8th International Olympiad on Astronomy and Astrophysics
Suceava - Gura Humorului - August 2014

ALL THE CORRECT SOLUTIONS WHICH ARE DIFFERENT FROM THE AUTHOR'S SOLUTION WILL BE EVALUATED AND MARKED ACORDINGLY TO THE MARKING SCHEME.

1. Eagles on the Caraiman Cross !

The tallest cross built on a mountain peak is located on a plateau situated on the top of the peak called Caraiman in Romania at altitude $H = 2300\,m$ from the sea level. Its height, including the base-support is $h = 39.3\,m$. The horizontal arms of the cross are oriented on the N-S direction. The latitude at which the Cross is located is $\varphi = 45^\circ$.

A. On the evening of 21st of March 2014, the vernal equinox day, two eagles stop from their flight, first near the monument, and the second, on the top of the Cross as seen in figure 1. The two eagles are in the same

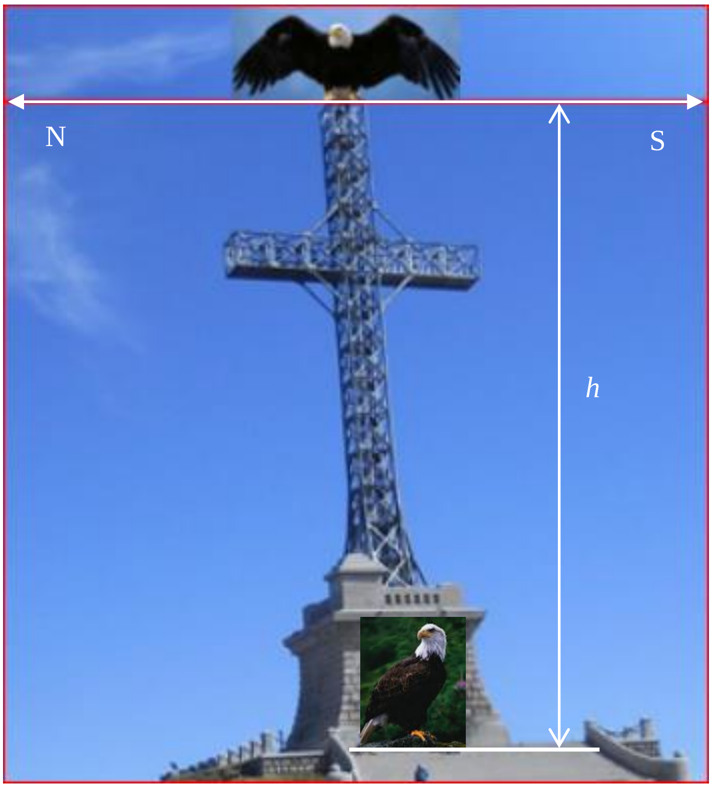


Figure 1

vertical direction. The sky was very clear, so the eagles could see the horizon and observe the Sun set. Each eagle began to fly right at the moment it observed that the Sun disappeared completely.

At the same time, an astronomer is located at the sea-level, at the base of the Bucegi Mountains. Assume that he is in the same vertical direction as the two eagles.

Assuming the atmospheric refraction to be negligible, solve the following questions:

1) *Calculate* the duration of the sunset, measured by the astronomer.

2) *Calculate* the durations of sunsets measured by each of the two eagles and indicate which of the eagles leaves the Cross first. What is the time interval between the moments of the flights of the two eagles.

The following information is necessary:

The duration of the sunset measurement starts when the solar disc is tangent to the horizon line and stops when the solar disc completely disappears.

The Earth's rotational period is $T_E = 24\,\text{h}$, the radius of the Sun $R_S = 6.96 \cdot 10^5\ \text{km}$, Earth – Sun distance $d_{ES} = 14.96 \cdot 10^7\ \text{km}$, the latitude of the Heroes Cross is $\varphi = 45^\circ$. $R_E = 6370\,\text{km}$

B) At a certain moment the next day, 22nd March 2014, the two eagles come back to the Heroes Cross. One of the eagles lands on the top of the vertical pillar of the Cross and the other one land on the horizontal plateau, just at the tip of the shadow of the vertical pillar of the Cross, at that moment of the day when the shadow length is minimum.

1) Calculate the distance between the two eagles and the second eagle's distance from the cross.

2) Calculate the length of the horizontal arms of the Cross l_b, if the shadow ~~on the plateau~~ of one of the arm of the cross at this moment has the length $u_b = 7\ \mathrm{m}$

C) At midnight, the astronomer visits the cross and, from its top, he identifies a bright star at the limit of the circumpolarity. He named this star "Eagles Star". Knowing that due to the atmospheric refraction the horizon lowering is $\xi = 34'$, calculate:

1) The "Eagles star" declination;

2) The "Eagles star" maximum height above the horizon.

Long problem 1. Marking scheme - Eagles on the Caraiman Cross

1)	**10**
2)	**10**
B1)	10
B2)	10
C1)	5
C2)	5

1 The following notations are used: D_S the diameter of the Sun, d_{ES} Earth-Sun distance, θ angular diameter of the Sun as seen from the Earth:

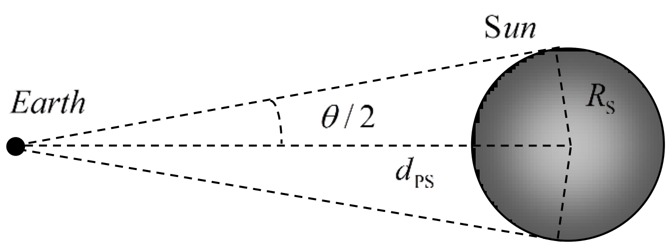


Fig. 1

According the fig. 1 the angular diameter of the Sun can be calculated as follows

$$\sin\frac{\theta}{2} = \frac{R_S}{d_{PS}} \approx \frac{\theta}{2};$$

$$\theta = \frac{2R_S}{d_{PS}} = \frac{D_S}{d_{PS}} = \frac{2 \cdot 6{,}96 \cdot 10^5 \text{ km}}{15 \cdot 10^7 \text{ km}} = 0{,}00928 \,\text{rad.}$$

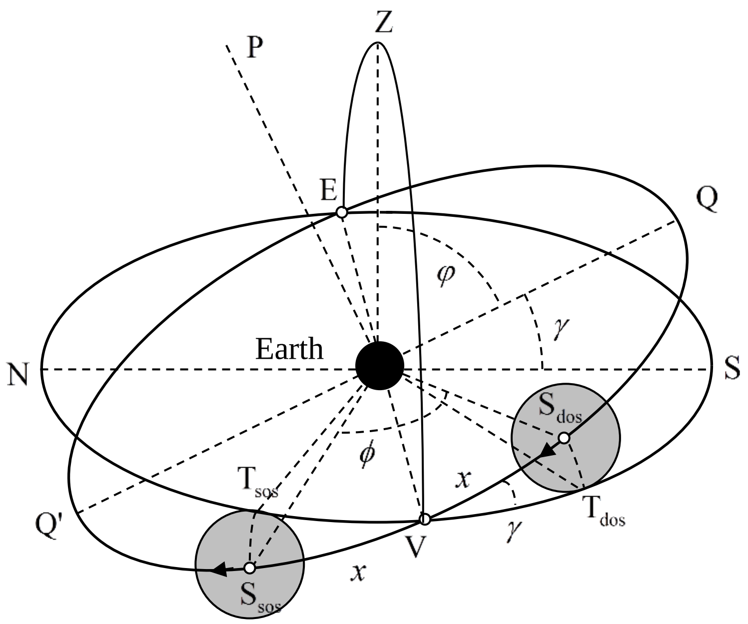
The figure 2 presents the Sun's evolution during sunset as seen by the astronomer. In an equinox day the Sun moves retrograde along the celestial equator. There are marked the following 3 positions of the Sun:

T_{dos} - The solar disc is tangent to the equatorial plane above the standard horizon – the start of the sunset;

S_{dos} - The center of the solar disc on the celestial equator in the moment of the sunset starts;

T_{sos} - The solar disc is tangent to the equatorial plane bellow the standard horizon – the end of the sunset

S_{sos} - The center of the solar disc on the celestial equator in the moment of the sunset ends;



The duration of the sunset is τ. During this time the center of the Sun moves along the equator from S_{dos} to S_{sos}. The vector-radius of the Sun rotates in equatorial plane with angle ϕ and in vertical plane with angle θ. i.e. the angular diameter of the Sun as seen from the Earth.

Considering that the Sun travels the distance $2x$ along the equatorial path with merely constant i.e. during time τ and that the spherical right triangle $S_{dos}T_{dos}V$ can be considered a plane one the following relations can be written:

$$\sin\gamma = \frac{R_S}{x};\; x = \frac{R_S}{\sin\gamma};\;\; 2x = \frac{2R_S}{\sin\gamma} = \frac{D_S}{\sin\gamma};$$

$$\tau = \frac{2x}{\text{v}} = \frac{2x}{\omega \cdot d_{PS}} = \frac{\dfrac{D_S}{\sin\gamma}}{\dfrac{2\pi}{T_P}\cdot d_{PS}} = \frac{\dfrac{D_S}{d_{PS}}}{\dfrac{2\pi}{T_P}\cdot \sin\gamma} = \frac{\theta \cdot T_P}{2\pi \cdot \sin\gamma};$$

$$\sin\gamma = \sin\left(90^\circ - \varphi\right) = \cos\varphi;$$

$$\tau = \frac{\theta \cdot T_P}{2\pi \cdot \cos\varphi};$$

$$\tau = \frac{0{,}00928\,\text{rad} \cdot 24\,\text{h}}{2 \cdot 3{,}14\,\text{rad} \cdot \cos(45^\circ 21')} = \frac{0{,}22272 \cdot 60}{2 \cdot 3{,}14 \cdot 0{,}707}\,\text{min} \approx 3\,\text{min}.$$

2) If the atmospheric refraction is negligible the eagle on the top of the cross V_1 on figure 3 is on the same latitude (φ), as the astronomer but at the altitude H. Thus from the point of view of the V_1 the horizon line is below the standard horizon line with an angle $\Delta\alpha_1$,

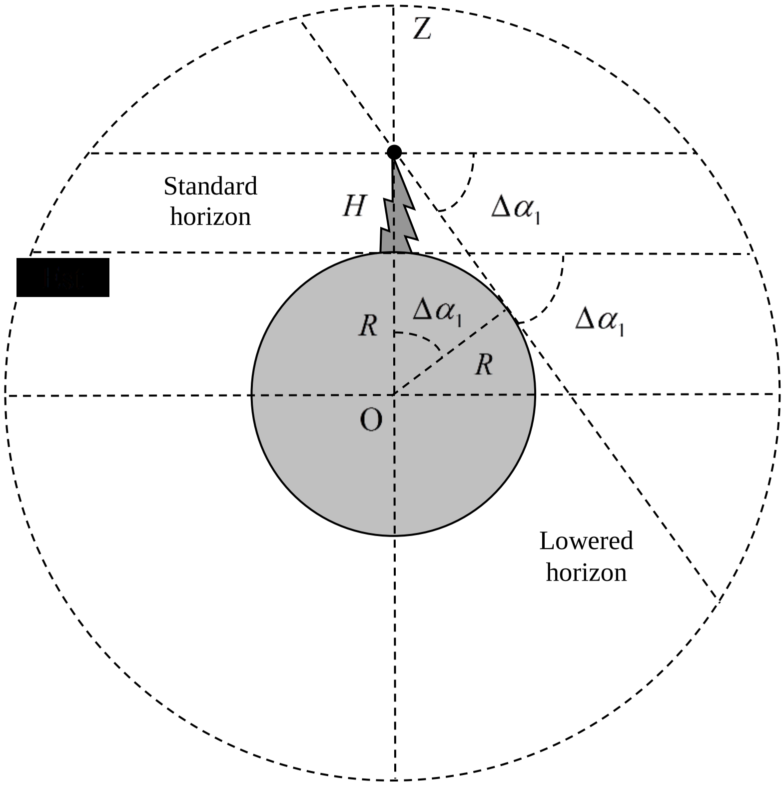


Fig. 3

$$\cos\Delta\alpha_1 = \frac{R}{R+H};$$

$$\sin\Delta\alpha_1 = \frac{\sqrt{(R+H)^2 - R^2}}{R+H} = \frac{\sqrt{2RH + H^2}}{R+H} \approx \frac{\sqrt{2RH}}{R} = \sqrt{\frac{2H}{R}} \approx \Delta\alpha_1;$$

$$\Delta\alpha_1 = \sqrt{\frac{2 \cdot 2{,}3\,\text{km}}{6380\,\text{km}}} \approx 0{,}02685\,\text{rad} \approx 1{,}54^\circ.$$

For the observer V_1 the Sun will go below the lowered horizon after moving down under the standard horizon with angle $\Delta\alpha_1$ and moving along the equator with an angle $\Delta\beta_1$, as seen in fig. 4

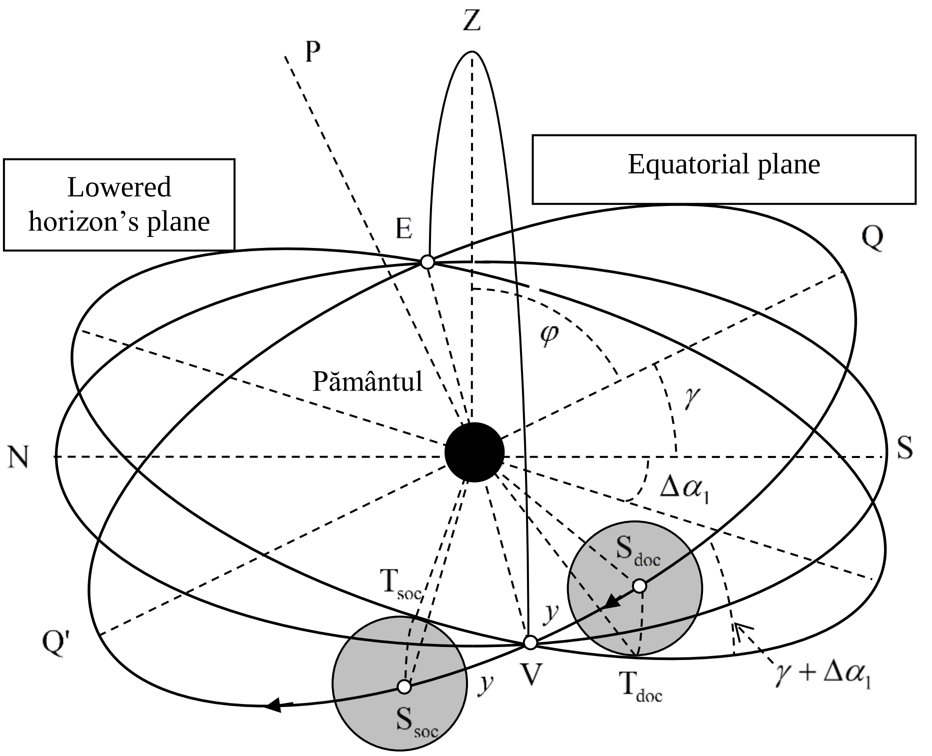


Fig.5

$$\sin(\gamma+\Delta\alpha_1)=\frac{R_S}{y};\ y=\frac{R_S}{\sin(\gamma+\Delta\alpha_1)};\ 2y=\frac{2R_S}{\sin(\gamma+\Delta\alpha_1)}=\frac{D_S}{\sin(\gamma+\Delta\alpha_1)};$$

$$\tau_1=\frac{2y}{\mathrm{v}}=\frac{2y}{\omega\cdot d_{PS}}=\frac{\dfrac{D_S}{\sin(\gamma+\Delta\alpha_1)}}{\dfrac{2\pi}{T_P}\cdot d_{PS}}=\frac{\dfrac{D_S}{d_{PS}}}{\dfrac{2\pi}{T_P}\cdot\sin(\gamma+\Delta\alpha_1)}=\frac{\theta\cdot T_P}{2\pi\cdot\sin(\gamma+\Delta\alpha_1)};$$

$$\gamma=90^\circ-\varphi;$$

$$\sin(\gamma+\Delta\alpha_1)=\sin(90^\circ-\varphi+\Delta\alpha_1)=\sin[90^\circ-(\varphi-\Delta\alpha_1)]=\cos(\varphi-\Delta\alpha_1);$$

$$\tau_1=\frac{\theta\cdot T_P}{2\pi\cdot\cos(\varphi-\Delta\alpha_1)};$$

$$\tau_1=\frac{0{,}00928\,\mathrm{rad}\cdot 24\,\mathrm{h}}{2\cdot 3{,}14\,\mathrm{rad}\cdot\cos(45^\circ-1{,}54^\circ)}=\frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}725}\,\mathrm{min}\approx 2{,}9350\,\mathrm{min},$$

Which represents the total duration of sunset for V_1 at altitude H.

Similarly for eagle V_2 at the same latitude (φ), but altitude $H+h$ (the top of the cross), the lowering effect on the horizon is measured by angle $\Delta\alpha_2$ thus

$$\cos\Delta\alpha_2=\frac{R}{R+H+h};$$

$$\sin\Delta\alpha_2=\frac{\sqrt{(R+H+h)^2-R^2}}{R+H+h}=\frac{\sqrt{2R(H+h)+(H+h)^2}}{R+H+h}\approx\frac{\sqrt{2R(H+h)}}{R}=\sqrt{\frac{2(H+h)}{R}}\approx\Delta\alpha_2;$$

$$\Delta\alpha_2=\sqrt{\frac{2\cdot(2{,}3+0{,}0393)\,\text{km}}{6380\,\text{km}}}\approx 0{,}02707\,\text{rad}\approx 1{,}55^\circ;$$

$$\frac{\sin(90^\circ-\varphi)}{\sin\Delta\alpha_2}=\frac{\sin 90^\circ}{\sin\Delta\beta_2};$$

$$\frac{\cos\varphi}{\Delta\alpha_2}=\frac{1}{\Delta\beta_2};\quad \Delta\beta_2=\frac{\Delta\alpha_2}{\cos\varphi};$$

$$\Delta\beta_2=\omega\cdot\Delta\tau_2=\frac{2\pi}{T_\text{P}}\cdot\Delta\tau_2;$$

$$\Delta\tau_2=\frac{\Delta\alpha_2}{\cos\varphi}\cdot\frac{T_\text{P}}{2\pi}=\frac{1{,}55^\circ}{\cos(45^\circ 21')}\cdot\frac{24\cdot 60\,\text{min}}{360^\circ}\approx 8{,}77\,\text{minute},$$

Which represents the delay of the start moment of the sunset for V_2 due to the altitude $H+h$.

Similar the total duration of the sunset for the observer V_2 :

$$\tau_2=\frac{\theta\cdot T_\text{P}}{2\pi\cdot\cos(\varphi-\Delta\alpha_2)};$$

$$\tau_2=\frac{0{,}00928\,\text{rad}\cdot 24\,\text{h}}{2\cdot 3{,}14\,\text{rad}\cdot\cos(45^\circ-1{,}55^\circ)}=\frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}726}\,\text{min}\approx 2{,}9309\,\text{min},$$

We may note the following:

- the horizon- lowering $\Delta\alpha$ is increased by the increase of the altitude;

$$\left(H<H+h\rightarrow\Delta\alpha_1<\Delta\alpha_2;\,H\uparrow\rightarrow\Delta\alpha\uparrow\right).$$

- the delay of the moment of sunset start is increased by the increase of the altitude:

$$\left(H<H+h\rightarrow\Delta\tau_1<\Delta\tau_2;\,H\uparrow\rightarrow\Delta\tau\uparrow\right).$$

- the total duration of sunset is reduced by the increase of the altitude:

$$\left(0<H<H+h\rightarrow\tau>\tau_1>\tau_2;\,H\uparrow\rightarrow\tau\downarrow\right).$$

Conclusions:

If we consider t_0 the moment of sunset star for the astronomer

- for V_1 the sunset starts at $t_0+8{,}71\,\text{min}$ and ends at $t_0+8{,}71\,\text{min}+2{,}9350\,\text{min}=t_0+11{,}6450\,\text{min}$
- for V_2 the sunset starts at $t_0+8{,}77\,\text{min}$ and ends at $t_0+8{,}77\,\text{min}+2{,}9309\,\text{min}=t_0+11{,}7009\,\text{min}$
- Thus eagle from the plateau leaves first the cross;
- The time between the leaving moments is:

$$\Delta t=t_0+11{,}7009\,\text{min}-t_0-11{,}6450\,\text{min}=0{,}0559\,\text{min}=3{,}354\,\text{s}.$$

b)

As seen in fig. 6 the length of the cross on the plateau will be minimum when the Sun passes the local meridian, i.e. the height of the Sun above the horizon will be maximum:

$\left(h_{\max}=\gamma=90^\circ-\varphi\right).$

Thus the shadow of the horizontal arms of the cross is superposed on the shadow of the vertical pillow.

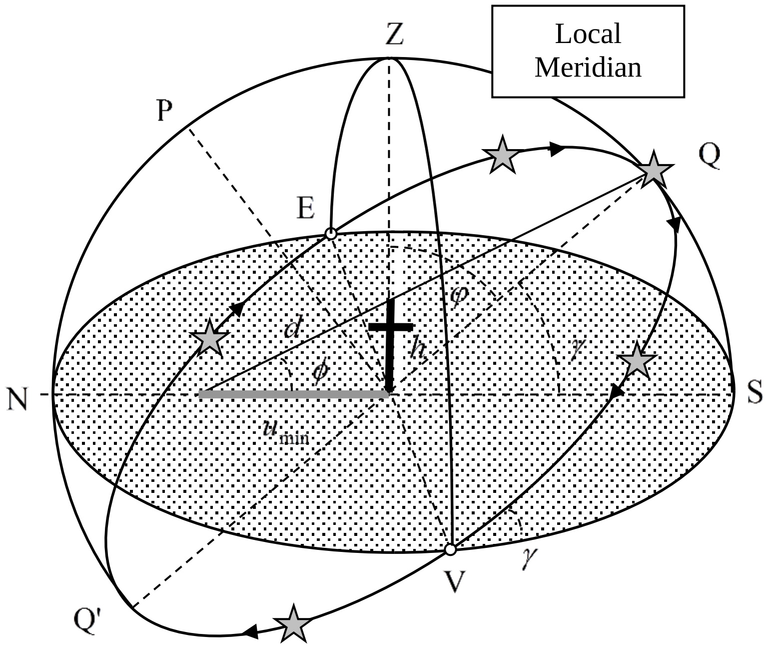


Fig. 6

In this conditions :

$$\sin\phi = \frac{h}{d};\ \phi \approx \gamma = 90^\circ - \varphi;$$

$$d = \frac{h}{\sin\phi} \approx \frac{h}{\sin\gamma} = \frac{h}{\sin(90^\circ - \varphi)} = \frac{h}{\cos\varphi} = \frac{39{,}3\,\mathrm{m}}{\cos 45^\circ} = \frac{39{,}3}{0{,}707}\,\mathrm{m} \approx 55{,}58\,\mathrm{m};$$

The distance between the two eagles is

$$u_{\min} = h\cdot\cot\phi \approx h\cdot\cot\varphi = h\cdot\cot(90^\circ - \varphi) = h\cdot\tan\varphi = 39{,}3\,\mathrm{m}.$$

2) In the above mentioned conditions the shadow of the arm oriented toward South is on the vertical pillow of the cross, as seen in fig. 7:

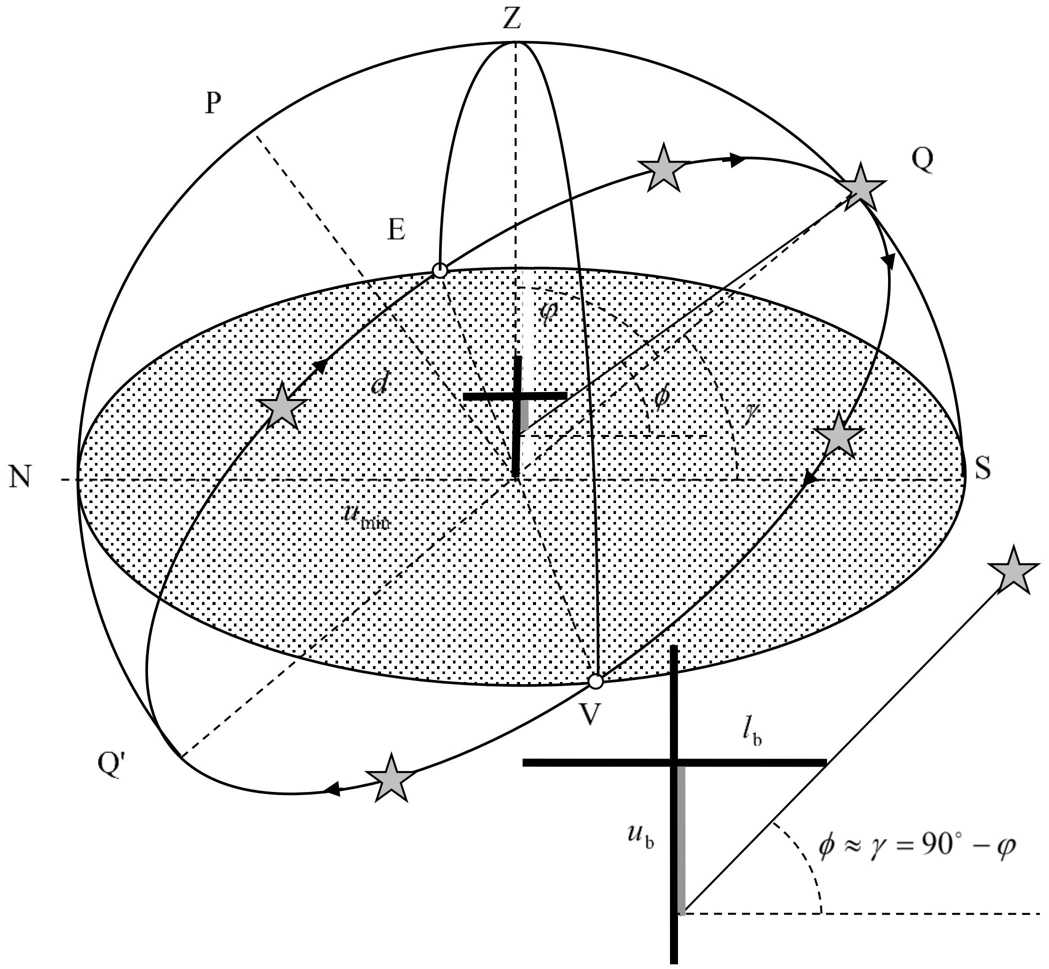


Fig. 7

$$\tan\varphi = \frac{l_b}{u_b};\ l_b = u_b \cdot \tan\varphi = 7\ \text{m} \cdot \tan 45^\circ = 7\ \text{m},$$

Which represents the length of the cross arm.

C)

1) For an observer situated in the center O of the celestial topocentric sphere, at latitude φ, at sea level, al the stars are circumpolar ones see fig. 8. Their diurnal parallels, parallel with the equatorial parallel, are above the real local horizon $\left(N_0S_0\right)$. The star σ_0 is at the circumpolar limit because its parallel touches the real local horizon in point N_0 but still remains above it. Thus σ_0 is a marginal circumpolar star. Without taking into account the atmospheric refraction:

From the isosceles triangle $O\sigma_0 N_0$ results the σ_0 declination:

$$\delta_{0,\min} + 90^\circ + \left(\varphi - \delta_{\min}\right) = 180^\circ;$$

$$\delta_{0,\min} = 90^\circ - \varphi.$$

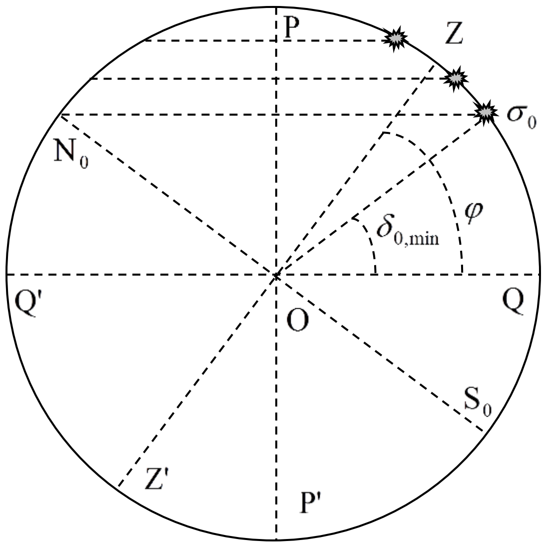


Fig. 8

By taking into account the atmospheric refraction the horizon line changes to line N'S', with angle $\theta' \approx 34'$, as seen in fig. 9. The star σ_0 remains a circumpolar one but above the limit. In this conditions the star σ' gather the limit conditions its declination been $\delta'_{\min} < \delta_{\min}$. In this conditions for an observer situated in the center of the topocentric celestial sphere, at latitude φ and altitude zero, the star σ', with declination $\delta'_{\min} < \delta_{0\min}$ is on the limit of the circumpolarity.

From the isosceles triangle $N'O\sigma'$ the declination σ' will be :

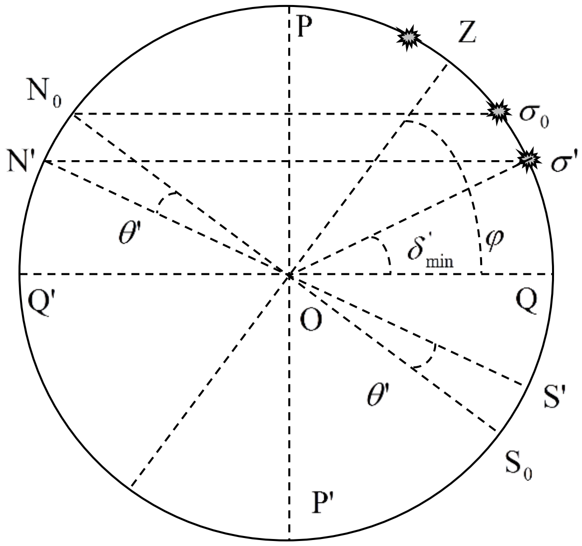


Fig. 9

$$2\delta'_{\min} + (\varphi - \delta'_{\min}) + (90^\circ + \theta') = 180^\circ;$$
$$\delta'_{\min} = 90^\circ - \varphi - \theta'.$$

For the observer at latitude φ, but at height h, taking into account the effect of lowering of the horizon the star σ'' will meet the problem requirements see figure 10. The new horizon is $N''S''$ and declination of σ'' is $\delta''_{\min} < \delta_{0\min}$. Star σ_0 remains a circumpolar one but above the limit.

From the isosceles triangle $N''O\sigma''$ the declination of star σ'' will be:

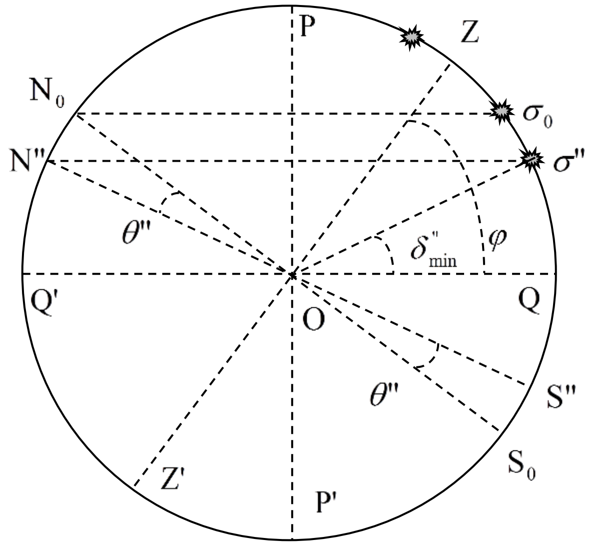


Fig. 10

$$2\delta''_{\min} + \left(\varphi - \delta''_{\min}\right) + \left(90^\circ + \theta''\right) = 180^\circ;$$
$$\delta''_{\min} = 90^\circ - \varphi - \theta''.$$

By taking into account the refraction effect and the altitude effect, from triangle $NO\sigma$ in figure 11, the declination will be

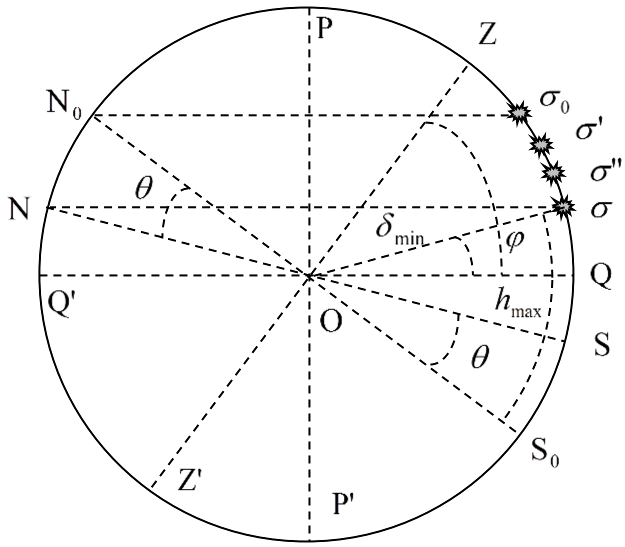
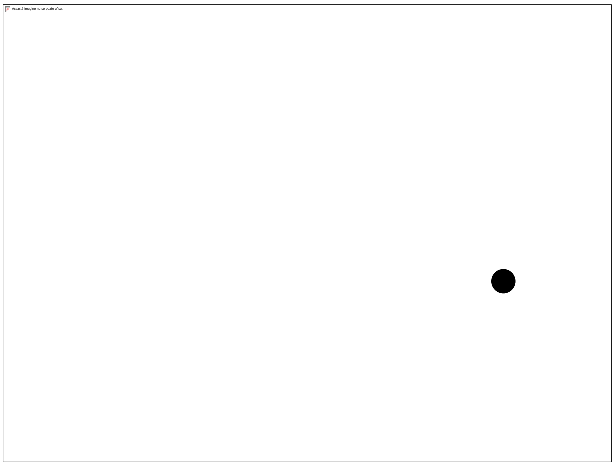


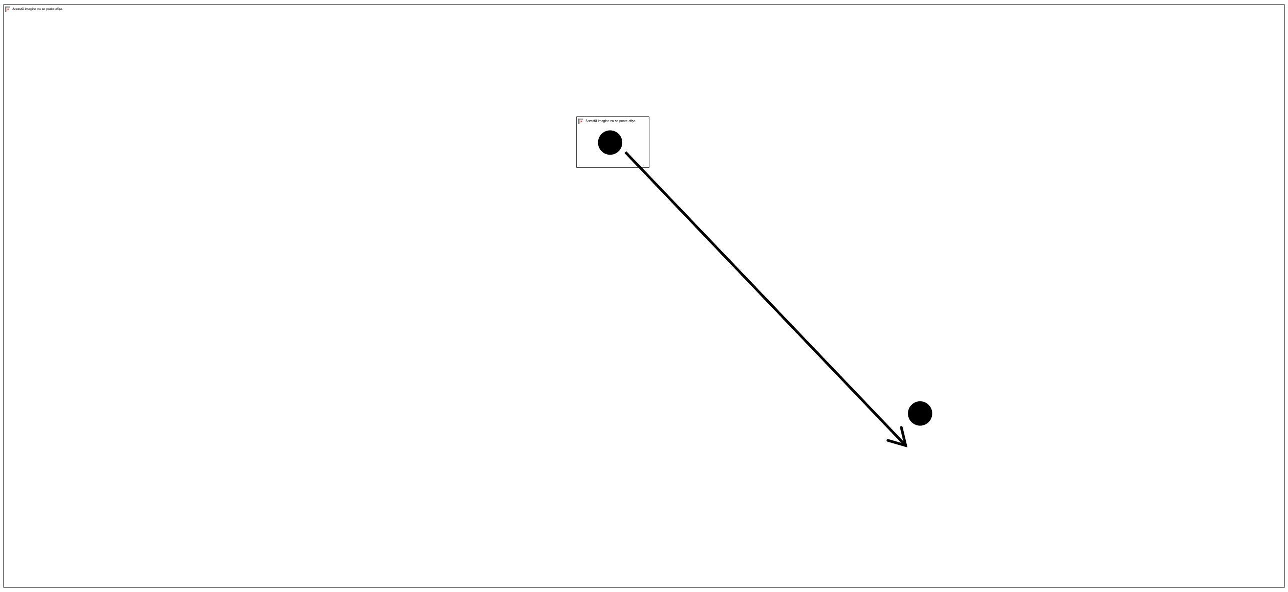
Fig. 11

$$2\delta_{\min} + \left(\varphi - \delta_{\min}\right) + \left(90^\circ + \theta\right) = 180^\circ;$$
$$\theta = \theta' + \theta''; \ \theta' = \xi = 34': \ \theta'' = \Delta\alpha_2 = 1{,}55^\circ;$$

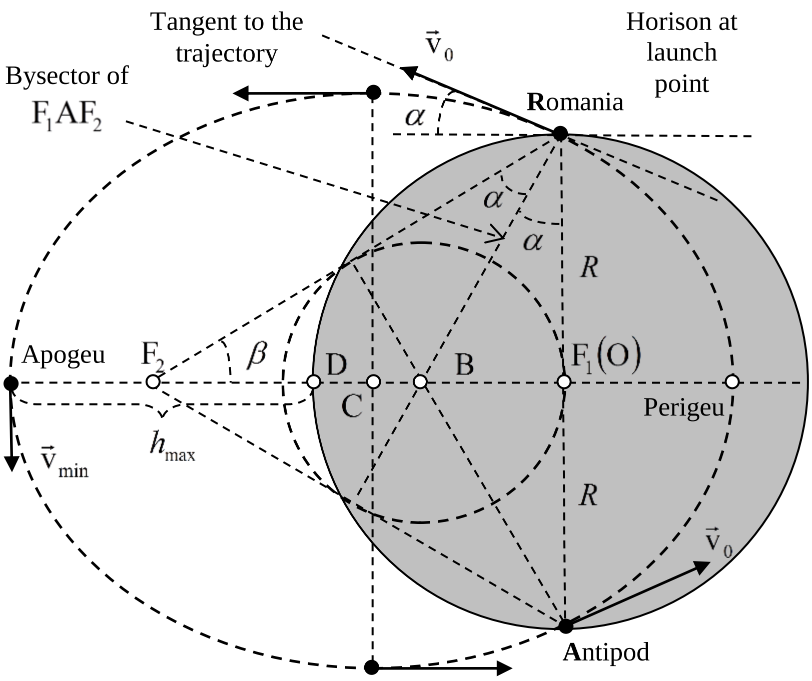
2) The maximum height above the horizon will be

2





b) The schetch of the trajectory



In order to hit the point the trajectory of the missile has to be an elypse with the Earth center in the center of the Earth. *Se știe că*:

$$F_2B = 2 \cdot F_1B;\ F_1F_2 = 3 \cdot F_1B.$$

Rezultă:

$$\tan 2\alpha = \frac{F_1F_2}{R};\ F_1F_2 = R \cdot \tan 2\alpha;$$

$$\tan\alpha = \frac{F_1B}{R};\ \ F_1B = R \cdot \tan\alpha;$$

$$R \cdot \tan 2\alpha = F_1B = R \cdot \tan\alpha;$$

$$\tan 2\alpha = 3 \cdot \tan\alpha;$$

$$\frac{\sin 2\alpha}{\cos 2\alpha} = 3\frac{\sin\alpha}{\cos\alpha};$$

$$\frac{2\sin\alpha \cdot \cos\alpha}{\cos 2\alpha} = 3\frac{\sin\alpha}{\cos\alpha};$$

$$2\cos^2\alpha = 3\cos 2\alpha;\ \ 2\cos^2\alpha = 3\left(\cos^2\alpha - \sin^2\alpha\right);$$

$$3\sin^2\alpha = \cos^2\alpha;\ \tan^2\alpha = \frac{1}{3};$$

$$\tan\alpha = \frac{\sqrt{3}}{3};\ \ \alpha = 30^\circ;\ \ 2\alpha = 60^\circ;\ \beta = 90^\circ - 2\alpha = 30^\circ;\ 2\beta = 60^\circ;$$

$$\Delta\left(RF_2A\right) \rightarrow \text{triunghi echilateral;}$$

$$RF_2 = AF_2 = RA = 2R;$$

$$RF_2 + RF_1 = 2a = 3R;$$

$$a = \frac{3}{2}R;$$

$$\mathrm{v}_0 = \sqrt{KM\left(\frac{2}{r} - \frac{1}{a}\right)};\ r = R;\ g_0 = K\frac{M}{R^2};$$

$$\mathrm{v}_0 = \sqrt{\frac{KM}{R^2} \cdot R^2\left(\frac{2}{R} - \frac{2}{3R}\right)} = 2\sqrt{\frac{g_0 R}{3}}.$$

c)

$$\mathrm{v}_{\mathrm{Antipod}} = \mathrm{v}_0\,.$$

d)

$$\mathrm{F_1F_2} = R \cdot \tan 2\alpha = 2c;\ c = \frac{R}{2} \cdot \tan 2\alpha = \frac{R}{2} \cdot \tan 60^\circ = \frac{\sqrt{3}}{2}R;$$

$$b = \sqrt{a^2 - c^2} = \sqrt{\frac{3}{2}}R;$$

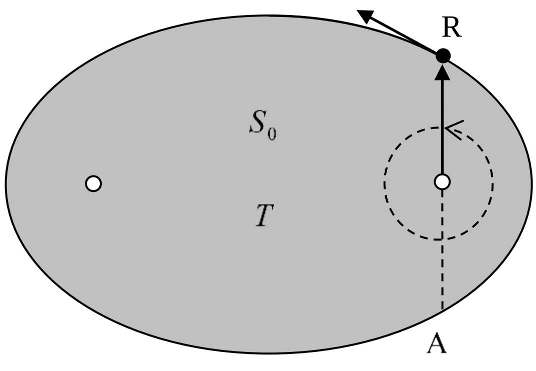
$$2a = 2r_{\min} + 2c;\ r_{\min} = a - c = \frac{1}{2}\left(3 - \sqrt{3}\right)R;$$

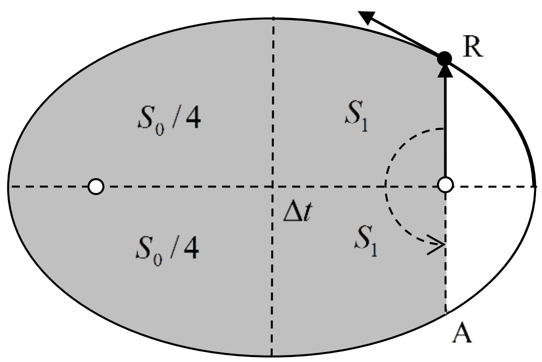
$$r_{\max} = 2a - r_{\min} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$\mathrm{v}_{\min} = \sqrt{KM\left(\frac{2}{r_{\max}} - \frac{1}{a}\right)};\ r_{\max} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$\mathrm{v}_{\min} = \sqrt{\frac{KM}{R^2} \cdot R^2\left(\frac{4}{\left(3 + \sqrt{3}\right)R} - \frac{2}{3R}\right)} = \sqrt{\frac{2g_0 R}{3} \cdot \frac{3 - \sqrt{3}}{3 + \sqrt{3}}}.$$

e) Accordin to Kepplers laws:





$$\Omega = \frac{\mathrm{d}S}{\mathrm{d}t} = \mathrm{constant};$$

$$\frac{S_0}{T} = \frac{2\dfrac{S_0}{4} + 2S_1}{\Delta t};\ \frac{S_0}{T} = \frac{\dfrac{S_0}{2} + 2S_1}{\Delta t};\ \frac{S_0}{T} = \frac{S_0 + 4S_1}{2 \cdot \Delta t};$$

$$S_0 = \pi ab;\ S_1 = \frac{ab}{2}\left[\sqrt{1-\frac{b^2}{a^2}}\cdot\frac{b}{a} + \arcsin\sqrt{1-\frac{b^2}{a^2}}\right];$$

$$\Delta t = \frac{S_0 + 4S_1}{2S_0}\cdot T = \left(\frac{1}{2} + 2\frac{S_1}{S_0}\right)\cdot T;$$

$$T = 2\pi\sqrt{\frac{a^3}{KM}};\ T = \frac{2\pi}{R}\sqrt{\frac{a^3}{g_0}};$$

$$\frac{2S_1}{S_0} = \frac{1}{\pi}\left(\frac{b}{a}\cdot\sqrt{1-\frac{b^2}{a^2}} + \arcsin\sqrt{1-\frac{b^2}{a^2}}\right);$$

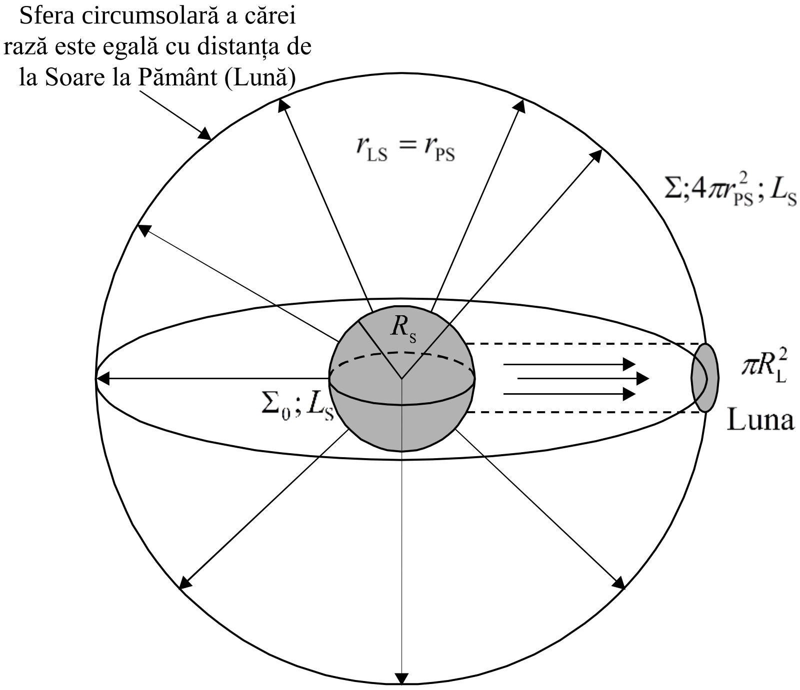
$$\sqrt{1-\frac{b^2}{a^2}} = e;\ \frac{2S_1}{S_0} = \frac{1}{\pi}\left(\frac{b}{a}\cdot e + \arcsin e\right);$$

$$\Delta t = \left(\frac{1}{2} + \frac{eb}{\pi a} + \frac{\arcsin e}{\pi}\right)\cdot T.$$

f) The integral luminosity of Sun:

$$L_{\mathrm{S}} = \frac{E_{\mathrm{emis,Soare}}}{t} = 3{,}86\cdot 10^{26}\ \mathrm{W},$$

Dacă For a circumsolar surface Σ with radius r_{PS}, see picture bellow the solar radiation enegy passing through the surface in one second is L_{S}.



Density of solar flux

$$\phi_{\mathrm{Soare},r_{\mathrm{PS}}} = \frac{E_{\mathrm{emis,Soare}}}{St} = \frac{\frac{E_{\mathrm{emis,Soare}}}{t}}{S} = \frac{L_{\mathrm{S}}}{S} = \frac{L_{\mathrm{S}}}{4\pi r_{\mathrm{PS}}^2} = \mathrm{constant}.$$

$$F_{\mathrm{incident},FullMoon} = \phi_{\mathrm{S}un,r_{\mathrm{PS}}} \cdot \pi R_{\mathrm{L}}^2.$$

Dacă α_{L} este albedoul Lunii, rezultă:

$$\alpha_{\mathrm{L}} = \frac{F_{\mathrm{reflectat},\,FullMoon}}{F_{\mathrm{incident}\,FullMoon}},$$

unde $F_{\mathrm{reflectat,\,Luna\,Plina}}$ – fluxul energetic al radiațiilor reflectate de Luna Plină spre observatorul de pe Pământ;

$$F_{\mathrm{reflectat},\,FullMoon} = \alpha_{\mathrm{L}} \cdot F_{\mathrm{incident}\,FullMoon} = \alpha_{\mathrm{L}} \cdot \phi_{\mathrm{Soa,,re},\,r_{\mathrm{PS}}} \cdot \pi R_{\mathrm{L}}^2.$$

În consecință, densitatea fluxului energetic ajuns la observator, după reflexia pe suprafața Lunii, este:

$$\phi_{moon,\ \mathrm{observator}} = \frac{F_{\mathrm{reflectat},\ FullMoon}}{2\pi r_{\mathrm{PL}}^2} = \alpha_{\mathrm{L}} \cdot \phi_{\mathrm{Soare},\,r_{\mathrm{PS}}} \cdot \frac{\pi R_{\mathrm{L}}^2}{2\pi r_{\mathrm{PL}}^2}.$$

Symilarly

$$\phi_{\mathrm{proiectil,\ observator}} = \frac{F_{\mathrm{reflectat,\ proiectil}}}{4\pi r_{\mathrm{D,proiectil}}^2} = \alpha_{\mathrm{proiectil}} \cdot \phi_{\mathrm{Soare},\,r_{\mathrm{PS}}} \cdot \frac{\pi R_{\mathrm{proiectil}}^2}{4\pi r_{\mathrm{D,proiectil}}^2}.$$

În expresia anterioară s-a avut în vedere faptul că densitatea fluxului energetic al proiectilului la observator rezultă din distribuirea prin suprafața sferei cu raza $r_{\mathrm{P,proiectil}}$.

Utilizând formula lui Pogson, vom compara magnitudinea aparentă vizuală a Lunii Pline cu magnitudinea aparentă vizuală a proiectilului balistic:

$$\log \frac{\phi_{\mathrm{Luna,observator}}}{\phi_{\mathrm{proiectil,observator}}} = -0{,}4\left(m_{\mathrm{Luna\,Plina}} - m_{\mathrm{proiectil}}\right);$$

$$\log \frac{\phi_{\mathrm{Luna,observator}}}{\phi_{\mathrm{proiectil,observator}}} = \log \frac{\alpha_{\mathrm{L}} \cdot \phi_{\mathrm{Soare},\,r_{\mathrm{PS}}} \cdot \frac{\pi R_{\mathrm{L}}^2}{2\pi r_{\mathrm{PL}}^2}}{\alpha_{\mathrm{proiectil}} \cdot \phi_{\mathrm{Soare},\,r_{\mathrm{PS}}} \cdot \frac{\pi R_{\mathrm{proiectil}}^2}{4\pi r_{\mathrm{D,proiectil}}^2}} = \log \frac{\alpha_{\mathrm{L}} \cdot \frac{R_{\mathrm{L}}^2}{r_{\mathrm{PL}}^2}}{\alpha_{\mathrm{proiectil}} \cdot \frac{R_{\mathrm{proiectil}}^2}{2 r_{\mathrm{D,proiectil}}^2}};$$

$$\log \frac{\alpha_{\mathrm{L}} \cdot \frac{R_{\mathrm{L}}^2}{r_{\mathrm{PL}}^2}}{\alpha_{\mathrm{proiectil}} \cdot \frac{R_{\mathrm{proiectil}}^2}{2 r_{\mathrm{D,proiectil}}^2}} = -0{,}4\left(m_{\mathrm{L}} - m_{\mathrm{proiectil}}\right);$$

$$\log \frac{\alpha_{\mathrm{L}}}{\alpha_{\mathrm{proiectil}}} \cdot \left(\frac{R_{\mathrm{L}}}{R_{\mathrm{proiectil}}}\right)^2 \cdot 2 \cdot \left(\frac{r_{\mathrm{D,proiectil}}}{r_{\mathrm{PL}}}\right)^2 = -0{,}4\left(m_{\mathrm{L}} - m_{\mathrm{proiectil}}\right);$$

$$\alpha_{\mathrm{L}} = 0{,}12;\ \alpha_{\mathrm{proiectil}} = 1;$$

$$R_{\mathrm{L}} = 1738\,\mathrm{km};\ R_{\mathrm{proiectil}} = 400\,\mathrm{mm};$$

$$r_{\mathrm{D,proiectil}} = r_{\mathrm{max,observator\text{-}proiectil}} = h_{\mathrm{max}} = r_{\mathrm{max}} - R;\ r_{\mathrm{max}} = \frac{1}{2}\left(3+\sqrt{3}\right)R;$$

$$h_{\mathrm{max}} = \frac{1}{2}\left(3+\sqrt{3}\right)R - R = \frac{1}{2}\left(1+\sqrt{3}\right)R \approx 8700\,\mathrm{km};$$

$$r_{\text{PL}} = r_{\text{observator,Luna}} = 384400\,\text{km};\; m_{\text{L}} = -12{,}7^{\text{m}};$$

$$\log\frac{\alpha_{\text{L}}}{\alpha_{\text{proiectil}}} + 2\log\frac{R_{\text{L}}}{R_{\text{proiectil}}} + \log 2 + 2\log\frac{r_{\text{D-proiectil}}}{r_{\text{PL}}} = -0{,}4\left(m_{\text{L}} - m_{\text{proiectil}}\right);$$

$$\log(0{,}12) + 2\log\frac{1738000\,\text{m}}{0{,}400\,\text{m}} + \log 2 + 2\log\frac{8700\,\text{km}}{384400\,\text{km}} = -0{,}4\left(m_{\text{L}} - m_{\text{proiectil}}\right);$$

$$\log(0{,}12) + 2\log\frac{1738000}{0{,}400} + \log 2 + 2\log\frac{8700}{384400} = -0{,}4\left(m_{\text{L}} - m_{\text{proiectil}}\right);$$

$$-0{,}920818754 + 13{,}27597956 + 0{,}301029995 - 3{,}290528253 = -0{,}4\left(m_{\text{L}} - m_{\text{proiectil}}\right);$$

$$23{,}4^{\text{m}} = 12{,}7^{\text{m}} + m_{\text{proiectil}};$$

$$m_{\text{proiectil}} = 10{,}7^{\text{m}};$$

$$m_{\max} \approx 6^{\text{m}};\; m_{\text{proiectil}} > m_{\max},$$

The projectile wasn't seen when it was at its apogee