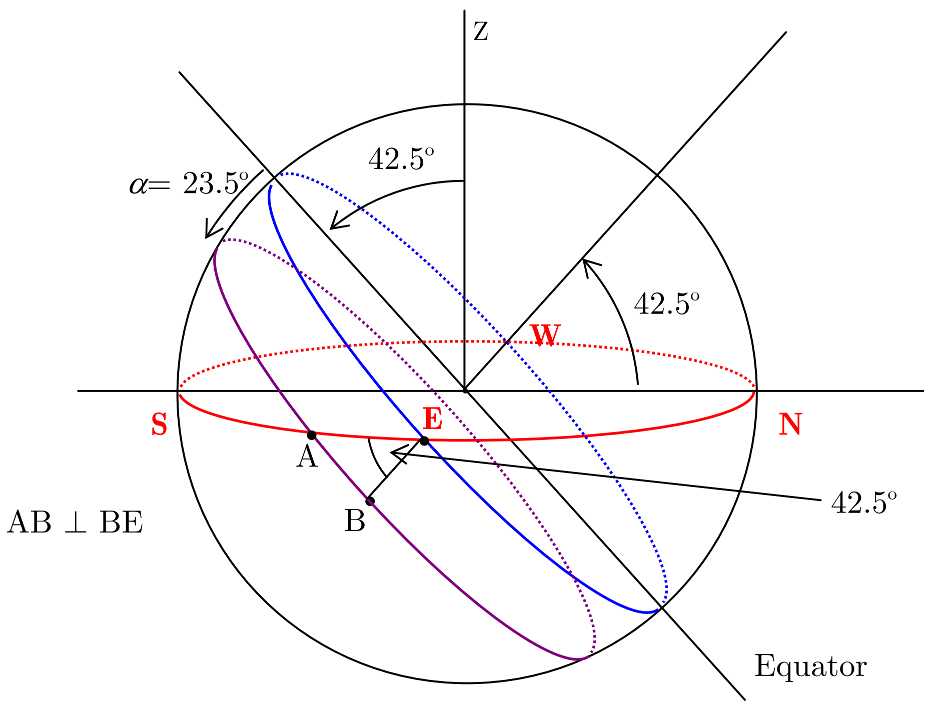
SOLUTION FOR QUESTION 1. (30 points for 15 short questions)

1.1



$$\mathrm{E} = \text{East point}$$

$$\mathrm{BE} = 23.5^\circ = |\text{declination of the Sun}|$$

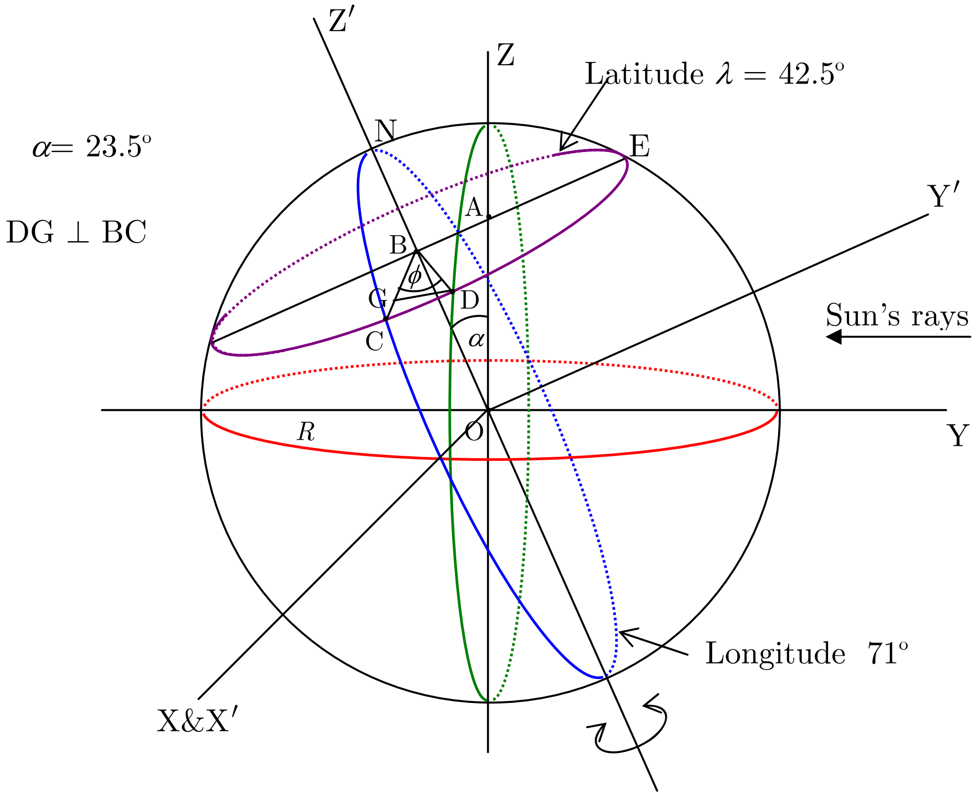
$$\frac{\mathrm{AB}}{\sin 42.5^\circ} = \frac{\mathrm{BE}}{\sin(90^\circ - 42.5^\circ)}$$

$$\mathrm{AB} = \mathrm{BE}\frac{\sin 42.5^\circ}{\cos 42.5^\circ} = 23.5^\circ \tan 42.5^\circ$$

local time = (23.5 tan 42.5° / 15) hrs after 6:00 = 7:26 am. The official time at 75° W should be 16 min. less.

Ans. 7:10 am.

Alternative solution (for 1.1)



The Earth's position relative to the Sun is shown in the figure.

Note that

$$\mathrm{OB} = R\sin\lambda$$

$$\mathrm{AB} = \mathrm{OB}\tan\alpha$$

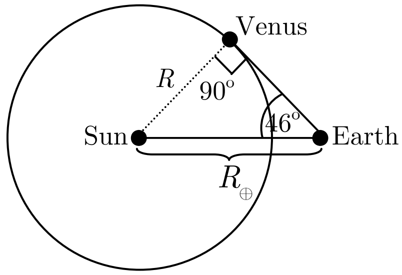
$$\mathrm{BC} = \mathrm{BD} = \mathrm{BE} = R\cos\lambda$$

$$\sin\phi = \frac{\mathrm{DG}}{\mathrm{BD}} = \frac{\mathrm{BA}}{\mathrm{BD}} = \frac{\mathrm{BO}\tan\alpha}{R\cos\lambda} = \tan\alpha\tan\lambda$$

$$= \tan\left(23.5^0\right)\tan\left(42.5^\circ\right) = 0.39843 = \sin(23.48^\circ)$$

Hence, the Sun will rise at $\left(71^\circ + 23.48^\circ\right)\times 4\,\text{min}. - 5$ hours. $= 77.92$ min. after 6 a.m.. This is at 7:18 a.m.

1.2



The angular separation is maximum when Sun, Venus and Earth form a right-angled triangle as shown.

Here
$$R = R_\oplus \sin 46^\circ$$
$$= (1\,\text{A.U.})\sin 46^\circ$$
$$= 0.72\ \text{A.U.}$$

1.3 If the same face of the Earth were to face the Sun all the time then the Earth would make <u>one</u> complete turn relative to fixed stars in one solar year (365.25 solar days).

This implies that in 365.25 solar days our actual Earth makes (365.25+1) complete turns relative to fixed stars.

Hence 365.25 solar days are the same time interval as 366.25 sidereal days;

and
$$183 \text{ solar days} \equiv \frac{183 \times 366.25}{365.25} \text{ sidereal days}$$
$$= 183.50 \quad \text{sidereal days}$$

<u>OR</u>

1 solar day = 24 x 60 x 60 s = 86400 s

1 sidereal day = 23 x 3600 + 56 x 60 + 4.1 = 86164.1 s

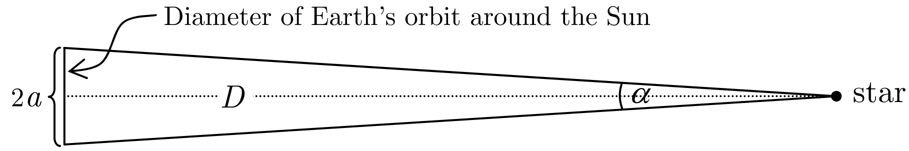
183 solar days = 183.50 sidereal days

1.4 During a full Moon we see the <u>whole face</u> of the Moon.

Hence
$$\frac{\text{Moon's diameter}}{\text{distance to Moon}} = \text{angle in radians}$$
$$= \frac{0.46 \times \pi}{180}$$

$$\text{Distance to the Moon} = (\text{Moon's diameter}) \times \frac{180}{0.46 \times \pi}$$
$$= (2 \times 1.7374 \times 10^6 \ \text{m}) \times \frac{180}{0.46 \times \pi}$$
$$= 4.328 \times 10^8 \ \text{m} = 4.3 \times 10^5 \ \text{km}$$

1.5



$$a = 1 \text{ A.U.} = 1.496\times10^{11} \text{ m}$$

$$1 \text{ pc} = 3.0856\times10^{16} \text{ m}$$

$$D = 100\times3.0856\times10^{16} \text{ m}$$

$$\alpha = \frac{2a}{D} = \frac{2\times1.496\times10^{11}}{100\times3.0856\times10^{16}} \text{ radian}$$
$$= 0.96966.....\times10^{-7} \text{ radian}$$
$$= 5.555779...\times10^{-6} \text{ degree}$$
$$= 0.02 \text{ arc second}$$

1.6 According to Kepler's third law we have

$$(\text{period})^2 = (\text{constant})(\text{semi-major axis})^3$$
$$T^2 = (\text{constant})a^3$$

This constant is $1\,\frac{(\text{year})^2}{(\text{A.U.})^3}$ when T is measured in years and a in A.U.'s.

For this comet we have

$$a = \frac{31.5+0.5}{2} = 16.0 \text{ A.U.}$$
$$T^2 = (16)^3 = 16\times4\times4\times16 = (64)^2$$
$$T = 64 \text{ years}$$

1.7 According to Kepler's second law we have;

The area is swept out at constant rate throughout the orbital motion.

Now, for this comet the area swept out in one orbital period is πab where a is the semi-major axis and b the semi-minor axis of the orbit.

$$a = 16.0 \text{ A.U.}$$

$$b^2 = a^2 - (\text{distance from a focus to centre of ellipse})^2$$

$$= (16.0)^2 - (16.0 - 0.5)^2$$
$$b = 3.968 \text{ A.U.}$$
$$\pi ab = 199.5 \ (\text{A.U.})^2$$

Hence, the area swept out per year is

$$\frac{199.5 \ (\text{A.U.})^2}{64 \text{ years}} = 3.1 \ (\text{A.U.})^2 \ / \text{ year}$$

1.8 From the Wien's displacement law

$$\lambda_{\max} T = 2.8977 \times 10^{-3} \text{ m.K}$$
$$\lambda_{\max} = \frac{2.8977 \times 10^{-3}}{4000} \text{ m} = 7.244 \times 10^{-7} \text{ m}$$
$$= 724 \text{ nanometers}$$

1.9 From the Stefan-Boltzmann law, the total power emitted is

$$L = (4\pi R^2)(\sigma T^4) = (4\pi R_\odot^2)(\sigma T_\odot^4)\left(\frac{R}{R_\odot}\right)\left(\frac{T}{T_\odot}\right)^4 = L_\odot (2.5)^2 \left(\frac{7500}{5800}\right)^4$$
$$\frac{L}{L_\odot} = (6.25)(2.796) = 17.47 = 17.5$$
$$L = 17.5 L_\odot$$

1.10

$$\text{flux density} = \frac{\text{luminosity}}{4\pi \times (\text{distance})^2}$$
$$\text{distance} = \sqrt{\frac{0.4 \times 3.826 \times 10^{26}}{4\pi \times 6.23 \times 10^{-14}}} = 1.398 \times 10^{19} \text{ m}$$
$$= \frac{1.398 \times 10^{19}}{3.0856 \times 10^{16}} \text{ pc} = 453 \text{ pc}$$

1.11 $$L_{\text{supernova}} = 10^{10} L_{\odot}$$

Let D be the distance to that supernova.

Then the flux intensity on Earth would be $I = \dfrac{10^{10} L_{\odot}}{4\pi D^2}$

This must be the same as $\dfrac{L_{\odot}}{4\pi \left(D_{\oplus-\odot}\right)^2}$

$$\frac{10^{10} L_{\odot}}{4\pi D^2} = \frac{L_{\odot}}{4\pi \left(1 \text{ A.U.}\right)^2}$$

$$\therefore \qquad D = 10^{\frac{10}{2}} \text{ A.U.} = 10^5 \text{ A.U.}$$

$$= \frac{1.496 \times 10^{16}}{3.0856 \times 10^{16}} \text{ pc} = 0.485 \text{ pc}$$

$$= 0.485 \times 3.2615 \text{ ly} = 1.58 \text{ ly}$$

1.12 The difference in frequencies is due to the relativistic Doppler shift. Since the observed frequency of emission from the gas cloud is higher than the laboratory frequency ν_0, the gas cloud must be approaching the observer.

Hence, $$\nu = \nu_0 \sqrt{\frac{c+v}{c-v}}$$

$$\frac{v}{c} = \frac{\left(\frac{\nu}{\nu_0}\right)^2 - 1}{\left(\frac{\nu}{\nu_0}\right)^2 + 1}$$

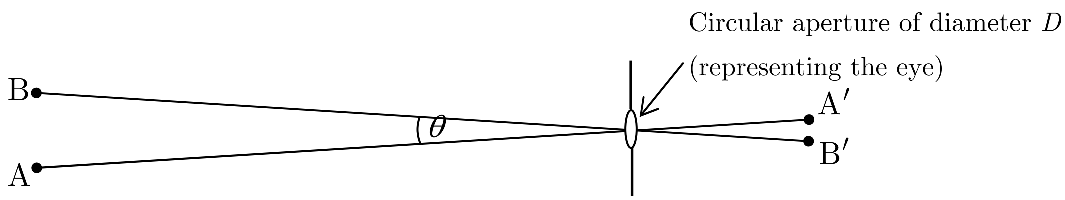
$$= \frac{0.001752379}{2.001752379} = 0.000875422$$

$$v = \left(0.000875422\right)\left(2.99792458 \times 10^8 \text{ m.s}^{-1}\right)$$

$$= 0.00262445 \times 10^8 \text{ m.s}^{-1}$$

$$= 262.445 \text{ km.s}^{-1}$$

1.13



A and B are two distant objects.

A′ and B′ are the central maxima of their diffracted images.

The angular position ϕ_1 of the first minimum relative to central maximum of each diffraction pattern is given by $\phi_1 = 1.22\dfrac{\lambda}{D}$.

According to Lord Rayleigh the minimum angle of resolution is ϕ_1.

Hence the images of A and B will be resolved if $\theta > \phi_1$, $D > 1.22\dfrac{\lambda}{\theta}$.

We may take distance AB to be the diameter of the crater. Hence the diameter D of the eye's aperture must be, at least, $D_{\min} = 1.22\dfrac{\lambda}{\theta}$.

$$\theta = \frac{80 \text{ km}}{\text{distance from Earth to Moon}}$$
$$= \frac{80\times10^{3} \text{ m}}{3.844\times10^{8} \text{ m}} = 2.081\times10^{-4} \text{ radian}$$

$$\lambda = 500\times10^{-9} \text{ m}$$

$$\therefore \quad D_{\min} = \frac{1.22\times500\times10^{-9}}{2.081\times10^{-4}} \text{ m} = 2.93\times10^{-3} \text{ m}$$
$$= 2.9 \text{ mm}$$

Hence, it is possible to resolve the 80 km-diameter crater with naked eye.

1.14 The spherical boundary at which the escape velocity becomes equal to the speed of light is of radius $R = \dfrac{2GM_\odot}{c^2}$

$$R = \frac{2\times6.672\times10^{-11}\times1.989\times10^{30}}{\left(2.99792458\times10^{8}\right)^2} \text{ m}$$
$$= \frac{2\times6.672\times1.989}{2.998\times2.998}\times10^{3} \text{ m}$$
$$= 2.95 \text{ km}$$

1.15 The flux ratio versus magnitude difference implies

$$m_1 - m_2 = -2.5\log\left(\frac{f_1}{f_2}\right)$$

$$\frac{f_1}{f_2} = 10^{\frac{(m_2 - m_1)}{2.5}}$$

So for a magnitude difference of $(-1.5) - 6 = -7.5$ we find a flux ratio of

$$\frac{f_{\min}}{f_{\max}} = 10^{\frac{-7.5}{2.5}} = 10^{-3}$$

And thus the visible stars range only over a factor of 1000 in brightness.

QUESTION 2 **A PLANET & ITS SURFACE TEMPERATURE**

SOLUTION

a.) Intensity $$I = \frac{L}{4\pi D^2}$$ (1 point)

b.) Absorption rate $$\mathcal{A} = (1-\alpha)\pi R^2 I$$

$$= (1-\alpha)\frac{LR^2}{4D^2}$$ (1 point)

c.) Light energy reflected by the planet per unit time is $\alpha\pi R^2 I = \frac{\alpha L R^2}{4D^2}$

(1 point)

Hence the planet's luminosity is

$$L_{\text{planet}} = \frac{\alpha L R^2}{4D^2}$$ (1 point)

d.) Here, we will neglect the planet's internal source of energy.

Let T be the black-body temperature of the planet's surface in kelvins.

Since the planet is rotating fast, we may assume that its surface is being heated up uniformly to approximately the same temperature T.

The total amount of black-body radiation emitted by the planet's surface is from Stefan-Boltzmann law: $4\pi R^2 \cdot \sigma T^4, \sigma$ being Stefan-Boltzmann constant.

At equilibrium, that is when the temperature remains steady, this emission rate must be equal to the absorption rate in b.). (1 point)

Hence

$$4\pi R^2 \cdot \sigma T^4 = (1-\alpha)\frac{LR^2}{4D^2}$$

$$T = \left[(1-\alpha)\frac{L}{16\pi\sigma D^2}\right]^{\frac{1}{4}}$$ (1 point)

e.) In this case the emitted black-body radiation is mostly from the planet's surface facing the star. The emitting surface area is now only $2\pi R^2$ and not $4\pi R^2$. Hence the surface temperature is given by T', where

$$2\pi R^2 \cdot \sigma (T')^4 = (1-\alpha)\frac{LR^2}{4D^2}$$ (1 point)

$$T' = \left[\left(1-\alpha\right)\frac{L}{8\pi\sigma D^2}\right]^{\frac{1}{4}} = \left(2\right)^{\frac{1}{4}}\cdot T \approx \left(1.19\right)T \qquad \text{(1 point)}$$

f.)

$$T = \left[\left(1-\alpha\right)\frac{L}{16\pi\sigma D^2}\right]^{\frac{1}{4}}$$

$$T = \left[\left(1-0.25\right)\times\frac{3.826\times10^{26}}{16\pi\times5.67\times10^{-8}\times\left(1.523\times1.496\times10^{11}\right)^2}\right]^{\frac{1}{4}}$$

$$= 209.8 \simeq 210 \text{ K} = \text{-63}^{\circ}\text{C} \qquad \text{(2 points)}$$

QUESTION 3 BINARY SYSTEM

Solution

a) The total angular momentum of the system is

$$L = I\omega = (M_1 r_1^2 + M_2 r_2^2)\omega \qquad \text{(0.5 points)}$$

We have also $M_1 r_1 = M_2 r_2$ and $D = r_1 + r_2$

which yield $$L = \frac{M_1 M_2}{M_1 + M_2} D^2 \omega \qquad \text{.................(1)} \qquad \text{(0.5 points)}$$

The kinetic energy of the system is

$$K.E. = \frac{1}{2} M_1 \left(r_1 \omega\right)^2 + \frac{1}{2} M_2 \left(r_2 \omega\right)^2 = \frac{1}{2}\left(M_1 r_1^2 + M_2 r_2^2\right)\omega^2 \qquad \text{(0.5 points)}$$

$$= \frac{1}{2}\frac{M_1 M_2}{(M_1 + M_2)} D^2 \omega^2 \qquad \text{...............(2)} \qquad \text{(0.5 points)}$$

b) From Newton's laws of motion we have

$$M_1 \omega^2 r_1 = M_2 \omega^2 r_2 = \frac{G M_1 M_2}{D^2} \qquad \text{(1 point)}$$

These equations together with those in a) yield

$$\omega^2 = \frac{G\left(M_1 + M_2\right)}{D^3} \qquad \text{...................(3)} \qquad \text{(1 point)}$$

c) In order to find the quantity $\Delta\omega$, we must also realize that

$$M_1 + M_2 = \text{constant} \qquad \text{................(4)} \qquad \text{(0.5 points)}$$

And that, since there is no external torque acting on the system, the total angular momentum must be conserved.

$$L = \frac{M_1 M_2}{M_1 + M_2} D^2 \omega = \text{constant}$$

that is, $$M_1 M_2 D^2 \omega = \text{constant} \qquad \text{(0.5 points)}$$

Now, after the mass transfer,

$$\omega \to \omega + \Delta\omega$$
$$M_1 \to M_1 + \Delta M_1$$
$$M_2 \to M_2 - \Delta M_1$$
$$D \to D + \Delta D$$

Hence, $M_1M_2D^2\omega = (M_1+\Delta M_1)(M_2-\Delta M_1)(D+\Delta D)^2(\omega+\Delta\omega)$

After using the approximation $(1+x)^n \sim 1+nx$ and rearranging, we get

$$\frac{\Delta\omega}{\omega}+2\frac{\Delta D}{D} = \left(\frac{M_1-M_2}{M_1M_2}\right)\Delta M_1 \quad \text{..........(5)} \qquad \text{(1 point)}$$

From equation (3), ω^2D^3 is also constant. That is,

$$\omega^2D^3 = (\omega+\Delta\omega)^2(D+\Delta D)^3$$

This gives,

$$\frac{\Delta D}{D} = -\frac{2}{3}\frac{\Delta\omega}{\omega} \quad \text{..................(6)} \qquad \text{(0.5 points)}$$

Hence $$\Delta\omega = -\frac{3(M_1-M_2)}{M_1M_2}\omega\,\Delta M_1 \quad \text{..........(7)} \qquad \text{(0.5 points)}$$

d) Given that

$M_1 = 2.9\ \mathrm{M}_\odot$, $M_2 = 1.4\ \mathrm{M}_\odot$

orbital period, $T = 2.49$ days

and that T has increased by 20 s in the past 100 years,

we have $\omega = \frac{2\pi}{T}$ and $\Delta\omega = -\frac{\omega}{T}\Delta T$ ....(8) (0.5 points)

$$\Delta M_1 = +\frac{1}{3}\left(\frac{M_1M_2}{M_1-M_2}\right)\frac{\Delta T}{T} \qquad \text{(0.5 points)}$$

$$\frac{\Delta M_1}{M_1\Delta t} = \frac{1}{3}\left(\frac{1.4}{(2.9-1.4)}\right)\left(\frac{20}{2.49\times24\times3600}\right)\left(\frac{1}{100}\right)$$

$$= 2.89\times10^{-7} \text{ per year} \qquad \text{(0.5 points)}$$

e) Mass is flowing from M_2 to M_1. (0.5 points)

f) From equations (6) and (8):

$$\frac{\Delta D}{D\Delta t} = -\frac{2}{3}\frac{\Delta\omega}{\omega} = +\frac{2}{3}\frac{\Delta T}{T} = 6.20\times10^{-7} \text{ per year} \quad \text{(1.0 points)}$$

QUESTION 4 GRAVITATIONAL LENSING

Solution

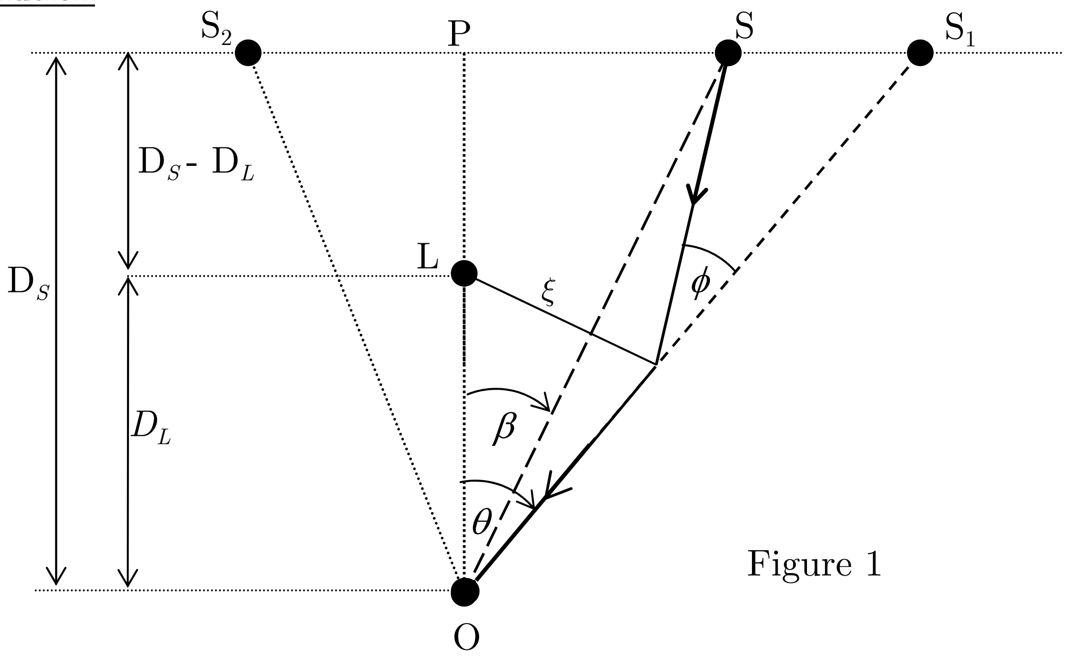


Figure 1

a). From figure 1, for small angles, $\tan\theta \approx \theta$, hence

$$PS_1 = PS + SS_1$$

$$\theta D_S = \beta D_S + (D_S - D_L)\phi$$

$$\theta - \beta = \frac{4GM}{\xi c^2}\frac{(D_S - D_L)}{D_S} \quad \text{......................(1)} \qquad (\text{ 1 point})$$

Note that $\theta = \dfrac{\xi}{D_L}$ also, hence,

$$\theta^2 - \beta\theta = \left(\frac{4GM}{c^2}\right)\left(\frac{D_S - D_L}{D_L D_S}\right) \quad \text{......................(2)}$$

For a perfect alignment in which $\beta = 0$, we have $\theta = \pm\theta_E$, where

$$\theta_E = \sqrt{\left(\frac{4GM}{c^2}\right)\left(\frac{D_S - D_L}{D_L D_S}\right)} \quad \text{......................(3)} \qquad (\text{ 1 point})$$

b). From equation (3), for a solar-mass lens with $D_S = 50$ kpc,

$D_L = 50 - 10 = 40$ kpc

$$\theta_E = \sqrt{\left(\frac{4GM}{c^2}\right)\left(\frac{D_S - D_L}{D_L D_S}\right)} = 0.956 \times 10^{-9} \text{ radian}$$

$$= 1.97 \times 10^{-4} \text{ arc second} \qquad (\text{ 1 point})$$

c.) The resolution of the Hubble space telescope having diameter of 2.4 m is,

$$\theta_{Hubble} = 1.22\frac{\lambda}{D} = 1.22\times\frac{5\times10^{-7}\text{ m}}{2.4\text{ m}} = 2.54\times10^{-7}\text{ radian}$$ for light of wavelength 500 nm. (1 point)

Hence the Hubble telescope could not resolve this Einstein ring.

(1 point)

d). The quadratic equation (2) has two distinct roots, namely,

$$\theta_1 = \frac{\beta}{2}+\sqrt{\left(\frac{\beta}{2}\right)^2+\theta_E^2}$$ (1 point)

$$\theta_2 = \frac{\beta}{2}-\sqrt{\left(\frac{\beta}{2}\right)^2+\theta_E^2}$$ (1 point)

where $\theta_E = \sqrt{\left(\frac{4GM}{c^2}\right)\left(\frac{D_S-D_L}{D_LD_S}\right)}$

This implies that there are two images for a single isolated source.

e).

$$\frac{\theta_{1,2}}{\beta} = \frac{\frac{\beta}{2}\pm\sqrt{\left(\frac{\beta}{2}\right)^2+\theta_E^2}}{\beta}$$

$$= \frac{1}{2}\pm\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{\eta}\right)^2} = \frac{1}{2}\left[1\pm\frac{\sqrt{\eta^2+4}}{\eta}\right]$$ (1 point)

f). From equation (2) $\theta^2-\beta\theta-\theta_E^2 = 0$

$$\left(\theta+\Delta\theta\right)^2-\left(\beta+\Delta\beta\right)\left(\theta+\Delta\theta\right)-\theta_E^2=0$$

$$\frac{\Delta\theta}{\Delta\beta} = \frac{\theta}{2\theta-\beta}$$ (1 point)

$$\left(\frac{\Delta\theta}{\Delta\beta}\right)_{\theta=\theta_{1,2}} = \frac{\theta_{1,2}}{2\theta_{1,2}-\beta} = \frac{\frac{1}{2}\eta\pm\sqrt{1+\frac{\eta^2}{4}}}{\pm2\sqrt{1+\frac{\eta^2}{4}}} = \frac{1}{2}\left[1\pm\frac{\eta}{\sqrt{\eta^2+4}}\right]$$ (1 point)