# 2016 International Astronomy Olympiad

## Practical round: solutions

The gravitational constant is γ or G = $6.67 \times 10^{-11}$ $m^3$ $kg^{-1}$ $s^{-2}$

**Problem β-6. The first gravitational wave detection.**

**6.1.** *1 pt*

The initial masses of the two black holes are 36+/-4$M_\odot$ and 28+/-4$M_\odot$. Their Schwarzschild radii are calculated using $R_s = 2GM/c^2$ , which gives 107+/-12 km and 83+/-12 km. The errors are calculated using ($\Delta R_s/R_s = \Delta M/M$).

**6.2.** *2.5 pts*

A good estimate of the precision of the measurements can be gained by comparing the Hanford and Livingston signals. At the time of the merger, we see two strong maxima and a minimum of the signal. For them, the time-axis intercepts for the Livingston signal are approximately at:

[0.4190, 0.4215, 0.4234, 0.4259] s

while for the Hanford signal, they are at:

[0.4193, 0.4217, 0.4239, 0.4259] s

These correspond to half-periods of:

[0.0025, 0.0019, 0.0025] sec
[0.0024, 0.0022, 0.0020] sec

for the two signals, respectively.

Assuming those measurements are both independent and representative for the time of the merger, taking the average and standard deviation of those, we get the half-period of the signal to be:

0.0023±0.0003s

An attempt at a more accurate determination can be made by building a P(t) diagram.

The signal has close to symmetric minima and maxima, and the trends of both the maxima and the minima are consistent across both the odd and even peaks. This symmetry implies that the signal is emitted from a binary system with two components of roughly equal masses, as confirmed by the LIGO results.

Therefore, from the symmetry of the configuration of the system, the signal must be identical irrespective of which of the initial black holes is closer to us while they orbit each other. Thus, the signal must have a period of exactly half of the orbital period of the black holes. Therefore, the orbital period at the time of the merger must be 4 times larger than the half-period of the signal quoted above.

Thus, we can conclude that the orbital period is:

$T = 0.0092 \pm 0.0012$ s

**6.3.** *2 pts*

Assuming circular orbits and Newtonian mechanics, we can apply Kepler's third law:

$a^3/T^2 = G(M_1+M_2)/(4\pi^2)$

where the semi-major axis at the moment of the merger is simply:

$a = R_{s1}+R_{s2} = 2G(M1+M2)/c^2$

We can combine the two equations above to solve for the total mass:

$M_1+ M_2 = Tc^3/(2^{5/2}\pi G)$

Using the value for the period we obtained in II.b, we find:

$M_1+M_2=105\pm13M_\odot$

**6.4.** *1.5 pts*

In Newtonian mechanics, the gravitational potential energy of the system is

$E_P = -GM_1M_2/a$.

For a circular orbit the kinetic energy is $E_K = |E_P|/2$, and so the mechanical energy is given by

$E = -GM_1M_2/(2a)$,

which at large separations tends to zero. Assuming, the orbits are close to circular, and using a semi-major axis of $a=R_{s1}+R_{s2}$ at the time of the merger, we obtain that the change in mechanical energy of the binary system until the moment of the merger is given by:

$\Delta E = (M_1M_2/(M_1+M_2))(c^2/4)$

which should equal the total energy emitted in gravitational waves. Plugging in the numerical results obtained so far, and the assumption M1=M2, we find:

$\Delta E = 6.6 \pm 0.8 M_\odot c^2$.

**6.5.** *2 pts*

We will find the period around t~0.32s. The two maxima are around that time are at:

0.316±0.005s and 0.344±0.005s

The period of the signal is then 0.028±0.005s (errors added in quadrature). This implies an orbital period which is twice larger (see II.b):

Ti=0.056±0.010s at 0.10 s before the outburst

Combining the expression for the mechanical energy:
$E = -GM_1M_2/(2a)$

with Kepler's third law:

$a^3/T^2=G(M_1+M_2)/(4\pi^2)$,

we can find the mechanical energy in the initial moment. We obtain:

$Ei = 2.0 \pm 0.5 M_\odot c^2$

Our previous estimate of E was made in 6.4.:

$E = 6.6 \pm 0.8 M_\odot c^2$ at t = 0.4226±0.0002s

Thus, the change of mechanical energy of the system is:
$\Delta E_{0.1} = Ei - E = 4.6 \pm 0.9 M_\odot c^2$

over a time interval of 0.1s

This corresponds to average power of emission for the chosen time interval, which equals:

$P = \Delta E_{0.1}/\Delta t = 8 \pm 2 \times 10^{48}$ W

**6.6.** *2 pts*

We are told that the flux (F) is proportional to $h^2$. We also know that for any wave, F is inversely proportional to the square of the distance (d). Then h and d must be inversely related ($F \sim h^2$, $F \sim 1/d^2 \rightarrow h \sim 1/d$). Thus, we can set up the following ratio:

$h_1/h_2=d_2/d_1$

for any two points outside the black holes.

Right next to the two black holes (at a distance $R_s \sim R_{s1} + R_{s2}$) from the center of masses), the strain is $h_0 = v^2/c^2$. We can estimate the distance d via

$h/h_0 = R_s/d \quad \rightarrow \quad d = R_s(v/c)^2/h$

where h is the strain measured here on Earth. From the figure, $h \sim 1\times10^{-21}$.

If we approximate the orbital velocity at the time of merging as circular, if would be

$v^2 \sim G(M_1+M_2)/(R_{s1}+R_{s2}) \rightarrow (v/c)^2 \sim 0.5$

Therefore, to an order of magnitude d ~ 1.7 Gpc.

**6.7.** *1pt*

Supermassive black holes span the range from $10^6 M_\odot$ to $10^{10} M_\odot$. Using a cosmological distance of $d_{min} \sim$ 1Gpc to such merging sources, combined with our result for how the strain scales with distance from the previous part of the problem:

$h \sim 0.5(R_{s1}+R_{s2})/d$

we get:

$h \sim$ from $10^{-17}$ to $10^{-13}$

for the signal from such merging SMBHs. To be detectable, we need a non-trivial signal-to-noise ratio, and thus we can put the minimum sensitivity requirement at

$h_{min} \sim$ from $10^{-18}$ to $10^{-14}$ for the range of masses, corresponding to SMBHs.