Theoretical round. Sketches for solutions

Note for jury and team leaders. The proposed sketches are not full; the team leaders have to give more detailed explanations for students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

1α. Atoll. On the equator, where the latitude is $\varphi = 0°$, the lower culmination of the Polar star is at the height $\varphi - 90° + \delta = -44'09''$. Refraction makes the horizon lower by approximately 35'. So, if we take refraction into account, on the equator at sea level the Polar star may be under the horizon only by 9'. For the Polar star to appear as circumpolar from the highest point of the atoll, it should be of such height that the horizon is lowered by 9'. The lowering of the physical horizon depends on the height of the observer according to the formula

$$\cos(\Delta\alpha) = R/(R+h),$$
$$\Delta\alpha = \arccos(R/(R+h)) \approx ((R+h)^2 - R^2)^{1/2}/R \approx (2h/R)^{1/2} \text{ (in radians)},$$
$$h \approx (\Delta\alpha)^2 \cdot R/2 \text{ (}\Delta\alpha\text{ is in radians)}.$$
For $\Delta\alpha \approx 9' \approx 0.0026$ rad we get $h \approx 22$ m.

1β. Temperature of star's core. Gas in the cores of stars consists of ions (completely ionized atoms) and electrons. The number of protons of each atom number is equal to number of electrons. As all atoms in the cores are ionized, the number of free electrons corresponds to the number of protons in its nucleus, that is Z. If n_H, n_{He} and n_C are concentrations of ions of hydrogen, helium and carbon, the concentrations of corresponding electrons are equal to $Z_H \cdot n_H$, $Z_{He} \cdot n_{He}$ and $Z_C \cdot n_C$.

The total pressure of the gas in stars is equal to the sum of pressure of the gas of ions p_i and the pressure of gas of electrons p_e,

$$p = p_i + p_e.$$

According to formulae for ideal gas

$$p_i = n_i kT \quad \text{and} \quad p_e = n_e kT,$$

where n_i and n_e are concentrations of ions and electrons respectively. The total pressure is equal to

$$p = (n_i + n_e) \cdot kT,$$

Then we write equations for the total pressures in the stars.

For the Sun:

$$p_S = (n_H + Z_H \cdot n_H + n_{He} + Z_{He} \cdot n_{He}) \cdot kT_S = (2 \cdot n_H + 3 \cdot n_{He}) \cdot kT_S = (2 \cdot (1-\alpha) \cdot n_S + 3 \cdot \alpha \cdot n_S) \cdot kT_S = (2 + \alpha) \cdot n_S \cdot kT_S.$$

For $\alpha = 0.08$:

$$p_S = 2.08 \cdot n_S \cdot kT_S.$$

For the carbon star:

$$p_C = (n_C + Z_C \cdot n_C) \cdot kT_C = 7 \cdot n_C \cdot kT_C.$$

Since these pressures are equal to each other (condition of the problem)

$$p_S = p_C$$

we may write

$$7 \cdot n_C \cdot kT_C = 2.08 \cdot n_S \cdot kT_S,$$

$$T_C = 0.297 \cdot (n_S/n_C) \cdot T_S.$$

Further we shall find a relation between the concentrations of ions in the solar core and the core of the carbon star, n_S and n_C. Practically all mass of matter is in the nuclei of atoms, therefore the densities in the cores of each star are equal to the concentrations of ions multiplied by the masses of ions which are proportional to the nuclear masses of the elements. As the density of substance in the solar core and the core of the carbon star are identical,

$$A_H \cdot n_H + A_{He} \cdot n_{He} = A_C \cdot n_C,$$

$$A_H \cdot (1-\alpha) \cdot n_S + A_{He} \cdot \alpha \cdot n_S = A_C \cdot n_C,$$

$$(1 \cdot 0.9 + 4 \cdot 0.1) \cdot n_S = 12 \cdot n_C, \quad 1.3 \cdot n_S = 12 \cdot n_C, \quad n_S = 9.23 \cdot n_C.$$

Thus,

$$T_C = 0.297 \cdot 9.23\ T_S = 2.74\ T_S.$$

$$T_C = 2.74 \cdot 1.5 \cdot 10^7\ K \approx 4.1 \cdot 10^7\ K.$$

(Since we talk about millions of degrees, especially, about an estimations, it is without importance whether the temperature given is in degrees Celsius or Kelvin.)

Answer: the temperature in the core of the carbon star is about 40 million degrees.

2αβ. Absolutely black cat. The statistically average cat, living in Grignano (Trieste, Italy) weighs about 4 kg and, if considered to be an absolutely black body, radiates basically in the IR part of the spectrum. For simplicity we shall compare the cat to the Sun, which absolute stellar magnitude is equal to $M_S = +4^m.8$. Let us consider that the surface area of a usual 4-kilogram cat is approximately equal to $s = 15\ dm^2 = 1.5 \cdot 10^{-1}\ m^2$, that is essentially less than the surface area of the Sun, $S = 4\pi R^2 = \pi D^2$, where $D = 1392000$ km is the diameter of the Sun, $S = 6.1 \cdot 10^{18}\ m^2$.

$$s/S \approx 2.5 \cdot 10^{-20}.$$

Also the surface temperature of the cat $t = 310$ K is essentially less than the temperature of the solar photosphere $T = 5780$ K.

$$t / T \approx 5.4 \cdot 10^{-2}, \quad (t / T)^4 \approx 8.3 \cdot 10^{-6}.$$

As the general power of radiation is proportional to $S \cdot T^4$, we may obtain, that the ration of the power of radiation of the cat to the power of radiation of the Sun is

$$w / L = (s/S) \times (t / T)^4 \approx 2 \cdot 10^{-25}.$$

That is, the cat radiates approximately $5 \cdot 10^{24}$ times less energy than the Sun. In stellar magnitudes this difference is

$$\Delta M = 5^m/2\ \lg (L/w) = 5^m/2\ \lg (5 \cdot 10^{24}) \approx 61^m.7.$$

Thus, the absolute stellar magnitude of the cat is equal to

$$M_{abc} = M_S + \Delta M = +4^m.8 + 61^m.7 = 66^m.5.$$

Of course, it is estimation only.

3αβ. Great opposition. Oppositions (including Great oppositions) of a planet may be only for observers at the inner planets. There is only one inner planet –Mercury – for the Venus. So any opposition of Venus may be observed only from Mercury.

The flux to Earth from Mars $\qquad F_M \sim \alpha_{Ma} \cdot D_{Ma}^2 \cdot (1/R_{E\text{-}Ma})^2 \cdot (1/R_{S\text{-}Ma})^2.$

(here and below α is albedo, D are diameters of the bodies and R are distances, indexes Ma, E, V, Me and S correspond to Mars, Earth, Venus, Mercury and Sun).

The flux to Mercury from Venus $\qquad F_V \sim \alpha_V \cdot D_V^2 \cdot (1/R_{Me\text{-}V})^2 \cdot (1/R_{S\text{-}V})^2.$

The ratio of fluxes $\qquad F_V/F_M = (\alpha_V/\alpha_{Ma}) \cdot (D_V^2/D_{Ma}^2) \cdot (R_{E\text{-}Ma} \cdot R_{S\text{-}Ma})^2/(R_{Me\text{-}V} \cdot R_{S\text{-}V})^2.$

And what does it mean the word "Great"? The Great oppositions appear for the situation when the distance to opposition planet is smaller than during usual (mean) oppositions. For Mars observing from Earth it is situation when the Mars is in the perihelion of its orbit, as the eccentricity of Mars orbit is quite larger than the eccentricity of Earth.

For Venus observing from Mercury the situation is opposite, the eccentricity of Mercury (observer's planet) orbit is quite larger than the eccentricity of Venus (the planet that observed). So the Great oppositions appear when the Mercury is in the aphelion of its orbit.

It means that for the comparison magnitudes (i.e. fluxes) for these two Great oppositions we have to use values $R_{S\text{-}Ma} = (1\text{-}e)A_{Ma}$ and $R_{S\text{-}Me} = (1\text{+}e)A_{Me}$ for positions of Mars and Mercury and may consider the orbits of Earth and Venus circular, $R_{S\text{-}E} = A_E$, $R_{S\text{-}V} = A_V$.

So the ratio of fluxes

$$F_V/F_M = (\alpha_V/\alpha_{Ma}) \cdot (D_V^2/D_{Ma}^2) \cdot ((1\text{-}e)A_{Ma} - A_E) \cdot (1\text{-}e)A_{Ma})^2/((A_V - (1\text{+}e)A_{Me}) \cdot A_V)^2.$$

Taking the necessary values from the data of Solar system we may calculate

$$F_V/F_M = (0.78/0.15)\cdot(12104/6794)^2\cdot(0.382\cdot1.382)^2/(0.256\cdot0.723)^2.$$

$$F_V/F_M \approx 134.$$

So Venus visible from Mercury on Great opposition is brighter than Mars visible from Earth on Great opposition, and the difference in stellar magnitudes is equal

$$\Delta m = -2^m.5\cdot\lg(F_V/F_M) \approx -5^m.3.$$

Stellar magnitude of Venus

$$m = -2^m.9 + \Delta m \approx -8^m.2.$$

Of course, it is not the only possible correct way for solution.

4αβ. Jump of bear. So, the bear makes a start and flies... On Spitsbergen (that is, on the Earth) the length of jump is equal to

$$L = V_{hor} \times t, \quad t = 2V_{vert} / g,$$

so

$$L = 2\cdot V_{hor}\cdot V_{vert} / g.$$

$$V_{hor} = V_0\cdot\cos\alpha, \; V_{vert} = V_0\cdot\sin\alpha.$$

V_0 here is the initial speed of jump of the bear from the surface, g – acceleration of gravity at the Earth. If we consider that this speed is not depend on the angle α (in general it is not absolutely true, but we shall use this assumption in our model), the maximum length of jump is reached when $\alpha = 45°$,

$$L = V_0^2 / g.$$

So, if L = 8 м, we may obtain, that speed with which the bear makes a start from the Earth is approximately equal to

$$V_0 = (L\cdot g)^{1/2} \approx 9 \text{ m/s}.$$

If this speed will be larger than the second cosmic speed for the asteroid, the bear will depart and surely become an independent object of the Kuiper belt. The second cosmic speed may be used for the solution. Nevertheless, the first cosmic speed is better in the situation. Even, if this speed will be less the second cosmic speed, but larger than the first cosmic speed, it would be quite uncomfortable for the bear after it has jumped of the surface: it will follow an orbit of suborbital flight during many hours.

The relationship between the first cosmic speed and the size and density of the asteroid is

$$V_I^2/R = G\cdot M / R^2 = G\cdot(4/3)\pi\cdot R^3\cdot\rho / R^2,$$

$$V_I^2 = G\cdot(4/3)\pi\cdot R^2\cdot\rho = \pi G D^2\rho/3.$$

The first cosmic speed for an asteroid of the Kuiper belt is equal

$$V_I = D(\pi G\rho/3)^{1/2}.$$

So

$$D = V_I / (\pi G\rho/3)^{1/2}.$$

Taking into account that the planned for settling by bears objects of the Kuiper belt consist basically of ice, their characteristic density is about 900 kg/m^3. So we have that the first cosmic speed V_I is equal to $V_0 = (L\cdot g)^{1/2} = 9$ m/s for the asteroids with diameter of order

$$D = V_0 / (\pi G\rho/3)^{1/2} = (L\cdot g)^{1/2} / (\pi G\rho/3)^{1/2} = (3Lg/\pi G\rho)^{1/2}.$$

$$D \approx 35 \text{ km}.$$

Thus, from the physicists' point of view it is possible to place polar bears comfortably on ice asteroids with diameters more than about 35 km.

Answer: $D > (3Lg/\pi G\rho)^{1/2} \approx 35$ km.

5αβ. Alternative theory. The simplified "direct solution".

The energy of every photon becomes smaller according to the law $E = E_0\cdot 2^{-t/T_0}$, which means that after a time $t = T_0$ the energy becomes smaller by a factor 2. $E_1 = E_0/2$, $h\nu_1 = h\nu_0/2$, $hc/\lambda_1 = (hc/\lambda_0)/2$, so $\lambda_1 = 2\lambda_0$. The theory of an expanding Universe explains it by terms of the Doppler effect due to that the source of the photon is receding from us with the velocity v.

There are two ways of solution presented later: in the classical theory and in the special theory of relativity. The solution according to the special theory of relativity is better but not all students know the formula of this theory *(1 pt difference recommended in the evaluation for Group β, no difference for Group α)*.

According to the classical Doppler effect,

$$\lambda_1 = \lambda_0(1+v/c),$$

so $$2\lambda_0 = \lambda_0(1+v/c), \quad v/c = 1, \quad v = c.$$

In other words the red shift $$z = 1.$$

According to the Hubble law $$v = H\cdot R, \quad R = v/H.$$

Since $v = c$, $$R = c/H.$$

The time passed since the emission of the photon is $t = R/c$, so

$$t = R/c = c/H/c = 1/H.$$

So $$T_0 = t = (1/70)\cdot s\cdot Mpc/km = (1/70)\cdot s\cdot 3.26\cdot 10^6\ years\cdot 3\cdot 10^5\ km/s\ /km \approx 14\cdot 10^9\ years.$$

In the theory of an expanding Universe this period is equal to cosmological age of the Universe.

According to the Doppler effect in special theory of relativity the red shift is also $z = 1$ but

$$\lambda_2 = \lambda_0(1+v/c)/(1-(v/c)^2)^{1/2}.$$

So

$$2\lambda_0 = \lambda_0(1+v/c)/(1-(v/c)^2)^{1/2},$$

$$2\cdot(1-(v/c)^2)^{1/2} = (1+v/c),$$

$$4\cdot(1-(v/c)^2) = (1+v/c)^2,$$

$$4\cdot(1-v/c)\cdot(1+v/c) = (1+v/c)^2,$$

$$4\cdot(1-v/c) = (1+v/c),$$

$$v/c = 0.6.$$

According to the Hubble law $$v = H\cdot R, \quad R = v/H.$$

Since $v = 0.6\cdot c$, $$R = 0.6\cdot c/H.$$

The time passed since the emission of the photon is $t = R/c$, so

$$t = R/c = 0.6\cdot c/H/c = 0.6/H.$$

So $$T_0 = t = 0.6\cdot(1/70)\cdot s\cdot Mpc/km = 0.6\cdot(1/70)\cdot s\cdot 3.26\cdot 10^6\ years\cdot 3\cdot 10^5\ km/s\ /km \approx 8.4\cdot 10^9\ years.$$

In the theory of an expanding Universe this time is roughly the cosmological age of the Universe.

Answer: in both cases (the classical theory or the special theory of relativity) the parameter of the so-called half-decay period of a photon is about 10^{10} (10 billion) years that is of the order of the cosmological age of the Universe according to the theory of an expanding Universe.

Solution with derivatives. This method is more precise than the "direct solution" but the math apparatus is not possible for most students.

For small values, if $\Delta\lambda \ll \lambda_0$ using derivatives of parts of equation ($E = E_0\cdot 2^{-t/T_0}$) one can find:

$$dE = d(E_0\cdot 2^{-t/T_0}) = dt\cdot(-\ln 2/T_0)\cdot(E_0\cdot 2^{-t/T_0}) = dt\cdot(-\ln 2/T_0)\cdot E,$$

$$dE/E = -\ln 2\cdot dt/T_0.$$

Using a small time t in comparison with T_0 we may write

$$\Delta E/E = -\ln 2\cdot t/T_0.$$

Since

$$\Delta E/E = -\Delta\lambda/\lambda = -v/c$$

$$-v/c = -\ln 2\cdot t/T_0.$$

According to the Hubble law $$v = H\cdot R,$$

so $$H\cdot R/c = \ln 2\cdot t/T_0.$$

Since the time passed since the emission of the photon is $t = R/c$,

finally we have $$T_0 = \ln 2/H.$$

So $$T_0 = \ln 2\cdot(1/70)\cdot s\cdot Mpc/km = 0.69\cdot(1/70)\cdot s\cdot 3.26\cdot 10^6\ years\cdot 3\cdot 10^5\ km/s\ /km =$$

$$= 9.7\cdot 10^9\ years \approx 10\cdot 10^9\ years.$$

In the theory of an expanding Universe this period is about cosmological age of the Universe.

Answer: the parameter of the so-called half-decay period of a photon is about 10^{10} (10 billion) years that is of order of cosmological age of the Universe according to the theory of expanding Universe.

Note: students of group β should take the Hubble constant H_0 themselves. It may be in the range from 50 till 100 km/s/Mpc. The answers for these values will appear $\sqrt{2}$ larger or $\sqrt{2}$ smaller.

Note: the results are depend on the objects (or times) that we use as references (like for the nearest universe (solution with derivatives) or objects with the red shift 1). Nevertheless it does mean that "the situation is impossible". It means only that at least one of the formulae (formula of Hubble $v = HR$ or formula of aging of photons with a "half-decay" $E = E_0\cdot 2^{-t/T_0}$) is not coincide reality.