2016 International Astronomy Olympiad

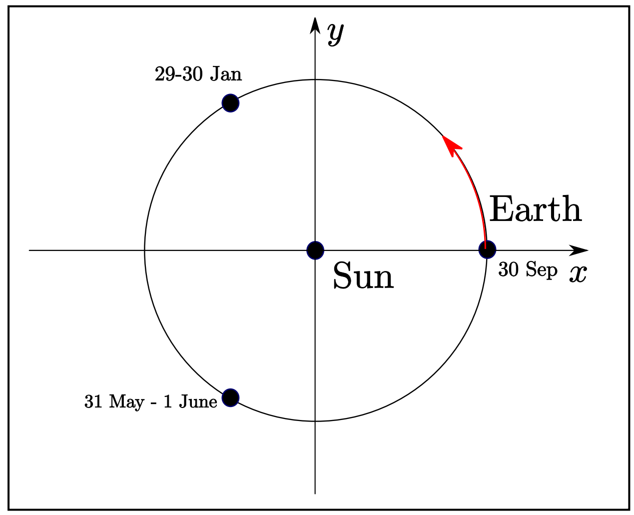
Practical round: solutions

The gravitational constant is γ or G = 6.67×10^{-11} $m^3\ kg^{-1}\ s^{-2}$

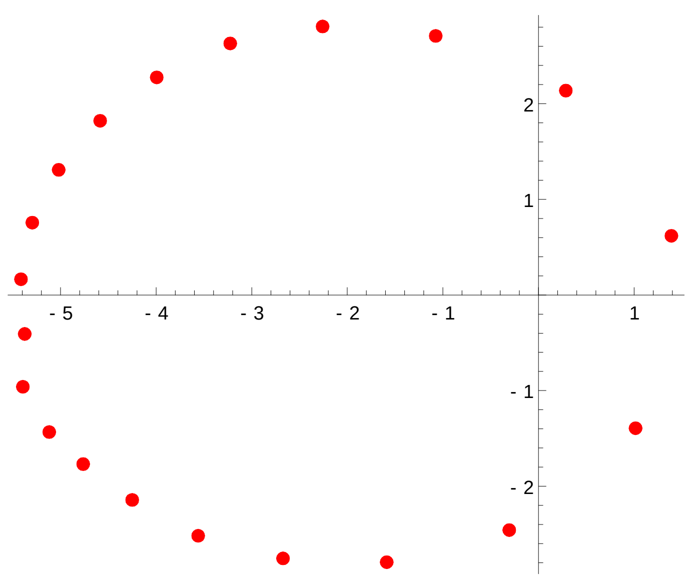
Problem α-6. Comet observer.

6.1. *2pts*

We start off by drawing the Earth's orbit. Using the tabular data we find that the suitable value for the radius is 2 cm. Because there are 3 observations per year, as noted in the text, there are just 3 positions of the Earth, where the observations take place: one for **30 September**, one for **29-30 January** and one for **31 May-1 June**. We mark these positions on the graph paper. Because 30 Sep 2004 is when Delta takes the largest tabular value (6.485 AU), it is convenient to place 30 Sep on the x-axis. The other two observational points are separated by an angle of 120° and are easily marked using simple geometry.



Next we draw all 20 points from the table on the graphing paper, where each point represents the comet's position for a particular observation with given N . To this end, we use only columns **Date(UT)**, **Delta** and **S-O-T** and disregard the rest. Note that **Delta** is the distance between the comet and the Earth. Therefore some care must be exercised to measure **Delta** from the correct position of the Earth defined by **Date(UT)**.



Next we draw a smooth closed curve around the points, which resembles an ellipse.

6.2. *3pts*

First by observing the curve we identify the direction of the semi-major axis. Then we draw a line along this direction, which approximately halves the ellipse. Next we mark the so-obtained perihelion and aphelion and measure the distance between them, which yields $2a$. We also measure the perihelion distance, which yields $a(1-e)$. Bearing in mind that 2 cm = 1 AU, we find that

$$a \approx 3.45 \text{ AU and } e \approx 0.58.$$

In fact, the real values are $a \approx 3.44$ AU and $e \approx 0.57$.

6.3. *1pt*

The first point ($N = 1$)and the last point ($N = 20$) almost lie on the x-axis meaning that for the time interval separating these observations, the comet has done a full cycle. Therefore we obtain

$$T \approx (20-1)\frac{1}{3} \text{years} \approx 6.33 \text{ years},$$

while the real value is $T \approx 6.38$ years.

6.4. *2pts*

From Kepler's second law we know that a line segment joining an orbitingbody and the Sun sweeps out equal areas ΔS during equal intervals of time Δt. This means that $\Delta S / \Delta t = S / T$, where S isthe area inside the ellipse, and T is the comet's orbital period. Note that at the perihelion and at the aphelion the following identities hold for very small Δt:

$$\frac{\Delta S}{\Delta t} = \frac{1}{2} v_p a(1-e) \text{ and } \frac{\Delta S}{\Delta t} = \frac{1}{2} v_a a(1+e).$$

For the derivation consider the area of an isosceles triangle of height $a(1-e)$ and base $v_p \Delta t$.

Thus we find

$$v_a = \frac{2S/T}{a(1-e)} \approx 31.5 \text{ km/s and } v_p = \frac{2S/T}{a(1+e)} \approx 8.4 \text{ km/s}.$$

We can express the speeds directly by a and T if we use the formula $S = \pi ab = \pi a^2 \sqrt{1-e^2}$:

$$v_a = \frac{2\pi a}{T}\sqrt{\frac{1-e}{1+e}} \approx 31.5 \text{ km/s and } v_p = \frac{2\pi a}{T}\sqrt{\frac{1+e}{1-e}} \approx 8.4 \text{ km/s.}$$

6.5. *2pts*

The solar mass M can be obtained directly from the identity (neglecting the comet's mass): $T^2 = \frac{4\pi^2 a^3}{\gamma M}$. Thus we have

$$M = \frac{4\pi^2 a^3}{\gamma T^2} \approx 2\text{x}10^{30} \text{ kg.}$$

6.6. *2pts*

From conservation of energy we have $\frac{mv^2}{2} - \frac{\gamma Mm}{r} = \frac{mv_p^2}{2} - \frac{\gamma Mm}{a(1-e)}$, which gives

$$v^2 = v_p^2 + 2\gamma M\left(\frac{1}{r} - \frac{1}{a(1-e)}\right).$$

In solution 6.4. we found $v_p = \frac{2\pi a}{T}\sqrt{\frac{1+e}{1-e}}$. From $T^2 = \frac{4\pi^2 a^3}{\gamma M}$ we obtain $v_p^2 = \frac{\gamma M}{a}\frac{1+e}{1-e}$.

Thus we arrive at

$$v = \sqrt{\gamma M\left(\frac{2}{r} - \frac{1}{a}\right)}.$$

For the observation with $N = 7$ we measure that $r = 3.25$ AU, so we get $v_7 \approx 17.2$ km/s.

The escape velocity *at this point* is obtained by taking the limit as a approaches ∞:

$$v_e = \sqrt{\frac{2\gamma M}{r}} \approx 23.7 \text{ km/s.}$$