac{l}{r}\cos\alpha\right) = Km_2M - \frac{F}{\cos\alpha}r^2\left(1 - 2\frac{l}{r}\cos\alpha\right);$$

$$m_2\left(\frac{KM}{r^3} + \frac{F\cos\alpha}{m_1 r}\right)r^3\left(1 - 3\frac{l}{r}\cos\alpha\right) = Km_2M - \frac{F}{\cos\alpha}r^2\left(1 - 2\frac{l}{r}\cos\alpha\right);$$

$$m_2\left(KM + \frac{F\cos\alpha}{m_1}r^2\right)\cdot\left(1 - 3\frac{l}{r}\cos\alpha\right) = Km_2M - \frac{F}{\cos\alpha}r^2\left(1 - 2\frac{l}{r}\cos\alpha\right);$$

$$F = \frac{3Km_2M\frac{l}{r}\cos\alpha}{\left(\frac{m_2}{m_1}\cos\alpha + \frac{1}{\cos\alpha}\right)r^2 - rl\left(\frac{m_2}{m_1}\cos^2\alpha + 1\right)};$$

$$\omega^2 = \frac{KM}{r^3} + \frac{F\cos\alpha}{m_1 r};$$

$$\omega^2 = \frac{KM}{r^3} + \frac{3K\frac{m_2}{m_1}M\frac{l}{r^2}\cos^2\alpha}{\left(\frac{m_2}{m_1}\cos\alpha + \frac{1}{\cos\alpha}\right)r^2 - rl\left(\frac{m_2}{m_1}\cos^2\alpha + 1\right)}.$$

Observation: If $m_2 << m_1$ and $l << r$, the results are already found :

$$\omega^2 = \frac{KM}{r^3} = \omega_0^2;$$

$$F = \frac{3Km_2M\frac{l}{r}\cos\alpha}{\left(\frac{1}{\cos\alpha}\right)r^2} = 3Km_2M\frac{l}{r^3}\cos^2\alpha.$$

Corresponding to the initial circulae orbit with radius r, the total mechanical energy of the system shuttle – Earth is:

$$E_0 = \frac{m_1 v_0^2}{2} - K\frac{m_1M}{r} = -K\frac{m_1M}{2r}.$$

After one complete rotation, the tangential component of the tension in the rod $\vec{F}_t$, acting on the shuttle will determine the change of the radius of the orbit i.e. $(r - \Delta r)$. The total energy of the system will be:

$$E = \frac{m_1 v^2}{2} - K\frac{m_1M}{r - \Delta r} = -K\frac{m_1M}{2(r - \Delta r)}.$$

$$\Delta E = E - E_0 = -K\frac{m_1M}{2(r - \Delta r)} + K\frac{m_1M}{2r} = -K\frac{m_1M}{2}\left(\frac{1}{r - \Delta r} - \frac{1}{r}\right);$$

$$\Delta E = -K\frac{m_1M}{2}\cdot\frac{r - r + \Delta r}{r(r - \Delta r)} = -K\frac{m_1M}{2}\cdot\frac{\Delta r}{r(r - \Delta r)};$$

$$r - \Delta r \approx r;$$

The variation of the mechanical energy after a complete rotation

$$\Delta E \approx -K\frac{m_1 M \cdot \Delta r}{2r^2} < 0,$$

Thus

$$\Delta E = L_t = -2\pi r \cdot F_t;\ F_t = F\sin\alpha;$$

$$-K\frac{m_1 M \cdot \Delta r}{2r^2} = -2\pi r \cdot F_t$$

The variation of the altitude due to the action of the satellite will be:

$$\Delta r = \frac{4\pi r^3 F_t}{Km_1 M} = \frac{4\pi r^3 F\sin\alpha}{Km_1 M},$$

The potentialenergy v

$$\Delta E_p = -K\frac{m_1 M}{r-\Delta r} - \left(-K\frac{m_1 M}{r}\right) = -Km_1 M\left(\frac{1}{r-\Delta r} - \frac{1}{r}\right);$$

$$\Delta E_p = -Km_1 M \cdot \frac{r - r + \Delta r}{r(r-\Delta r)} = -Km_1 M\frac{\Delta r}{r(r-\Delta r)};$$

$$r - \Delta r \approx r;$$

$$\Delta E_p \approx -K\frac{m_1 M \cdot \Delta r}{r^2} < 0,$$

Thus

$$\Delta E_p \approx -K\frac{m_1 M \cdot \Delta r}{2r^2} \cdot 2 = \Delta E \cdot 2 = -2\pi r F_t \cdot 2 = -4\pi r F_t;$$

$$\Delta E_c = \frac{m_1 \mathrm{v}^2}{2} - \frac{m_1 \mathrm{v}_0^2}{2};$$

$$\frac{m_1 \mathrm{v}_0^2}{r} = K\frac{m_1 M}{r^2};\ m_1 \mathrm{v}_0^2 = K\frac{m_1 M}{r};$$

$$\frac{m_1 \mathrm{v}^2}{r-\Delta r} = K\frac{m_1 M}{(r-\Delta r)^2};\ m_1 \mathrm{v}^2 = K\frac{m_1 M}{r-\Delta r};$$

$$\Delta E_c = K\frac{m_1 M}{2(r-\Delta r)} - K\frac{m_1 M}{2r} = K\frac{m_1 M}{2}\left(\frac{1}{r-\Delta r} - \frac{1}{r}\right);$$

$$\Delta E_c = K\frac{m_1 M}{2}\left(\frac{1}{r-\Delta r} - \frac{1}{r}\right) = K\frac{m_1 M}{2}\left(\frac{1}{r-\Delta r} - \frac{1}{r}\right);$$

$$\Delta E_c = K\frac{m_1 M}{2}\left(\frac{1}{r-\Delta r} - \frac{1}{r}\right) = K\frac{m_1 M}{2} \cdot \frac{r - r + \Delta r}{r(r-\Delta r)};$$

$$\Delta E_c = K\frac{m_1 M}{2} \cdot \frac{\Delta r}{r(r-\Delta r)};$$

$$r - \Delta r \approx r;$$

$$\Delta E_c \approx K\frac{m_1 M \cdot \Delta r}{2r^2} > 0,$$

reprezentând variația energiei cinetice a navei, după o rotație completă în jurul Pământului;

$$\Delta E_c > 0;\ v > v_0,$$

rezultat care constituie "paradoxul sateliților", adică, atunci când altitudinea orbitei navei scade, viteza navei crește;

$$\Delta E = \Delta E_c + \Delta E_p;$$

$$\Delta E_c = \Delta E - \Delta E_p = -2\pi r F_t - \left(-4\pi r F\right) = 2\pi r F_t > 0;$$

$$\Delta E_c = \frac{m_1 v^2}{2} - \frac{m_1 v_0^2}{2} = \frac{m_1}{2}\left(v^2 - v_0^2\right) = \frac{m_1}{2}\left(v - v_0\right)\left(v + v_0\right);$$

$$v = v_0 + \Delta v;\ v - v_0 = \Delta v;\ v + v_0 = 2v_0 + \Delta v \approx 2v_0;$$

$$\Delta E_c \approx \frac{m_1}{2} 2v_0 \cdot \Delta v = m_1 v_0 \Delta v = 2\pi r F_t;$$

$$\Delta v = \frac{2\pi r F_t}{m_1 v_0};\ v_0 = \sqrt{K\frac{M}{r}};$$

$$\Delta v = \frac{2\pi r F_t}{m_1} \cdot \sqrt{\frac{r}{KM}}.$$

d) Manevra tehnică propusă este reprezentată în figura alăturată: o forță reactivă, egală în modul și de sens contrar cu tensiunea din tijă:

$$\vec{F}_r = -\vec{F},$$

astfel încât forța rezultantă care acționează asupra navei să rămână forța de atracție gravitațională din partea Pământului:

$$\vec{F}_{\text{Nava}} = \vec{F}_1 + \vec{F} + \vec{F}_r = \vec{F}_1.$$

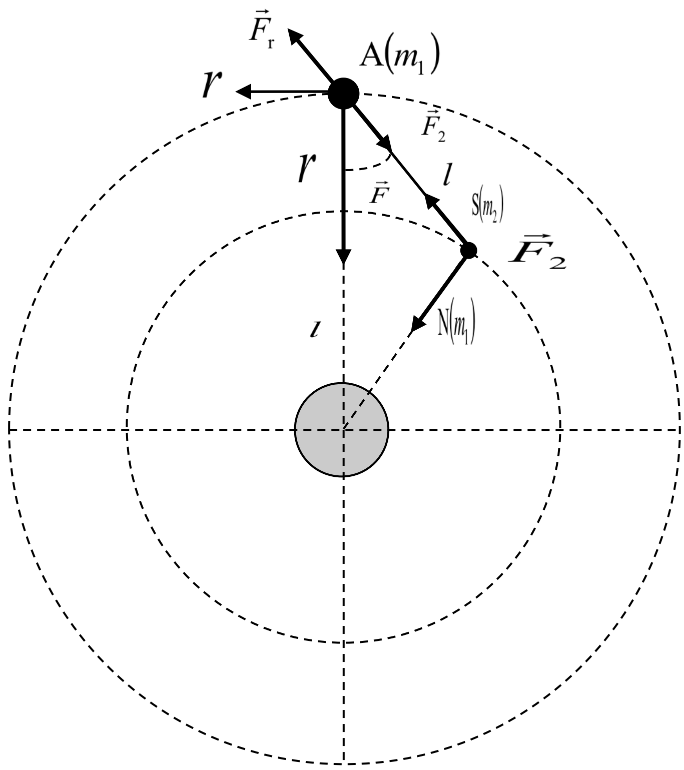


Fig.

18. Long problem 3. From Romania …. to Antipod! … a ballistic messenger

The 8th IOAA organizers plan to send to the **antipode** the official flag using a ballistic projectile. The projectile will be launched from Romania, and the rotation of the Erath will be neglected.

a) Calculate the coordinates of the target-point if the launch-point coordinates are: $\varphi_{\text{Romania}} = 44^\circ$ *North*; $\lambda_{\text{Romania}} = 30^\circ$ Est.

b) Determine the elements of the launching-speed vector, regardless to the center of the Earth, in order that the projectile should hit the target.

c) Calculate the velocity of the projectile when it hits the target.

d) Calculate the minimum velocity of the projectile.

e) Calculate the flying –time of the projectile, from the launch-moment to the impact one. You will use the value of the gravitational acceleration at Erath surface $g_0 = 9{,}81\,\text{ms}^{-2}$;the Earth radius $R = 6370\,\text{km}$.

f) Evaluate the possibility that the projectile to be seen with the naked eye in the moment that it passes at the maximum distance from the Earth. You will use the following values: The Moon albedo $\alpha_M = 0{,}12$; The Moon radius $R_M = 1738\,Km$; the Earth –Moon distance $r_{EL} = 384400$km; the apparent magnitude of the full moon $m_M = -12{,}7^{\mathrm{m}}$. You assume that the projectile is perfectly metallic sphere with radius $r_{projectile} = 400 x 10^{-3}\ m$ and with perfectly reflective surface.

Long problem 3. Marking scheme - From Romania …. to Antipod! … a ballistic messenger

a)	10
b)	10
c)	10
d)	10
e)	10
f)	10

a) The two places are represented in the figure.

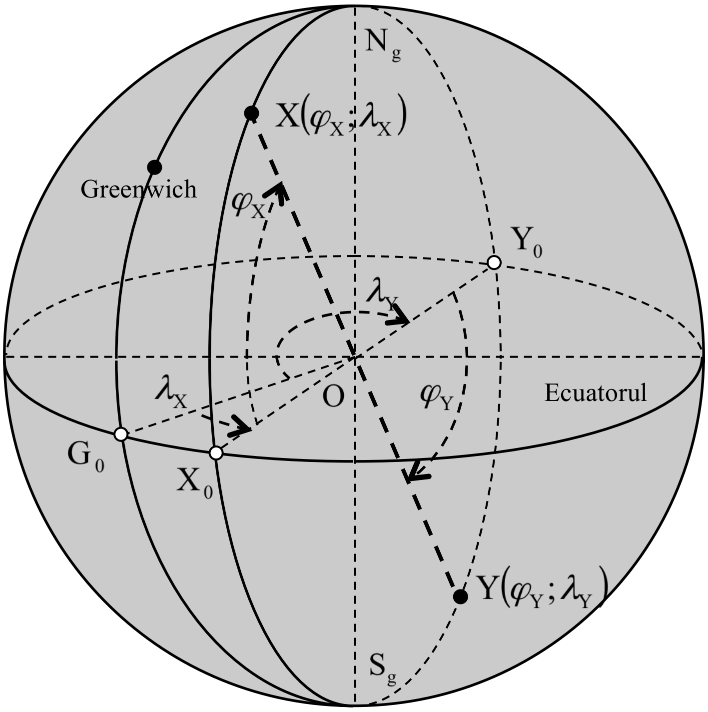


Fig.

$$\varphi_Y = \varphi_{Y,\text{Sud}} = \varphi_{X,\text{Nord}} = \varphi_X;$$

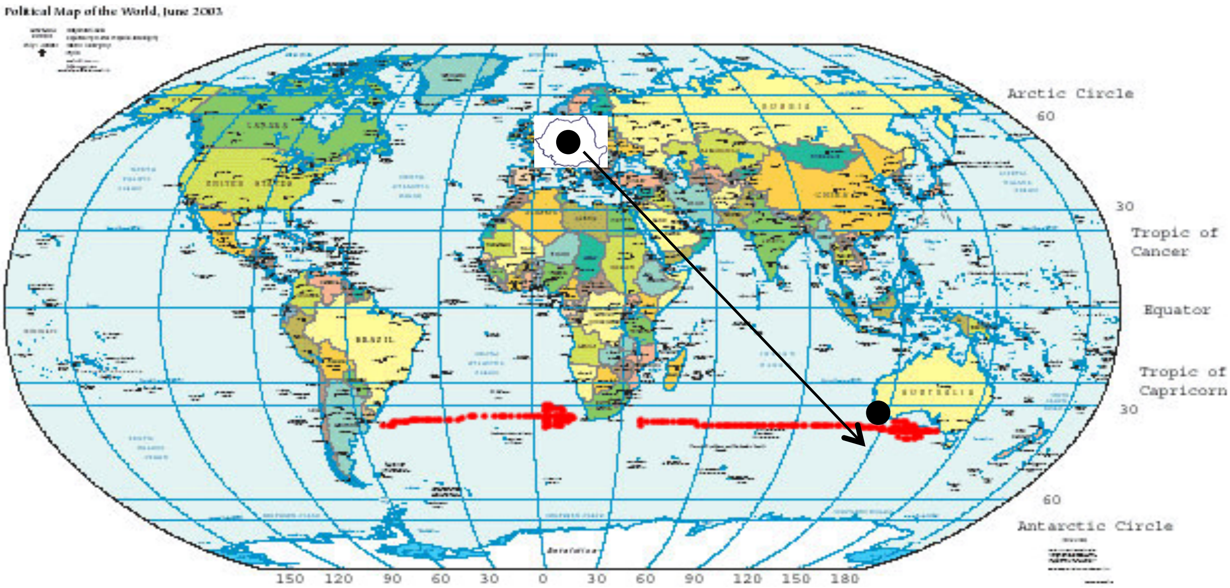
$$\lambda_{Y,\text{Vest}} + \lambda_{X,\text{Eest}} = 180^\circ;\ \lambda_Y + \lambda_X = 180^\circ.$$

$$\varphi_{\text{Romania}} = 43^\circ \text{ Nord};\ \lambda_{\text{Romania}} = 30^\circ \text{ Est},$$

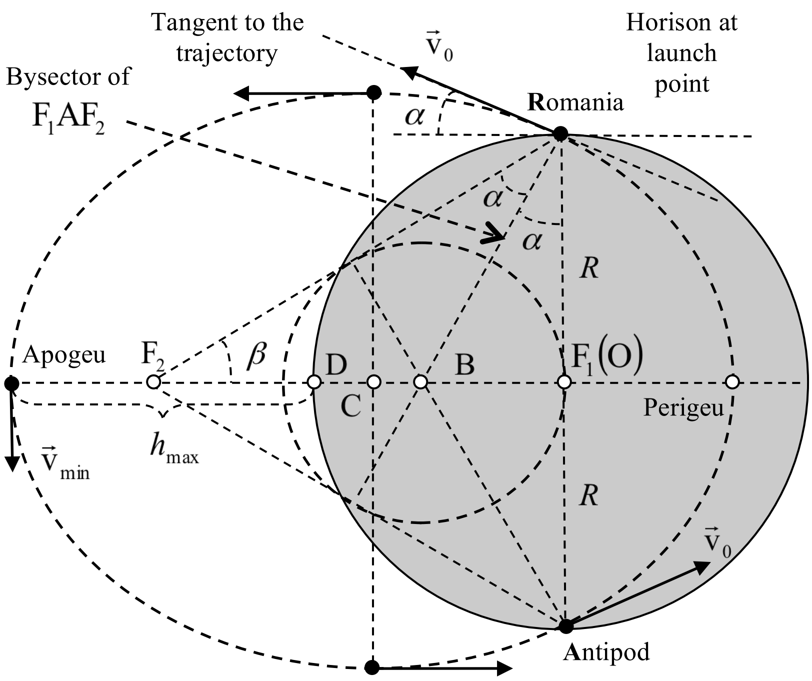
The landing point is

$$\varphi_{\text{Antipod}} = 43^\circ \text{ Sud};\ \lambda_{\text{Antipod}} = 150^\circ \text{ Vest},$$

Somewhere South EEst from Tasmania (South from Australia).



b) The schetch of the trajectory



In order to hit the point the trajectory of the missile has to be an elypse with the Earth center in the center of the Earth. *Se știe că*:

$$F_2B = 2 \cdot F_1B;\ F_1F_2 = 3 \cdot F_1B.$$

Rezultă:

$$\tan 2\alpha = \frac{F_1F_2}{R};\ F_1F_2 = R \cdot \tan 2\alpha;$$

$$\tan \alpha = \frac{F_1B}{R};\ F_1B = R \cdot \tan \alpha;$$

$$R \cdot \tan 2\alpha = F_1B = R \cdot \tan \alpha;$$

$$\tan 2\alpha = 3 \cdot \tan \alpha;$$

$$\frac{\sin 2\alpha}{\cos 2\alpha} = 3\frac{\sin \alpha}{\cos \alpha};$$

$$\frac{2\sin \alpha \cdot \cos \alpha}{\cos 2\alpha} = 3\frac{\sin \alpha}{\cos \alpha};$$

$$2\cos^2 \alpha = 3\cos 2\alpha;\ 2\cos^2 \alpha = 3\left(\cos^2 \alpha - \sin^2 \alpha\right);$$

$$3\sin^2 \alpha = \cos^2 \alpha;\ \tan^2 \alpha = \frac{1}{3};$$

$$\tan \alpha = \frac{\sqrt{3}}{3};\ \alpha = 30^\circ;\ 2\alpha = 60^\circ;\ \beta = 90^\circ - 2\alpha = 30^\circ;\ 2\beta = 60^\circ;$$

$$\Delta(RF_2A) \rightarrow \text{triunghi echilateral;}$$

$$RF_2 = AF_2 = RA = 2R;$$

$$RF_2 + RF_1 = 2a = 3R;$$

$$a = \frac{3}{2}R;$$

$$v_0 = \sqrt{KM\left(\frac{2}{r} - \frac{1}{a}\right)};\ r = R;\ g_0 = K\frac{M}{R^2};$$

$$v_0 = \sqrt{\frac{KM}{R^2} \cdot R^2\left(\frac{2}{R} - \frac{2}{3R}\right)} = 2\sqrt{\frac{g_0 R}{3}}.$$

c)

$$v_{\text{Antipod}} = v_0 .$$

d)

$$F_1F_2 = R \cdot \tan 2\alpha = 2c;\ c = \frac{R}{2} \cdot \tan 2\alpha = \frac{R}{2} \cdot \tan 60^\circ = \frac{\sqrt{3}}{2}R;$$

$$b = \sqrt{a^2 - c^2} = \sqrt{\frac{3}{2}}R;$$

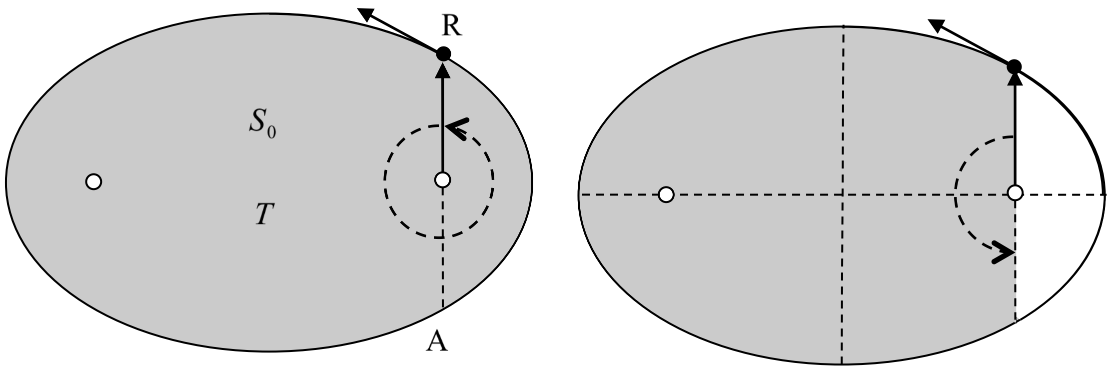
$$2a = 2r_{\min} + 2c;\ r_{\min} = a - c = \frac{1}{2}\left(3 - \sqrt{3}\right)R;$$

$$r_{\max} = 2a - r_{\min} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$v_{\min} = \sqrt{KM\left(\frac{2}{r_{\max}} - \frac{1}{a}\right)};\ r_{\max} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$v_{\min} = \sqrt{\frac{KM}{R^2} \cdot R^2\left(\frac{4}{\left(3 + \sqrt{3}\right)R} - \frac{2}{3R}\right)} = \sqrt{\frac{2g_0R}{3} \cdot \frac{3 - \sqrt{3}}{3 + \sqrt{3}}}.$$

e) Accordin to Kepplers laws:



$$\Omega = \frac{\mathrm{d}S}{\mathrm{d}t} = \text{constant};$$

$$\frac{S_0}{T} = \frac{2\frac{S_0}{4} + 2S_1}{\Delta t};\ \frac{S_0}{T} = \frac{\frac{S_0}{2} + 2S_1}{\Delta t};\ \frac{S_0}{T} = \frac{S_0 + 4S_1}{2 \cdot \Delta t};$$

$$S_0 = \pi ab;\ S_1 = \frac{ab}{2}\left[\sqrt{1-\frac{b^2}{a^2}}\cdot\frac{b}{a} + \arcsin\sqrt{1-\frac{b^2}{a^2}}\right];$$

$$\Delta t = \frac{S_0 + 4S_1}{2S_0}\cdot T = \left(\frac{1}{2} + 2\frac{S_1}{S_0}\right)\cdot T;$$

$$T = 2\pi\sqrt{\frac{a^3}{KM}};\ T = \frac{2\pi}{R}\sqrt{\frac{a^3}{g_0}};$$

$$\frac{2S_1}{S_0} = \frac{1}{\pi}\left(\frac{b}{a}\cdot\sqrt{1-\frac{b^2}{a^2}} + \arcsin\sqrt{1-\frac{b^2}{a^2}}\right);$$

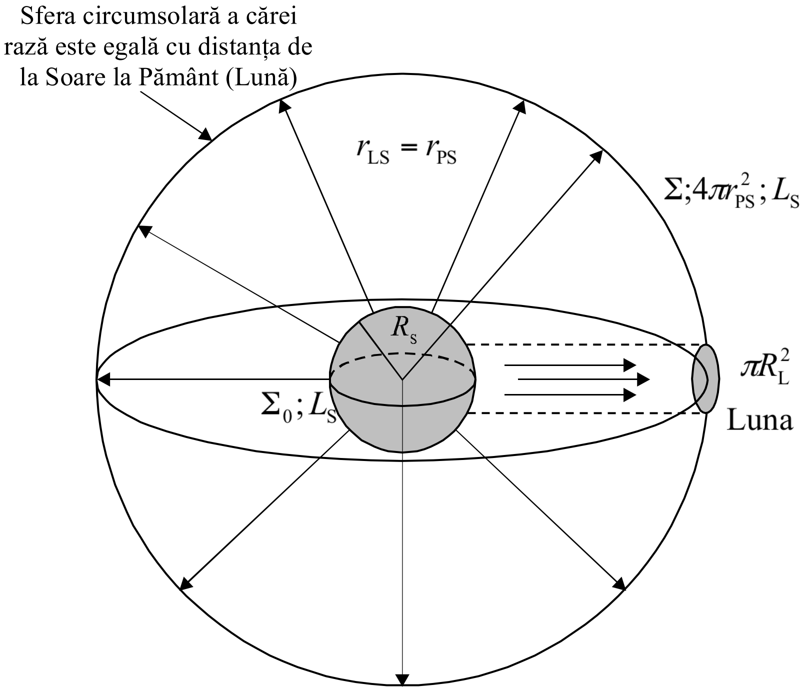
$$\sqrt{1-\frac{b^2}{a^2}} = e;\ \frac{2S_1}{S_0} = \frac{1}{\pi}\left(\frac{b}{a}\cdot e + \arcsin e\right);$$

$$\Delta t = \left(\frac{1}{2} + \frac{eb}{\pi a} + \frac{\arcsin e}{\pi}\right)\cdot T.$$

f) The integral luminosity of Sun:

$$L_S = \frac{E_{\text{emis,Soare}}}{t} = 3{,}86\cdot 10^{26}\ \text{W},$$

Dacă For a circumsolar surface Σ with radius r_{PS}, see picture bellow the solar radiation enegy passing through the surface in one second is L_{S}.



Density of solar flux

$$\phi_{\mathrm{Soare},r_{\mathrm{PS}}} = \frac{E_{\mathrm{emis,Soare}}}{St} = \frac{\frac{E_{\mathrm{emis,Soare}}}{t}}{S} = \frac{L_{\mathrm{S}}}{S} = \frac{L_{\mathrm{S}}}{4\pi r_{\mathrm{PS}}^2} = \mathrm{constant}.$$

$$F_{\mathrm{incident},FullMoon} = \phi_{\mathrm{S}un,r_{\mathrm{PS}}} \cdot \pi R_{\mathrm{L}}^2.$$

Dacă α_{L} este albedoul Lunii, rezultă:

$$\alpha_{\mathrm{L}} = \frac{F_{\mathrm{reflectat},FullMoon}}{F_{\mathrm{incident},FullMoon}},$$

unde $F_{\mathrm{reflectat,Luna\,Plina}}$ – fluxul energetic al radiațiilor reflectate de Luna Plină spre observatorul de pe Pământ;

$$F_{\mathrm{reflectat},FullMoon} = \alpha_{\mathrm{L}} \cdot F_{\mathrm{incident},FullMoon} = \alpha_{\mathrm{L}} \cdot \phi_{\mathrm{Soa,,re},r_{\mathrm{PS}}} \cdot \pi R_{\mathrm{L}}^2.$$

În consecință, densitatea fluxului energetic ajuns la observator, după reflexia pe suprafața Lunii, este:

$$\phi_{moon,\ \mathrm{observator}} = \frac{F_{\mathrm{reflectat},\ FullMoon}}{2\pi r_{\mathrm{PL}}^2} = \alpha_{\mathrm{L}} \cdot \phi_{\mathrm{Soare},r_{\mathrm{PS}}} \cdot \frac{\pi R_{\mathrm{L}}^2}{2\pi r_{\mathrm{PL}}^2}.$$

Symilarly

$$\phi_{\mathrm{proiectil,\ observator}} = \frac{F_{\mathrm{reflectat,\ proiectil}}}{4\pi r_{\mathrm{D,proiectil}}^2} = \alpha_{\mathrm{proiectil}} \cdot \phi_{\mathrm{Soare},r_{\mathrm{PS}}} \cdot \frac{\pi R_{\mathrm{proiectil}}^2}{4\pi r_{\mathrm{D,proiectil}}^2}.$$

În expresia anterioară s-a avut în vedere faptul că densitatea fluxului energetic al proiectilului la observator rezultă din distribuirea prin suprafața sferei cu raza $r_{\mathrm{P,proiectil}}$.

Utilizând formula lui Pogson, vom compara magnitudinea aparentă vizuală a Lunii Pline cu magnitudinea aparentă vizuală a proiectilului balistic:

$$\log \frac{\phi_{\mathrm{Luna,\,observator}}}{\phi_{\mathrm{proiectil,\,observator}}} = -0{,}4\left(m_{\mathrm{Luna\,Plina}} - m_{\mathrm{proiectil}}\right);$$

$$\log \frac{\phi_{\mathrm{Luna,\,observator}}}{\phi_{\mathrm{proiectil,observator}}} = \log \frac{\alpha_{\mathrm{L}} \cdot \phi_{\mathrm{Soare},r_{\mathrm{PS}}} \cdot \dfrac{\pi R_{\mathrm{L}}^2}{2\pi r_{\mathrm{PL}}^2}}{\alpha_{\mathrm{proiectil}} \cdot \phi_{\mathrm{Soare},r_{\mathrm{PS}}} \cdot \dfrac{\pi R_{\mathrm{proiectil}}^2}{4\pi r_{\mathrm{D,proiectil}}^2}} = \log \frac{\alpha_{\mathrm{L}} \cdot \dfrac{R_{\mathrm{L}}^2}{r_{\mathrm{PL}}^2}}{\alpha_{\mathrm{proiectil}} \cdot \dfrac{R_{\mathrm{proiectil}}^2}{2r_{\mathrm{D,proiectil}}^2}};$$

$$\log \frac{\alpha_{\mathrm{L}} \cdot \dfrac{R_{\mathrm{L}}^2}{r_{\mathrm{PL}}^2}}{\alpha_{\mathrm{proiectil}} \cdot \dfrac{R_{\mathrm{proiectil}}^2}{2r_{\mathrm{D,proiectil}}^2}} = -0{,}4\left(m_{\mathrm{L}} - m_{\mathrm{proiectil}}\right);$$

$$\log \frac{\alpha_{\mathrm{L}}}{\alpha_{\mathrm{proiectil}}} \cdot \left(\frac{R_{\mathrm{L}}}{R_{\mathrm{proiectil}}}\right)^2 \cdot 2 \cdot \left(\frac{r_{\mathrm{D,proiectil}}}{r_{\mathrm{PL}}}\right)^2 = -0{,}4\left(m_{\mathrm{L}} - m_{\mathrm{proiectil}}\right);$$

$$\alpha_{\mathrm{L}} = 0{,}12;\ \alpha_{\mathrm{proiectil}} = 1;$$

$$R_{\mathrm{L}} = 1738\,\mathrm{km};\ R_{\mathrm{proiectil}} = 400\,\mathrm{mm};$$

$$r_{\mathrm{D,proiectil}} = r_{\mathrm{max,observator-proiectil}} = h_{\max} = r_{\max} - R;\ r_{\max} = \frac{1}{2}\left(3+\sqrt{3}\right)R;$$

$$h_{\max} = \frac{1}{2}\left(3+\sqrt{3}\right)R - R = \frac{1}{2}\left(1+\sqrt{3}\right)R \approx 8700\,\mathrm{km};$$

$$r_{PL} = r_{observator,Luna} = 384400\,\text{km};\ m_L = -12{,}7^m;$$

$$\log\frac{\alpha_L}{\alpha_{proiectil}} + 2\log\frac{R_L}{R_{proiectil}} + \log 2 + 2\log\frac{r_{D-proiectil}}{r_{PL}} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$\log(0{,}12) + 2\log\frac{1738000\ \text{m}}{0{,}400\ \text{m}} + \log 2 + 2\log\frac{8700\ \text{km}}{384400\ \text{km}} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$\log(0{,}12) + 2\log\frac{1738000}{0{,}400} + \log 2 + 2\log\frac{8700}{384400} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$-0{,}920818754 + 13{,}27597956 + 0{,}301029995 - 3{,}290528253 = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$23{,}4^m = 12{,}7^m + m_{proiectil};$$

$$m_{proiectil} = 10{,}7^m;$$

$$m_{max} \approx 6^m;\ m_{proiectil} > m_{max},$$

The projectile wasn't seen when it was at its apogee